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On dilation and commuting liftings of n -tuples of commuting Hilbert space contractions

Abstract. The n -tuples of commuting Hilbert space contractions are considered. We give a model of a commuting lifting of one contraction and investigate conditions under which a commuting lifting theorem holds for an n -tuple. A series of such liftings leads to an isometric dilation of the n -tuple. The method is tested on some class of triples motivated by Parrotts example. It provides also a new proof of the fact that a positive definite n -tuple has an isometric dilation.

1. Introduction

By the dilation theory, started by Szökefalvi-Nagy, every contraction has a unitary dilation. The dilation provides a simpler proof of the von-Neumann inequality. It is a bit striking that the result does not extend to n -tuples of (commuting) operators. Precisely, a single contraction on a Hilbert space has the minimal, unique up to isomorphism, unitary dilation (Nagy [15, 16]), a pair of commuting contractions has a minimal unitary dilation but it is not necessarily unique (Andô [1]), while for $n \geq 3$ a unitary dilation may not exist (Parrott [11], Varopoulos [17]). More precisely, the dilation does exist but it may fail to commute. Parrott gave the first example of a triple of commuting contractions not admitting a (commuting) unitary dilation. Varopoulos showed that for any $n \geq 3$ the von Neumann inequality may not be satisfied and hence the dilation may not exist. On the other hand, an example of four contractions not admitting a unitary dilation and satisfying von-Neumann inequality can be found in [5].

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The dilation theory is well developed, to mention for example results on row contractions. We refer the interested reader for example to [12, Chap. 4, 5]. On the other hand, the theory still remains of interest as indicated by the following list of recent results [2, 3, 4, 6, 7, 9, 13, 14].

The paper is devoted to the problem of existence of an isometric (equivalently unitary) dilation of three or more contractions. We approach the problem via commuting liftings. We give an equivalent condition for a triple to satisfy commuting lifting theorem (i.e. there is a commuting triple consisting of an isometric dilation of one operator and liftings of the two remaining). The result may be used for n -tuples. The condition is not simple enough to be treated as a solution of the problem. However, we show some examples in which it works well. In particular, we apply it to some class of triples motivated by the Parrott's example. Moreover, we show that a positive definite n -tuple has a unitary dilation. Such a result is known, but our proof gives a direct construction of the dilation.

2. Preliminaries

Let $\mathcal{B}(\mathcal{H}, \mathcal{H}')$ denote the space of all bounded linear operators from \mathcal{H} to \mathcal{H}' , where $\mathcal{H}, \mathcal{H}'$ are Hilbert spaces, and let $\mathcal{B}(\mathcal{H}) = \mathcal{B}(\mathcal{H}, \mathcal{H})$ be the algebra of bounded linear operators on \mathcal{H} . A subspace $\mathcal{H}_0 \subset \mathcal{H}$ is a closed linear manifold. Let $P_{\mathcal{H}_0}$ stand for the orthogonal projection onto \mathcal{H}_0 (i.e. $P_{\mathcal{H}_0}^2 = P_{\mathcal{H}_0} = P_{\mathcal{H}_0}^*$).

Let $\mathbf{T} = (T_1, \dots, T_n)$ be an n -tuple of commuting contractions, where $T_i \in \mathcal{B}(\mathcal{H})$, and let $\mathbf{T}^\alpha = \prod_{i=1}^n T_i^{\alpha_i}$, where $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n$ and $T_i^{\alpha_i} = T_i^{*\lvert\alpha_i\rvert}$ for $\alpha_i < 0$. The n -tuple $\mathbf{U} = (U_1, \dots, U_n)$ on a Hilbert space \mathcal{K} is called a *dilation* of \mathbf{T} if and only if

- (i) $\mathcal{H} \subset \mathcal{K}$,
- (ii) \mathbf{U} is a commuting n -tuple,
- (iii) $\mathbf{T}^\alpha = P_{\mathcal{H}} \mathbf{U}^\alpha|_{\mathcal{H}}$ for any $\alpha \in (\mathbb{Z}_+ \cup \{0\})^n$.

If the projection in condition (iii) may be canceled (i.e. $\mathbf{T}^\alpha = \mathbf{U}^\alpha|_{\mathcal{H}}$) the dilation is called an *extension*. A dilation is called *isometric* or *unitary* if operators in \mathbf{U} are of the respective type. Since an n -tuple of isometries admits a unitary extension, an arbitrary n -tuple of commuting contractions admits a unitary dilation if and only if it admits an isometric dilation.

Note that a dilation is assumed to be a commuting n -tuple. If n -tuple fails to admit a unitary dilation it is due to the commutativity requirement.

A unitary dilation is called *regular* if $\mathbf{T}^{*\alpha_-} \mathbf{T}^{\alpha_+} = P_{\mathcal{H}} \mathbf{U}^\alpha|_{\mathcal{H}}$ for any $\alpha \in \mathbb{Z}^n$, where $\alpha_- = (-\min\{0, \alpha_1\}, \dots, -\min\{0, \alpha_n\})$, $\alpha_+ = (\max\{0, \alpha_1\}, \dots, \max\{0, \alpha_n\})$. The n -tuple \mathbf{T} is called *positive definite* if

$$\sum_{v \subset u} (-1)^{|\alpha(v)|} \mathbf{T}^{*\alpha(v)} \mathbf{T}^{\alpha(v)} \geq 0$$

for any $u \subset \{1, \dots, n\}$, where $\alpha(v) = (\chi_v(1), \dots, \chi_v(n))$, $|\alpha(v)| = \sum_{i=1}^n \chi_v(i)$ and χ_v is the characteristic function of $v \subset \{1, \dots, n\}$. An n -tuple admits a regular dilation if and only if it is positive definite [16, Theorem 9.1, Chap.1].

As we mentioned, a contraction $T \in \mathcal{B}(\mathcal{H})$ admits a unique isometric dilation where uniqueness is up to the unitary equivalence. For the model of such dilation recall that $D_T = \sqrt{I - T^*T}$ and $\mathcal{D}_T = \overline{\text{ran}(D_T)}$ are called the *defect operator* and the *defect space of a contraction* T , respectively. Let $\mathbb{D} \subset \mathbb{C}$ be the unit disk with boundary \mathbb{T} and let $L^2(\mathbb{T})$ and $H^2(\mathbb{T})$ denote the space of scalar valued, square integrable functions and the Hardy space, respectively. The space of square integrable functions valued in a separable Hilbert space \mathcal{H} with the inner product induced by the norm $\|f\| = (\int_{\mathbb{T}} \|f(z)\|_{\mathcal{H}}^2 dm(z))^{1/2}$ (m - normalized Lebesgue measure) is unitarily equivalent to $L^2(\mathbb{T}) \otimes \mathcal{H}$ and is denoted with this symbol. Similarly, $H^2(\mathbb{T}) \otimes \mathcal{H}$ denotes the space of analytic, square integrable, \mathcal{H} -valued functions. Set $P_n \in \mathcal{B}(H^2(\mathbb{T}))$ for the projection onto the subspace $\mathbb{C}z^n$ and $P_n \otimes I$ for the projection onto $z^n \otimes \mathcal{H}$ for $n \geq 0$, where z^n stands for the monomial $\{z \mapsto z^n\}$ in this context. In particular, $1 \otimes \mathcal{H}$ denotes constant functions in $H^2(\mathbb{T}) \otimes \mathcal{H}$. Let then

$$\mathcal{K}_T = \mathcal{H} \oplus (H^2(\mathbb{T}) \otimes \mathcal{D}_T)$$

be the Cartesian product Hilbert space with $\langle \cdot, \cdot \rangle_{\mathcal{K}_T} = \langle \cdot, \cdot \rangle_{\mathcal{H}} + \langle \cdot, \cdot \rangle_{H^2(\mathbb{T}) \otimes \mathcal{D}_T}$ and the respective norms. By the Nagy result the minimal isometric dilation of T may be defined as an operator on \mathcal{K}_T given by the matrix

$$V_T = \begin{bmatrix} T & 0 \\ ED_T & M_z \end{bmatrix}, \tag{1}$$

where $E: \mathcal{D}_T \ni h \mapsto 1 \otimes h \in H^2(\mathbb{T}) \otimes \mathcal{D}_T$ is the embedding operator and M_z is an operator of multiplication by the independent variable.

