sciendo
Ann. Univ. Paedagog. Crac. Stud. Math. 22 (2023), 5-11
DOI: 10.2478/aupcsm-2023-0001

FOLIA 385

Annales Universitatis Paedagogicae Cracoviensis
Studia Mathematica 22 (2023)

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Necessary and sufficient conditions for the inclusion relation between two summability methods

Abstract. In this paper, a general theorem gives necessary and sufficient conditions for the inclusion relation between $\varphi - |A, \beta; \delta|_k$ and $\varphi - |B, \beta; \delta|_k$ methods is proved.

1. Introduction

Let $\sum a_n$ be an infinite series with its partial sums (s_n). Let $A = (a_{nv})$ be a normal matrix, i.e. a lower triangular matrix of nonzero diagonal entries. Then A defines the sequence-to-sequence transformation, mapping the sequence $s = (s_n)$ to $As = (A_n(s))$, where

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v, \qquad n = 0, 1, \ldots.$$

Let (φ_n) be any sequence of positive real numbers. The series $\sum a_n$ is said to be summable $\varphi - |A, \beta; \delta|_k$, $k \geq 1$, $\delta \geq 0$ and β is a real number, if (see [16]),

$$\sum_{n=1}^{\infty} \varphi_n^{\beta(\delta k + k - 1)} |A_n(s) - A_{n-1}(s)|^k < \infty.$$

For $\beta = 1$ and $\delta = 0$, we get $\varphi - |A|_k$ summability method (see [13]).

AMS (2020) Subject Classification: 40D25, 40F05, 40G99.
Keywords and phrases: relative strength, absolute matrix summability, infinite series.
ISSN: 2081-545X, e-ISSN: 2300-133X.

Let $A = (a_{nv})$ be a normal matrix, then two lower semimatrices $\bar{A} = (\bar{a}_{nv})$ and $\hat{A} = (\hat{a}_{nv})$ are defined as follows:

$$\bar{a}_{nv} = \sum_{i=v}^{n} a_{ni}, \qquad n, v = 0, 1, \ldots, \tag{1}$$

$$\hat{a}_{00} = \bar{a}_{00} = a_{00}, \quad \hat{a}_{nv} = \bar{a}_{nv} - \bar{a}_{n-1,v}, \qquad n = 1, 2, \ldots \tag{2}$$

and

$$\bar{\Delta} A_n(s) = A_n(s) - A_{n-1}(s) = \sum_{v=0}^{n} \hat{a}_{nv} a_v. \tag{3}$$

If A is a normal matrix, then $A' = (a'_{nv})$ denotes the inverse of A, and $\hat{A} = (\hat{a}_{nv})$ is a normal matrix and it has two-sided inverse $\hat{A}' = (\hat{a}'_{nv})$ which is also normal (see [6]).

2. Known Results

There are many papers focused on sufficient or necessary conditions for absolute summability of infinite series, equivalence theorems for summability and the relative strength of absolute summability methods. In [1], Bor obtained the relative strength of two absolute summability methods. Bor, Srivastava and Sulaiman [5] achieved the sufficient conditions for summability of an infinite series by using generalized power increasing sequences. Özarslan and Özgen [18] proved a theorem gives necessary conditions for absolute matrix summability. Sezer and Çanak [20], Bor [2] obtained equivalence theorems on summability. Özgen [19], Özarslan and Karakaş [14], Özarslan and Kartal [15], [17], Sonker and Munjal [21], [22], Karakaş [7], Kartal [8], [9], Özarslan [10],[11], Bor and Agarwal [3], Bor and Mohapatra [4] obtained theorems on absolute Riesz, Cesàro and matrix summability of infinite series. Furthermore, the following theorem on the relative strength of two absolute matrix summability methods has been proved in [13].

Theorem 1
Let $k > 1$. Let $A = (a_{nv})$ and $B = (b_{nv})$ be two positive normal matrices. In order that

$$\varphi - |A|_k \Rightarrow \varphi - |B|_k \tag{4}$$

it is necessary that

$$b_{nn} = O(a_{nn}). \tag{5}$$

If we suppose that

$$b_{n-1,v} \geq b_{nv} \qquad for\ n \geq v + 1, \tag{6}$$

$$\bar{a}_{n0} = 1, \quad \bar{b}_{n0} = 1, \qquad n = 0, 1, \ldots, \tag{7}$$

$$a_{vv} - a_{v+1,v} = O(a_{vv} a_{v+1,v+1}), \tag{8}$$

$$\sum_{v=1}^{n-1} (b_{vv} \hat{b}_{n,v+1}) = O(b_{nn}), \tag{9}$$

$$\sum_{n=v+1}^{m+1} (\varphi_n b_{nn})^{k-1} \hat{b}_{n,v+1} = O(\varphi_v^{k-1} b_{vv}^{k-1}), \tag{10}$$

$$\sum_{n=v+1}^{m+1} (\varphi_n b_{nn})^{k-1} |\Delta_v(\hat{b}_{nv})| = O(\varphi_v^{k-1} b_{vv}^{k}), \tag{11}$$

$$\sum_{v=r+2}^{n} \hat{b}_{nv} |\hat{a}'_{vr}| = O(\hat{b}_{n,r+1}), \tag{12}$$

then (5) *is also sufficient.*

Lemma 1 ([1])
Let $k \geq 1$ and $A = (a_{nv})$ be an infinite matrix. In order that $A \in (l^k; l^k)$ it is necessary that

$$a_{nv} = O(1) \qquad (all \quad n, v). \tag{13}$$

3. Main Result

The aim of this paper is to generalize Theorem 1 as in the following form.

Theorem 2
Let $k > 1$. Let $A = (a_{nv})$ and $B = (b_{nv})$ be two positive normal matrices. In order that

$$\varphi - |A, \beta; \delta|_k \Rightarrow \varphi - |B, \beta; \delta|_k \tag{14}$$

condition (5) *is necessary. If we suppose that* (6)-(9), (12) *and*

$$\sum_{n=v+1}^{m+1} \varphi_n^{\beta(\delta k+k-1)} b_{nn}^{k-1} \hat{b}_{n,v+1} = O(\varphi_v^{\beta(\delta k+k-1)} b_{vv}^{k-1}), \tag{15}$$

$$\sum_{n=v+1}^{m+1} \varphi_n^{\beta(\delta k+k-1)} b_{nn}^{k-1} |\Delta_v(\hat{b}_{nv})| = O(\varphi_v^{\beta(\delta k+k-1)} b_{vv}^{k}), \tag{16}$$

then (5) *is also sufficient, where $\delta \geq 0$ and $-\beta(\delta k + k - 1) + k > 0$.*

Proof. Necessity. Let (I_n) and (U_n) denote A-transform and B-transform of the series $\sum a_n$, respectively. By (3), we get

$$\bar{\Delta} I_n = \sum_{v=0}^{n} \hat{a}_{nv} a_v \quad \text{and} \quad \bar{\Delta} U_n = \sum_{v=0}^{n} \hat{b}_{nv} a_v.$$

Then, we write $a_v = \sum_{r=0}^{v} \hat{a}'_{vr} \bar{\Delta} I_r$ and $\bar{\Delta} U_n = \sum_{v=0