An operator $L_A \in \mathcal{B}(\mathcal{K}, \mathcal{K}')$ is called a *lifting* of $A \in \mathcal{B}(\mathcal{H}, \mathcal{H}')$ if $P_{\mathcal{H}'}L_A = AP_{\mathcal{H}}$, where $\mathcal{H} \subset \mathcal{K}$ and $\mathcal{H}' \subset \mathcal{K}'$. Note that $\|A\| \leq \|L_A\|$. Let us recall the Nagy-Foias lifting theorem [16, Theorem 2.3, Chap. 2]

THEOREM 2.1

Let T, T' be contractions on the Hilbert spaces $\mathcal{H}, \mathcal{H}'$, and let $V_T, V_{T'}$ be their minimal isometric dilations on the spaces $\mathcal{K}, \mathcal{K}'$, respectively. For every bounded operator $S \in \mathcal{B}(\mathcal{H}, \mathcal{H}')$ satisfying

$$T'S = ST$$

there exists a bounded operator $C_S \in \mathcal{B}(\mathcal{K}, \mathcal{K}')$ satisfying the conditions:

- (i) $V_{T'}C_S = C_SV_T$,
- (ii) $S = P_{\mathcal{H}'}C_S|_{\mathcal{H}}$,
- (iii) $S^* = C_S^*|_{\mathcal{H}'}$,
- (iv) $\|S\| = \|C_S\|$.

By (ii) and (iii) the operator C_S is a lifting of S .

Parrott noticed in [11] that the lifting theorem is equivalent to the existence of a unitary dilation for a pair of contractions. Indeed, Theorem 2.1 for $\mathcal{H} = \mathcal{H}'$ and $T = T' \in \mathcal{B}(\mathcal{H})$ provides a lifting of any operator in the commutant of T . By conditions (i) - (iv) of Theorem 2.1 the pair (V_T, C_S) is a dilation of (T, S) . It is

not yet an isometric dilation, but V_T is an isometry. However, if we lift V_T with respect to the isometric dilation of C_S , since a lifting of an isometry is an isometry, we get a pair of isometries (C_{V_T}, V_{C_S}) , so an isometric dilation of (T, S) .

A similar construction could be used for an n -tuple of commuting contractions provided the respective contractions can be lifted to commuting operators. Indeed, for a given n -tuple of commuting contractions $\mathbf{T} = (T_1, \dots, T_n)$ let $k_0 \leq n + 1$ be the maximal integer, such that T_i is an isometry for $i < k_0$ ($k_0 = 1$ if T_1 is not an isometry). Let $\mathbf{C}_{\mathbf{T}} = (C_{T_1}, \dots, C_{T_n})$ consist of an isometric dilation of T_{k_0} and corresponding commuting liftings of the remaining contractions. Since a lifting of an isometry is an isometry, C_{T_i} are isometries for $i < k_0 + 1$, so at least C_{T_1} is an isometry. If k_1 is the maximal integer such that C_{T_i} are isometries for $i < k_1$, then $k_0 + 1 \leq k_1$. If we repeat the construction with $\mathbf{C}_{\mathbf{T}}$ we get $\mathbf{C}_{\mathbf{C}_{\mathbf{T}}}$ which is a lifting of $\mathbf{C}_{\mathbf{T}}$, so also a lifting of \mathbf{T} and $C_{C_{T_i}}$ are isometries for $i < k_1 + 1$, so at least $C_{C_{T_1}}, C_{C_{T_2}}$ are isometries. Repeating the construction at most n -times we get an n -tuple of commuting isometries which is an isometric dilation of \mathbf{T} . However, the lifting theorem does not extend to n -tuples of operators. For $S, R \in \mathcal{B}(\mathcal{H})$ commuting with T there are C_S, C_R by Theorem 2.1 but not necessarily commuting, which is required in the construction of an isometric dilation described above. Another idea is to construct a dilation of a pair and add a lifting of the third operator afterwards. However, if we replace T, T' by n -tuples \mathbf{T}, \mathbf{T}' the operator C_S may not exist even if \mathbf{T}, \mathbf{T}' admit regular dilations - see [10] for details.

Summing up, the problem of the existence of a dilation (isometric, unitary) reduces to the problem of the existence of commuting liftings. More precisely, we focus on conditions under which the lifting theorem holds for an n -tuple of contractions to be understood as the existence of commuting liftings of $n - 1$ contractions with respect to the dilation of the remaining one. A triple is representative for n -tuples.

3. A lifting of a contraction

The lifting theorem does not hold for an arbitrary n -tuple. However, for an n -tuple for which commuting liftings exist and the liftings may be lifted to commuting contractions and so on, there is an isometric dilation of the n -tuple as it was described in the last but one paragraph of the previous section. Lifting C_S may be constructed of the same norm as S by Theorem 2.1. However, for the purposes of the construction of an isometric dilation it is enough if C_S is a contraction. Hence, if needed, we may use a wider class of liftings than constructed in the proof of Theorem 2.1. Let us define:

DEFINITION 3.1

Let $T, S \in \mathcal{B}(\mathcal{H})$ be two commuting contractions and $V_T \in \mathcal{B}(\mathcal{K})$ be the minimal isometric dilation of T . The operator C_S satisfying conditions

- (i) $V_T C_S = C_S V_T$,
- (ii) $S = P_{\mathcal{H}} C_S|_{\mathcal{H}}$,
- (iii) $S^* = C_S^*|_{\mathcal{H}}$,

(iv) $\|C_S\| \leq 1$.

is called a *contractive lifting* of S with respect to V_T or simply a contractive lifting of S if the operator V_T is clear.

The starting point for a construction of C_S is a characterization of 2 by 2 block matrices in [8, Chap. 4]. Since $\mathcal{H} \subset \mathcal{K}$, we may assume that $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}_1$ and

$$C_S = \begin{bmatrix} S & 0 \\ A & B \end{bmatrix}.$$

Indeed, the matrix of C_S is lower triangular by $S^* = C_S^*|_{\mathcal{H}}$. Since C_S is a contraction, by [8, Theorem 3.1, Chap. 4] and C_S is lower triangular we get

$$C_S = \begin{bmatrix} S & 0 \\ YD_S & D_{Y^*}\Gamma \end{bmatrix}, \quad (2)$$

where $Y \in \mathcal{B}(\mathcal{D}_S, \mathcal{H}_1)$, $\Gamma \in \mathcal{B}(\mathcal{H}_1, \mathcal{D}_{Y^*})$ are contractions. Moreover, by [8, Lemma 1.1, Corollary 3.6, Chap. 4] it follows that for any contraction Y there is a contraction Γ such that C_S is a contraction. Unfortunately, such C_S does not necessarily commute with V_T . In Lemma 3.2 and Example 3.4 we show that for a given Y there may not exist Γ such that C_S given by (2) commutes with V_T , but if such Γ exists, then it is unique. Hence any contractive lifting C_S is determined by the contraction $Y \in \mathcal{B}(\mathcal{D}_S, \mathcal{H}_1)$ and may be denoted by $C_{S,Y}$. In other words, there is a one-to-one correspondence between contractive liftings of S commuting with V_T and some subfamily of contractions in $\mathcal{B}(\mathcal{D}_S, \mathcal{H}_1)$.

LEMMA 3.2

Let $S, T \in \mathcal{B}(\mathcal{H})$ be a pair of commuting contractions and let V_T be an isometric dilation of T as in (1). Any lifting of S commuting with V_T is of the form

$$C_{S,Y} = \begin{bmatrix} S & 0 \\ YD_S & S_Y \end{bmatrix}, \quad (3)$$

where $Y \in \mathcal{B}(\mathcal{D}_S, H^2(\mathbb{T}) \otimes \mathcal{D}_T)$ determines S_Y by

$$S_Y(z^n \otimes D_T h) = M_z^n (ED_T S + M_z Y D_S - Y D_S T) h \quad (4)$$

for any $n \geq 0$, $h \in \mathcal{H}$.

To be clear, the lemma considers an existing lifting and investigates its form. Hence (4) describes the existing contraction S_Y on a dense subset of its domain. By the continuity, S_Y is determined by (4) on the whole domain.

Proof. By (2) only the formula for S_Y needs an explanation. By the commutativity $C_{S,Y} V_T = V_T C_{S,Y}$ we get

$$\begin{bmatrix} ST & 0 \\ YD_S T + S_Y E D_T & S_Y M_z \end{bmatrix} = \begin{bmatrix} TS & 0 \\ E D_T S + M_z Y D_S & M_z S_Y \end{bmatrix}.$$

By the equality of (2,1) entries the operator S_Y is defined on $1 \otimes \mathcal{D}_T$ by

$$S_Y ED_T = ED_T S + M_z Y D_S - Y D_S T. \quad (5)$$

However, by equality of (2,2) entries S_Y commutes with M_z . Hence

$$\begin{aligned} S_Y(z^n \otimes D_T h) &= S_Y M_z^n (1 \otimes D_T h) = M_z^n S_Y (ED_T h) \\ &= M_z^n (ED_T S + M_z Y D_S - Y D_S T) h. \end{aligned}$$

We have already emphasized that Lemma 3.2 does not show the existence of a contractive lifting, but only investigates its form. Indeed, if we take an arbitrary Y and try to use (4) as a definition of S_Y two problems appear. The first one is that S_Y acts on $H^2(\mathbb{T}) \otimes \mathcal{D}_T$, so among others on vectors of the form $z^n \otimes D_T h$ for $h \in \mathcal{H}$, while the right hand side of (4) acts directly on $h \in \mathcal{H}$. Hence (4) properly defines a linear operator if the right hand side does not depend on the choice of $h \in \mathcal{H}$ representing a certain $k \in D_T \mathcal{H}$. In other words, the right hand side of (4) has to vanish on $\ker D_T$. Then S_Y is a properly defined operator on $z^n \otimes D_T \mathcal{H}$ and as the right hand side is a bounded, so a continuous operator, the definition extends to $z^n \otimes \mathcal{D}_T$. However, this does not imply that S_Y defined by (4) is a bounded operator, while it should be a contraction. Indeed, the norm of a value of the right hand side operator in (4) on some h is bounded by $\beta \|h\|$ for some constant $\beta > 0$. For boundedness of S_Y the considered norm should be bounded by $\delta \|z^n \otimes D_T h\|_{H^2(\mathbb{T}) \otimes \mathcal{D}_T} = \delta \|D_T h\|$ for some $\delta > 0$. This does not follow if D_T is not bounded below.

Summing up, Lemma 3.2 does not provide a proof independent on Theorem 2.1 that there is Y properly defining a contractive lifting. However, since by Theorem 2.1 a lifting of S exists, by Lemma 3.2 it has to be of the form (3) and so there is at least one operator Y properly defining S_Y and so $C_{S,Y}$.

REMARK 3.3

A contraction Y properly defines by (4) a linear operator S_Y if and only if

$$(ED_T S + M_z Y D_S - Y D_S T)|_{\ker D_T} = 0.$$

Then the lifting $C_{S,Y}$ is a contraction if and only if S_Y is a contraction.

In the following example $Y = 0$ does not define a lifting. In Remark 3.7 we describe all contractions Y defining a lifting for this example. It is inspired by Parrotts example [11]. The example is the case when $ED_T S + M_z Y D_S - Y D_S T$ does not vanish on $\ker D_T$.

EXAMPLE 3.4

Let $T, S \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ be such that

$$T = \begin{bmatrix} 0 & 0 \\ V & 0 \end{bmatrix}, \quad S = \begin{bmatrix} 0 & 0 \\ B & 0 \end{bmatrix},$$

where $V, B \in \mathcal{B}(\mathcal{H})$, V is an isometry, $B \neq 0$ is a contraction. Then $TS = ST = 0$ and

$$D_T = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}, \quad D_S = \begin{bmatrix} D_B & 0 \\ 0 & I \end{bmatrix}.$$

Note that $\ker D_T = \{(h, 0) : h \in \mathcal{H}\}$ is not invariant under S since $S(h, 0) = (0, Bh) \notin \ker D_T$ for $h \notin \ker B$. Hence and by (5) for $Y = 0$ we get

$$S_0 0 = S_0 E D_T(h, 0) = E D_T S(h, 0) = 1 \otimes (0, Bh) \neq 0,$$

so S_0 is not a linear operator.

From (2) a contractive lifting $C_{S,Y}$ is an extension of S if and only if $Y D_S = 0$. Since \mathcal{D}_S is the domain of Y , the latter means that $Y = 0$. Hence, the extension of S with respect to V_T is $C_{S,0}$ and it exists if and only if S_0 defined by (4) is a well defined contraction. In particular, if S is an isometry, then $\mathcal{D}_S = \{0\}$ and so $Y = 0$ is the only choice. Let us investigate conditions under which an arbitrary contraction S commuting with T admits an extension with respect to V_T .

PROPOSITION 3.5

Let $S, T \in \mathcal{B}(\mathcal{H})$ be commuting contractions and let V_T be the minimal unitary dilation of T . A contraction S admits a contractive extension with respect to V_T if and only if $\|D_T S h\| \leq \|D_T h\|$ for any $h \in \mathcal{H}$.

Moreover, the extension is of the form

$$\begin{bmatrix} S & 0 \\ 0 & I \otimes \tilde{S} \end{bmatrix},$$

where $\tilde{S} \in \mathcal{B}(\mathcal{D}_T)$ is the continuous extension of $\{D_T h \mapsto D_T S h\}$.

Proof. By Lemma 3.2 the extension is given by

$$C_{S,0} = \begin{bmatrix} S & 0 \\ 0 & S_0 \end{bmatrix},$$

where, by (4), $S_0(z^n \otimes D_T h) = z^n \otimes D_T S h = (I \otimes \tilde{S})(z^n \otimes D_T h)$. Hence S_0 is a well defined contraction if and only if \tilde{S} is a well defined contraction, which in turn is equivalent to $\|D_T S h\| \leq \|D_T h\|$ for any $h \in \mathcal{H}$. This fact and Remark 3.3 finish the proof.

In particular, the only lifting of an isometry is an extension.

COROLLARY 3.6

An isometry S commuting with a contraction T admits a unique lifting with respect to V_T and the lifting is an isometry. The lifting is an extension of the form

$$\begin{bmatrix} S & 0 \\ 0 & I \otimes \tilde{S} \end{bmatrix},$$

where $\tilde{S} \in \mathcal{B}(\mathcal{D}_T)$ is the continuous extension of $\{D_T h \mapsto D_T S h\}$.

Proof. Since S is an isometry, $D_S = 0$ and by (2) the only lifting is an extension. Recall that $\|D_T h\|^2 = \|h\|^2 - \|T h\|^2$. Hence $\|D_T h\| = \|D_T S h\|$ and so \tilde{S} is a well defined isometry. Since S and \tilde{S} are isometries, C_S is an isometry.

Since not all contractions Y create a commuting lifting, it is natural to ask which of them do. Let us investigate Y more thoroughly. Since Y maps \mathcal{D}_S to $H^2(\mathbb{T}) \otimes \mathcal{D}_T$, there is a natural decomposition $Y = \sum_{n=0}^{\infty} P_{z^n \otimes \mathcal{D}_T} Y$. On the other hand, $P_{z^n \otimes \mathcal{D}_T} Y: \mathcal{D}_S \mapsto z^n \otimes \mathcal{D}_T$ may be identified with an operator $Y_n: \mathcal{D}_S \mapsto \mathcal{D}_T$. Then $P_{z^n \otimes \mathcal{D}_T} Y = z^n \otimes Y_n$ and we get

$$Y = \sum_{n=0}^{\infty} z^n \otimes Y_n, \quad (6)$$

where the convergence is in the strong operator topology. Subsequently (4) may be reformulated as

$$S_Y(z^n \otimes D_T h) = z^n \otimes (D_T S - Y_0 D_S T)h + \sum_{i=1}^{\infty} z^{i+n} \otimes (Y_{i-1} D_S - Y_i D_S T)h. \quad (7)$$

Let us describe liftings of S in Example 3.4.

REMARK 3.7

Let T, S be as in Example 3.4 and let V_T be the isometric dilation of T given by (1). Our aim is to describe all possible liftings of S . By Lemma 3.2 it is enough to describe all possible contractions $Y \in \mathcal{B}(\mathcal{D}_S, H^2(\mathbb{T}) \otimes \mathcal{D}_T)$ generating liftings.

Recall that

$$\mathcal{D}_T = \ker T = \{0\} \oplus \mathcal{H}, \quad D_T = P_{\{0\} \oplus \mathcal{H}}$$

and

$$\mathcal{D}_S = \mathcal{D}_B \oplus \mathcal{H}, \quad \mathcal{D}_T \subset \ker S, \quad D_S = \begin{bmatrix} D_B & 0 \\ 0 & I \end{bmatrix}.$$

Hence, by (7), we get

$$\begin{aligned} S_Y(z^n \otimes (0, k)) &= S_Y(z^n \otimes D_T(0, k)) = \sum_{i=1}^{\infty} z^{i+n} \otimes Y_{i-1} D_S(0, k) \\ &= M_z^{n+1} \sum_{i=0}^{\infty} z^i \otimes Y_i(0, k) = M_z^{n+1} Y(0, k) \end{aligned} \quad (8)$$

for an arbitrary contraction Y generating a lifting. On the other hand, $S_Y(z^n \otimes D_T(h, 0)) = 0$ and $D_S T(h, 0) = (0, Vh)$, which, by (7), implies

$$Y_0(0, Vh) = Y_0 D_S T(h, 0) = D_T S(h, 0) = D_T(0, Bh) = (0, Bh)$$

and

$$Y_i(0, Vh) = Y_i D_S T(h, 0) = Y_{i-1} D_S(h, 0) = Y_{i-1}(D_B h, 0)$$

for $i \geq 1$ and any $h \in \mathcal{H}$. Hence we get

$$\begin{aligned} Y(0, Vh) &= \sum_{i=0}^{\infty} z^i \otimes Y_i(0, Vh) = 1 \otimes (0, Bh) + \sum_{i=1}^{\infty} z^i \otimes Y_{i-1}(D_B h, 0) \\ &= 1 \otimes (0, Bh) + M_z Y(D_B h, 0). \end{aligned}$$

Summing up,

$$\begin{aligned} Y(h, k) &= Y(h, 0) + Y(0, k) = Y(h, 0) + Y(0, VV^*k) + Y(0, P_{\ker V^*}k) \\ &= Y(h, P_{\ker V^*}k) + 1 \otimes (0, BV^*k) + M_z Y(D_B V^*k, 0). \end{aligned}$$

In other words, an arbitrary contraction Y defining a lifting of S is determined by its restriction $Y|_{\mathcal{D}_B \oplus \ker V^*}$. However, an arbitrary contraction $\tilde{Y}: \mathcal{D}_B \oplus \ker V^* \mapsto H^2(\mathbb{T}) \otimes \mathcal{D}_T$ defines a bounded operator $Y: \mathcal{D}_B \otimes \mathcal{H} \mapsto H^2(\mathbb{T}) \otimes \mathcal{D}_T$,

$$Y(h, k) = \tilde{Y}(h, P_{\ker V^*}k) + 1 \otimes (0, BV^*k) + M_z \tilde{Y}(D_B V^*k, 0), \quad (9)$$

which may not be a contraction. More precisely, if $\|B\| < 1$, then \tilde{Y} of sufficiently small norm (any contraction multiplied by a sufficiently small constant) generates by (9) a contraction Y . On the other hand, since $1 \otimes (0, BV^*k)$ is orthogonal to $M_z \tilde{Y}(D_B V^*k, 0)$, we get

$$\begin{aligned} &\|1 \otimes (0, BV^*k) + M_z \tilde{Y}(D_B V^*k, 0)\|_{H^2(\mathbb{T}) \otimes \mathcal{D}_T}^2 \\ &= \|1 \otimes (0, BV^*k)\|_{H^2(\mathbb{T}) \otimes \mathcal{D}_T}^2 + \|M_z \tilde{Y}(D_B V^*k, 0)\|_{H^2(\mathbb{T}) \otimes \mathcal{D}_T}^2 \\ &\leq \|(0, BV^*k)\|_{\mathcal{H} \oplus \mathcal{H}}^2 + \|(D_B V^*k, 0)\|_{\mathcal{H} \oplus \mathcal{H}}^2 \\ &= \|BV^*k\|_{\mathcal{H}}^2 + \|D_B V^*k\|_{\mathcal{H}}^2 = \|V^*k\|_{\mathcal{H}}^2 \\ &= \|(0, V^*k)\|_{\mathcal{H} \oplus \mathcal{H}}^2 \end{aligned}$$

and the estimate may not be improved if $\|B\| = 1$ as $\text{ran } V^* = \mathcal{H}$. Consequently,

$$\begin{aligned} \|Y(h, k)\|_{H^2(\mathbb{T}) \otimes \mathcal{D}_T} &\leq \|\tilde{Y}(h, P_{\ker V^*}k)\|_{H^2(\mathbb{T}) \otimes \mathcal{D}_T} + \|(0, V^*k)\|_{\mathcal{H} \oplus \mathcal{H}} \\ &\leq \|\tilde{Y}\| \|(h, P_{\ker V^*}k)\|_{\mathcal{H} \oplus \mathcal{H}} + \|(0, VV^*k)\|_{\mathcal{H} \oplus \mathcal{H}} \\ &\leq (1 + \|\tilde{Y}\|^2)^{1/2} \|(h, k)\|_{\mathcal{H} \oplus \mathcal{H}}. \end{aligned}$$

Hence, if $\|B\| = 1$, then Y may not be a contraction even if $\|\tilde{Y}\|$ is very small.

By a direct calculation one may check that any operator of the form (9) satisfies

$$(ED_T S + M_z Y D_S - Y D_S T)(h, 0) = 0.$$

Hence, since $\ker D_T = \mathcal{H} \oplus \{0\}$ and by (8) we get by Remark 3.3 a one-to-one correspondence between liftings of S and contractions in $\mathcal{B}(\mathcal{D}_B \oplus \ker V^*, H^2(\mathbb{T}) \otimes \mathcal{D}_T)$ such that (9) defines a contraction.

Let us give a few hints how to find the proper contractions \tilde{Y} . The case $\|B\| < 1$ was explained. For an arbitrary contraction B , $\tilde{Y} = 0$ generates $Y \neq 0$, which in turn generates the lifting $C_{S, Y}$ which is not an extension. The other way is to get orthogonality of summands in (9), which may be obtained using decomposition (6) of \tilde{Y} . For example, let $\tilde{Y}_n = 0$ for odd n or simply $\tilde{Y}_n = 0$ for all n except one $n_0 \neq 0$.

4. Triples

In Section 2 there was described a construction of an isometric dilation of an n -tuple, which in the case of the triple $R, S, T \in \mathcal{B}(\mathcal{H})$ starts with the dilation of T to the isometry V_T and liftings of R, S to contractions C_R, C_S commuting with V_T . The problem that appeared was that C_R, C_S may not commute with each other and if so, the further construction fails. We investigate the condition under which C_R, C_S commute.

PROPOSITION 4.1

Let $R, S, T \in \mathcal{B}(\mathcal{H})$ be a triple of commuting contractions and let

$$C_{R,X} = \begin{bmatrix} R & 0 \\ XD_R & R_X \end{bmatrix}, \quad C_{S,Y} = \begin{bmatrix} S & 0 \\ YD_S & S_Y \end{bmatrix}, \quad V_T = \begin{bmatrix} T & 0 \\ ED_T & M_z \end{bmatrix}$$

be the liftings of R, S and an isometric dilation of T , respectively.

Operators $C_{R,X}, C_{S,Y}$ commute if and only if

$$YD_S R + S_Y X D_R = X D_R S + R_X Y D_S. \quad (10)$$

Proof. The commutativity between $C_{R,X}$ and $C_{S,Y}$ leads to the equality

$$\begin{bmatrix} SR & 0 \\ YD_S R + S_Y X D_R & S_Y R_X \end{bmatrix} = \begin{bmatrix} RS & 0 \\ X D_R S + R_X Y D_S & R_X S_Y \end{bmatrix}.$$

The equality of entries (2, 1) yields the condition (10) to be necessary. To show it is a sufficient condition we need to show that if it holds, then $R_X S_Y = S_Y R_X$. However, since R_X, S_Y commute with M_z it is enough to show commutativity on $1 \otimes \mathcal{D}_T$. Note that $1 \otimes \mathcal{D}_T = \overline{\text{ran}(ED_T)}$. On the other hand, from (5) we get

$$\begin{aligned} R_X S_Y ED_T &= R_X (ED_T S + M_z Y D_S - Y D_S T) \\ &= R_X ED_T S + R_X M_z Y D_S - R_X Y D_S T \\ &= (ED_T R + M_z X D_R - X D_R T) S + M_z R_X Y D_S - R_X Y D_S T \\ &= ED_T RS + M_z (X D_R S + R_X Y D_S) - (X D_R S + R_X Y D_S) T \end{aligned}$$

and, similarly,

$$S_Y R_X ED_T = ED_T SR + M_z (Y D_S R + S_Y X D_R) - (Y D_S R + S_Y X D_R) T.$$

Hence, indeed (10) yields $R_X S_Y ED_T = S_Y R_X ED_T$.

Let us now describe precisely a dilation of the triple (R, S, T) via liftings. We start with the dilation to a pair of contractions and an isometry (C_R, C_S, V_T) . Next denote by V_{C_S} the isometric dilation of C_S . Since V_T is an isometry, by Corollary 3.6 it admits the extension C_{V_T} with respect to V_{C_S} . Assume that C_R may be lifted to a contraction C_{C_R} with respect to V_{C_S} commuting with C_{V_T} . Then we get the commuting triple $(C_{C_R}, V_{C_S}, C_{V_T})$. Since V_{C_S}, C_{V_T} are isometries, they admit by Corollary 3.6 extensions with respect to $V_{C_{C_R}}$ – the isometric dilation

of C_{C_R} . Hence $(V_{C_{C_R}}, C_{V_{C_S}}, C_{C_{V_T}})$ is the isometric dilation of the triple (R, S, T) , where the operators commute by Proposition 4.1. Indeed, extensions are defined by $X = 0, Y = 0$, so they satisfy the condition (10). In other words, the dilation is constructed in three steps where the third one is always possible.

The second step of the construction makes the case when one contraction is an isometry interesting.

COROLLARY 4.2

The triple $R, S, T \in \mathcal{B}(\mathcal{H})$ of commuting contractions, where R is an isometry, admits a unitary dilation if and only if there is a contraction $Y = \sum_{n=0}^{\infty} z^n \otimes Y_n$ (each $Y_n: \mathcal{D}_S \mapsto \mathcal{D}_T$) defining a lifting of S with respect to a dilation of T as described in Lemma 3.2 and such that $Y_n D_S R = \tilde{R} Y_n D_S$ for each n , where $\tilde{R}: D_T h \mapsto D_T R h$ is an isometry.

Proof. With the notation of Proposition 4.1, by Corollary 3.6 we have $X = 0$ and $R_X = I \otimes \tilde{R}$, where $\tilde{R}: D_T h \mapsto D_T R h$ is an isometry. Since $D_R = 0$, the condition (10) simplifies to

$$Y D_S R = (I \otimes \tilde{R}) Y D_S.$$

The remaining part follows by the decomposition of Y .

As mentioned before, the last step of the described construction of an isometric dilation of a triple, which under the assumptions of Corollary 4.2 is the second step, is always possible. This follows from Corollary 3.6 and the fact that extensions always commute. Indeed, in this case we get an isometric dilation of a contraction and extensions of two remaining isometries.

The interesting case is when the triple admits a dilation to an isometry and liftings to contractions by some X, Y , where $X \neq 0$ or $Y \neq 0$ and it is not possible to get an isometry and two extensions. Instead of a single example of such a case we examine a class of triples based on the Parrott's idea. The class is defined in Example 4.3 below, where the Parrott's example is obtained by taking $V = I$ and assuming B to be an isometry (denoted by V in the Parrott's work) not commuting with A . We are not going to give an equivalent condition for a triple in the considered class to admit a dilation to an isometry and two contractions. Our aim is rather to show some constructive approach to the dilation problem following from the method described in Section 3. We give a necessary condition which covers the Parrott's result. Some sufficient condition is also presented.

EXAMPLE 4.3

Let $A, B \in \mathcal{B}(\mathcal{H})$ be contractions commuting with an isometry $V \in \mathcal{B}(\mathcal{H})$ but not necessarily with each other. Let us check conditions under which there exist commuting liftings of R, S with respect to the isometric dilation of T , where

$$R = \begin{bmatrix} 0 & 0 \\ A & 0 \end{bmatrix}, \quad S = \begin{bmatrix} 0 & 0 \\ B & 0 \end{bmatrix}, \quad T = \begin{bmatrix} 0 & 0 \\ V & 0 \end{bmatrix}.$$

With the notation of Proposition 4.1 let the liftings of R and S be determined by contractions

$$X = \sum_{i=0}^{\infty} z^i \otimes X_i \quad \text{and} \quad Y = \sum_{j=0}^{\infty} z^j \otimes Y_j,$$

respectively. As noted in Remark 3.7, $\mathcal{D}_T = \{0\} \oplus \mathcal{H}$ and so (8) describes S_Y on a (linearly) dense subset of its domain. In particular, since $\text{ran}(X_i) \subset \mathcal{D}_T$ we get $S_Y(z^i \otimes X_i(\cdot, \cdot)) = M_z^{i+1} Y X_i(\cdot, \cdot)$. Taking advantage of Remark 3.7 we get the left hand side of the condition (10) as

$$\begin{aligned}
& (YD_S R + S_Y X D_R)(h, k) \\
&= \sum_{j=0}^{\infty} z^j \otimes Y_j(0, Ah) + S_Y \sum_{i=0}^{\infty} z^i \otimes X_i(D_A h, k) \\
&= \sum_{j=0}^{\infty} z^j \otimes Y_j(0, Ah) + \sum_{i=0}^{\infty} S_Y(z^i \otimes X_i(D_A h, k)) \\
&\stackrel{(8)}{=} \sum_{j=0}^{\infty} z^j \otimes Y_j(0, Ah) + \sum_{i=0}^{\infty} M_z^{i+1} Y X_i(D_A h, k) \\
&= \sum_{j=0}^{\infty} z^j \otimes Y_j(0, Ah) + \sum_{i=0}^{\infty} M_z^{i+1} \sum_{j=0}^{\infty} z^j \otimes Y_j X_i(D_A h, k) \\
&= \sum_{j=0}^{\infty} z^j \otimes Y_j(0, Ah) + \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} z^{i+j+1} \otimes Y_j X_i(D_A h, k) \\
&= \sum_{j=0}^{\infty} z^j \otimes Y_j(0, Ah) + \sum_{k=0}^{\infty} z^{k+1} \otimes \sum_{l=0}^k Y_l X_{k-l}(D_A h, k) \\
&= 1 \otimes Y_0(0, Ah) + \sum_{j=1}^{\infty} z^j \otimes \left(Y_j(0, Ah) + \sum_{l=0}^{j-1} Y_l X_{j-l-1}(D_A h, k) \right).
\end{aligned}$$

Similarly, the right hand side has the form

$$\begin{aligned}
& (X D_R S + R_X Y D_S)(h, k) \\
&= 1 \otimes X_0(0, Bh) + \sum_{i=1}^{\infty} z^i \otimes \left(X_i(0, Bh) + \sum_{l=0}^{i-1} X_l Y_{i-l-1}(D_B h, k) \right).
\end{aligned}$$

Hence X, Y generate commuting liftings if and only if they satisfy

$$Y_0(0, Ah) = X_0(0, Bh) \quad (11)$$

and

$$Y_i(0, Ah) + \sum_{l=0}^{i-1} Y_l X_{i-l-1}(D_A h, k) = X_i(0, Bh) + \sum_{l=0}^{i-1} X_l Y_{i-l-1}(D_B h, k) \quad (12)$$

for $i \geq 1$. In particular, (12) for $i = 1$ and $h = 0$ yields

$$Y_0 X_0(0, k) = X_0 Y_0(0, k). \quad (13)$$

It turns out that (11) and (13) are not only necessary but also sufficient conditions for the existence of commuting liftings of R, S with respect to the dilation of T . More precisely, if X' and Y' are contractions generating liftings of R and S , respectively, and satisfying (11) and (13), then the liftings do not necessarily commute. However, the other two contractions X, Y defined as $X = 1 \otimes X_0$, $Y = 1 \otimes Y_0$, where $X_0(h, k) = X'_0(0, k)$, $Y_0(h, k) = Y'_0(0, k)$ (so in particular X_0, Y_0 vanish on the first coordinate) generate liftings of R and S , respectively, which do commute. Indeed, by Remark 3.7, X', Y' satisfy (9) for the respective \tilde{X}', \tilde{Y}' and are contractions. Then one can check that also X, Y satisfy (9) with $\tilde{X} = 1 \otimes \tilde{X}_0$ and $\tilde{Y} = 1 \otimes \tilde{Y}_0$, where $\tilde{X}_0(h, k) = \tilde{X}'_0(0, k)$, $\tilde{Y}_0(h, k) = \tilde{Y}'_0(0, k)$ and $\|X\| \leq \|X'\|$, $\|Y\| \leq \|Y'\|$, so X, Y are contractions defining liftings of R, S . Since X, Y satisfy also (12) (note that $X_i = 0, Y_i = 0$ for $i \geq 1$), the liftings commute.

Summing up, without loss of generality we may assume that $X = 1 \otimes X_0$ and $Y = 1 \otimes Y_0$, where $X_0 \in \mathcal{B}(\mathcal{D}_A \oplus \mathcal{H}, \mathcal{D}_T)$, $Y_0 \in \mathcal{B}(\mathcal{D}_B \oplus \mathcal{H}, \mathcal{D}_T)$ and both operators vanish on the first coordinate. By virtue of (9) our aim is to construct \tilde{X}_0, \tilde{Y}_0 such that

$$X_0(0, k) = \tilde{X}_0(0, P_{\ker V^*}k) + (0, AV^*k)$$

and

$$Y_0(0, k) = \tilde{Y}_0(0, P_{\ker V^*}k) + (0, BV^*k)$$

satisfy (11) and (13). However, note that (11) is precisely

$$X_0Y_0(0, Vh) = Y_0X_0(0, Vh).$$

Hence we look for a commuting pair X_0, Y_0 of the form above. In particular, since A, B commute with V , the condition (13) for $k = V^2h$ where $h \in \mathcal{H}$ is arbitrary yields commutativity of A and B .

Conclusion: A necessary condition for the existence of commuting liftings of R, S with respect to the dilation of T is commutativity of A, B .

Let us finish the remark by giving also a sufficient condition. Obviously we assume that A, B commute. If AV^* commutes with BV^* , then taking $\tilde{X}_0 = \tilde{Y}_0 = 0$ we get commuting liftings. Let us generalize this condition. Note that since $\mathcal{H} = V^*\mathcal{H}$, the commutativity $AV^*BV^* - BV^*AV^* = 0$ is equivalent to $AV^*B - BV^*A = 0$.

If we take $\tilde{X}_0(h, k) = (0, V^*Ak)$ and $\tilde{Y}_0(h, k) = (0, V^*Bk)$, then

$$\begin{aligned} X_0(h, k) &= (0, V^*AP_{\ker V^*}k) + (0, AV^*k) \\ &= (0, V^*A(I - VV^*)k + AV^*k) = (0, V^*Ak) \end{aligned}$$

and similarly

$$Y_0(h, k) = (0, V^*Bk).$$

Such X_0 and Y_0 are clearly contractions and they commute if V^*A commutes with V^*B . The latter is equivalent to $V^*(AV^*B - BV^*A) = 0$ which is more general than previous commutativity of AV^* and BV^* (check $B = V$).

In particular, if V is a unitary operator, commutativity of A and B is an equivalent condition for the existence of commuting liftings of R and S with respect to the dilation of T .

5. Dilation of a positive definite n -tuple

In this section we show that an isometric dilation of an n -tuple of commuting contractions may be obtained via extensions if and only if the n -tuple is positive definite.

LEMMA 5.1

Let $\mathbf{T} = (T_1, \dots, T_n)$ be an n -tuple of commuting contractions on a Hilbert space \mathcal{H} . The following conditions are equivalent:

(i) $\{D_{T_1}h \mapsto D_{T_1}T_ih\}$ defines contractions \tilde{T}_i on \mathcal{D}_{T_1} and

$$V_{T_1} = \begin{bmatrix} T_1 & 0 \\ ED_{T_1} & M_z \end{bmatrix}, \quad C_{T_i} = \begin{bmatrix} T_i & 0 \\ 0 & I \otimes \tilde{T}_i \end{bmatrix} \quad \text{for } i = 2, \dots, n,$$

are the dilation of T_1 and commuting extensions of T_2, \dots, T_n ;

(ii) operators T_i satisfy

$$\|h\|^2 + \|T_iT_1h\|^2 \geq \|T_ih\|^2 + \|T_1h\|^2 \quad \text{for } i = 2, \dots, n.$$

Moreover, if the conditions above are satisfied and $\mathbf{C}_T = (C_{T_2}, \dots, C_{T_n})$, then

$$\begin{aligned} \sum_{v \subset u} (-1)^{|\alpha(v)|} \|C_T^{\alpha(v)}k\|^2 &= \sum_{v \subset u} (-1)^{|\alpha(v)|} \|\mathbf{T}^{\alpha(v)}h\|^2 \\ &+ \sum_{i=0}^{\infty} \sum_{v \subset u \cup \{1\}} (-1)^{|\alpha(v)|} \|\mathbf{T}^{\alpha(v)}h_i\|^2 \end{aligned} \quad (14)$$

for any $k = (h, 0) \oplus \sum_{i=0}^{\infty} (0, z^i \otimes D_{T_1}h_i)$ and $u \subset \{2, \dots, n\}$, where $\mathbf{C}_T^{\alpha(v)} = \prod_{i=2}^n C_{T_i}^{\chi_v(i)}$.

Proof. Note that operators C_{T_i} considered in (i) are contractions. Hence we require $\|C_{T_i}\| \leq 1$ but not necessarily $\|C_{T_i}\| = \|T_i\|$. Recall that $\|D_{T_1}h\|^2 = \|h\|^2 - \|T_1h\|^2$. Therefore

$$\|D_{T_1}h\|^2 - \|D_{T_1}T_ih\|^2 = \|h\|^2 - \|T_1h\|^2 - \|T_ih\|^2 + \|T_iT_1h\|^2 \quad (15)$$

for $i = 2, \dots, n$. If (i) holds, then \tilde{T}_i are contractions. Hence

$$\|D_{T_1}h\|^2 - \|D_{T_1}T_ih\|^2 \geq 0$$

and by (15) we get (ii).

Conversely, condition (ii), by (15), implies $\|D_{T_1}T_ih\| \leq \|D_{T_1}h\|$ and so $\ker D_{T_1}$ is invariant under T_i . Hence \tilde{T}_i is a well defined contraction which may be extended to \mathcal{D}_{T_1} . We get

$$V_{T_1} = \begin{bmatrix} T_1 & 0 \\ ED_{T_1} & M_z \end{bmatrix}, \quad C_{T_i} = \begin{bmatrix} T_i & 0 \\ 0 & I \otimes \tilde{T}_i \end{bmatrix},$$

the well defined dilation and extensions, commuting by Proposition 4.1. It is clear that $\|C_{T_i}\| = \max\{\|T_i\|, \|\tilde{T}_i\|\} \leq 1$.

Let us show the second part. Fix $u \subset \{2, \dots, n\}$ and denote $\tilde{\mathbf{T}} = (\tilde{T}_2, \dots, \tilde{T}_n)$. One can check that

$$\begin{aligned} \sum_{v \subset u} (-1)^{|\alpha(v)|} \|C_{\tilde{\mathbf{T}}}^{\alpha(v)} k\|^2 &= \sum_{v \subset u} (-1)^{|\alpha(v)|} \|\mathbf{T}^{\alpha(v)} h\|^2 \\ &\quad + \sum_{i=0}^{\infty} \sum_{v \subset u} (-1)^{|\alpha(v)|} \|\tilde{\mathbf{T}}^{\alpha(v)} D_{T_1} h_i\|^2. \end{aligned}$$

On the other hand, by the definition of \tilde{T}_i we have

$$\begin{aligned} \sum_{i=0}^{\infty} \sum_{v \subset u} (-1)^{|\alpha(v)|} \|\tilde{\mathbf{T}}^{\alpha(v)} D_{T_1} h_i\|^2 &= \sum_{i=0}^{\infty} \sum_{v \subset u} (-1)^{|\alpha(v)|} \|D_{T_1} \mathbf{T}^{\alpha(v)} h_i\|^2 \\ &= \sum_{i=0}^{\infty} \sum_{v \subset u} (-1)^{|\alpha(v)|} (\|\mathbf{T}^{\alpha(v)} h_i\|^2 - \|\mathbf{T}^{\alpha(v)} T_1 h_i\|^2) \\ &= \sum_{i=0}^{\infty} \sum_{v \subset u} ((-1)^{|\alpha(v)|} \|\mathbf{T}^{\alpha(v)} h_i\|^2 + (-1)^{|\alpha(v \cup \{1\})|} \|\mathbf{T}^{\alpha(v \cup \{1\})} h_i\|^2) \\ &= \sum_{i=0}^{\infty} \sum_{v \subset u \cup \{1\}} (-1)^{|\alpha(v)|} \|\mathbf{T}^{\alpha(v)} h_i\|^2. \end{aligned}$$

From the second part of Lemma 5.1 follows that:

COROLLARY 5.2

If an n -tuple \mathbf{T} satisfies conditions of Lemma 5.1, then \mathbf{T} is positive definite if and only if $C_{\mathbf{T}}$ defined as in Lemma 5.1 is positive definite.

Proof. Assume \mathbf{T} is not positive definite, then

$$\sum_{v \subset u} (-1)^{|\alpha(v)|} \|\mathbf{T}^{\alpha(v)} h\|^2 < 0$$

for some $u \subset \{1, \dots, n\}$ and $h \in \mathcal{H}$. If $1 \notin u$, we have

$$\sum_{v \subset u} (-1)^{|\alpha(v)|} \|C_{\mathbf{T}}^{\alpha(v)}(h, 0)\|^2 = \sum_{v \subset u} (-1)^{|\alpha(v)|} \|\mathbf{T}^{\alpha(v)} h\|^2 < 0.$$

If $1 \in u$, we obtain

$$\sum_{v \subset u \setminus \{1\}} (-1)^{|\alpha(v)|} \|C_{\mathbf{T}}^{\alpha(v)}(0, z \otimes D_{T_1} h)\|^2 = \sum_{v \subset u} (-1)^{|\alpha(v)|} \|\mathbf{T}^{\alpha(v)} h\|^2 < 0.$$

Hence, in the both cases $\mathbf{C}_{\mathbf{T}}$ is not positive definite.

Assume \mathbf{T} is positive definite. Then

$$\sum_{v \subset u} (-1)^{|\alpha(v)|} \|\mathbf{T}^{\alpha(v)} h\|^2 \geq 0 \quad \text{and} \quad \sum_{v \subset u \cup \{1\}} (-1)^{|\alpha(v)|} \|\mathbf{T}^{\alpha(v)} h_i\|^2 > 0$$

for each $h, h_i \in \mathcal{H}$ and $u \subset \{2, \dots, n\}$. Hence the right hand side of (14) is nonnegative and so

$$\sum_{v \subset u} (-1)^{|\alpha(v)|} \|\mathbf{C}_{\mathbf{T}}^{\alpha(v)} k\|^2 \geq 0$$

for any $k \in \{(h, 0) \oplus \sum_{i=0}^{\infty} (0, z^i \otimes D_{T_1} h_i), h, h_i \in \mathcal{H}\}$. Since the last set is dense in the domain of $\mathbf{C}_{\mathbf{T}}$, the tuple $\mathbf{C}_{\mathbf{T}}$ is positive definite.

It turns out that the construction of an isometric dilation of an n -tuple via liftings, which we described earlier may be obtained via extensions instead of liftings (which clearly satisfy the condition (10)) if and only if the n -tuple is positive definite. First, we describe more precisely the construction of an isometric dilation for an n -tuple $\mathbf{T} = (T_1, \dots, T_n)$. Assume the contractions T_2, \dots, T_n admit the extensions with respect to the isometric dilation of T_1 and denote $\mathbf{T}_1 = (T_{1,1}, T_{2,1}, \dots, T_{n,1})$, where $T_{1,1}$ denotes the isometric dilation of T_1 and $T_{i,1}$ is an extension of T_i with respect to $T_{1,1}$ for $i = 2, \dots, n$. By Proposition 4.1 the operators in \mathbf{T}_1 commute. Next dilate $T_{2,1}$ to an isometry $T_{2,2}$, extend the isometry $T_{1,1}$ to an isometry $T_{2,1}$ with respect to $T_{2,2}$ (which can be done by Corollary 3.6) and assume that $T_{3,1}, \dots, T_{n,1}$ may be extended to contractions $T_{3,2}, \dots, T_{n,2}$ with respect to $T_{2,2}$. Put $\mathbf{T}_2 = (T_{1,2}, T_{2,2}, \dots, T_{n,2})$. One may proceed similarly and construct $\mathbf{T}_3, \dots, \mathbf{T}_n$. Note that $T_{i,k}$ is an isometry for $i \leq k$. In particular, \mathbf{T}_n is an isometric dilation of \mathbf{T} . Since the construction requires the assumption that contractions admit the respective extensions, it may not be done in the general case. As we already mentioned such a construction is possible if and only if \mathbf{T} is positive definite.

THEOREM 5.3

Let $\mathbf{T} = (T_1, \dots, T_n)$ be an n -tuple of commuting contractions on \mathcal{H} . The following conditions are equivalent:

- (i) \mathbf{T} admits an isometric dilation \mathbf{T}_n constructed as the last element of a sequence $\mathbf{T}_k = (T_{1,k}, \dots, T_{n,k})$ on \mathcal{K}_k for $k = 0, \dots, n$, where
 - (a) $\mathcal{K}_0 = \mathcal{H}$ and $\mathbf{T}_0 = \mathbf{T}$,
 - (b) $\mathcal{K}_k = \mathcal{K}_{k-1} \oplus (H^2(\mathbb{T}) \otimes \mathcal{D}_k)$ for $k = 1, \dots, n$, where D_k and \mathcal{D}_k are the defect operator and the defect space of $T_{k,k-1}$, respectively,
 - (c) $T_{k,k} = \begin{bmatrix} T_{k,k-1} & 0 \\ ED_k & M_z \end{bmatrix}$ and $T_{i,k} = \begin{bmatrix} T_{i,k-1} & 0 \\ 0 & I \otimes \tilde{T}_{i,k-1} \end{bmatrix}$, where the operators $\tilde{T}_{i,k-1}$ given by $\{D_k h \mapsto D_k T_{i,k-1} h\}$ are well defined and extend to contractions on \mathcal{D}_k for $i \neq k$.

- (ii) \mathbf{T} is positive definite.

Proof. Since extensions are obtained by zero operators which clearly satisfy the condition (10), each n -tuple \mathbf{T}_k in (i) is a commuting n -tuple. Hence indeed, \mathbf{T}_n is an isometric dilation. Moreover, if \mathbf{T}_k is defined, then $T_{i,k}$ are isometries for $i \leq k$. In particular, the last but one tuple \mathbf{T}_{n-1} consists of isometries $T_{i,n-1}$ (for $i \leq n-1$) and a contraction $T_{n,n-1}$. Hence, if the tuple \mathbf{T}_{n-1} is obtained, each $T_{i,n-1}$ may be extended with respect to the isometric dilation of $T_{n,n-1}$ by Corollary 3.6. In other words, the condition (i) holds if and only if the sequence \mathbf{T}_k may be constructed up to the term $n-1$. The last term may always be obtained.

We use induction on n . The base step $n=2$ follows by Lemma 5.1. Indeed, by the first paragraph of the proof, condition (i) is equivalent to the existence of \mathbf{T}_1 , which in turn is the condition (i) of Lemma 5.1. On the other hand, since T_1, T_2 are contractions, the pair (T_1, T_2) is positive definite if and only if $\sum_{v \in \{1,2\}} (-1)^{|\alpha(v)|} \|\mathbf{T}^{\alpha(v)} h\|^2 \geq 0$, which is condition (ii) of Lemma 5.1.

The inductive step. Assume the conditions are equivalent for $n-1$ tuples and consider an n -tuple \mathbf{T} . If \mathbf{T}_1 exists, then \mathbf{T} satisfies conditions of Lemma 5.1, where $\mathbf{C}_{\mathbf{T}} = (T_{2,1}, \dots, T_{n,1})$ is an $(n-1)$ -subtuple of \mathbf{T}_1 . Moreover, if \mathbf{T} satisfies (i) then $\mathbf{C}_{\mathbf{T}}$ satisfies (i) with $(\mathbf{C}_{\mathbf{T}})_k = (T_{2,k+1}, \dots, T_{n,k+1})$. Then, by the assumption of the inductive step, $\mathbf{C}_{\mathbf{T}}$ is positive definite. Hence, by Corollary 5.2, \mathbf{T} is positive definite.

Conversely, if \mathbf{T} is positive definite, then in particular condition (ii) in Lemma 5.1 is satisfied. Hence we may construct $\mathbf{T}_1 = (V_{T_1}, C_{T_2}, \dots, C_{T_n})$. Moreover, by Corollary 5.2 the $(n-1)$ -tuple $\mathbf{C}_{\mathbf{T}} = (C_{T_2}, \dots, C_{T_n})$ is positive definite. Thus by the assumption of the inductive step, there are $(\mathbf{C}_{\mathbf{T}})_k = ((C_T)_{2,k}, \dots, (C_T)_{n,k})$ for $k=1, \dots, n-1$. Note that $(C_T)_{2,1}$ is an isometric dilation of C_{T_2} and $(C_T)_{i,1}$ are extensions of C_{T_i} with respect to this dilation for $i=3, \dots, n$. Since V_{T_1} is an isometry, it also has a unique extension with respect to $(C_T)_{2,1}$ by Corollary 3.6. If we denote it by $T_{1,2}$, then $\mathbf{T}_2 = (T_{1,2}, (C_T)_{2,1}, \dots, (C_T)_{n,1})$ has all the properties required in (i). Since $T_{1,2}$ is an isometry, we may construct $T_{1,3}$ as an extension of $T_{1,2}$ with respect to $(C_T)_{3,2}$, which is the isometric dilation of $(C_T)_{3,1}$ and get a commuting n -tuple $\mathbf{T}_3 = (T_{1,3}, (C_T)_{2,2}, \dots, (C_T)_{n,2})$. By recurrence, having defined $\mathbf{T}_k = (T_{1,k}, (C_T)_{2,k-1}, \dots, (C_T)_{n,k-1})$ we get $\mathbf{T}_{k+1} = (T_{1,k+1}, (C_T)_{2,k}, \dots, (C_T)_{n,k})$, where $T_{1,k+1}$ is an extension of $T_{1,k}$ with respect to $(C_T)_{k+1,k}$ – the isometric dilation of $(C_T)_{k+1,k-1}$ for $k=3, \dots, n-1$. Hence we get a complete sequence required in (i).

We can conclude from Theorem 5.3 that a positive definite n -tuple admits an isometric, so also a unitary dilation. As we mentioned in Introduction such a result is known ([16, Theorem 9.1, Chap.1]) in a more precise version where an n -tuple is showed to admit a regular dilation if and only if it is positive definite. It is not our aim to provide a new proof of [16, Theorem 9.1, Chap.1], so we do not check whether the constructed dilation is regular or minimal. Our aim was to show the construction of the dilation and to emphasise that using extensions instead of liftings (which is a convenient choice by virtue of Proposition 4.1) has limited usability.

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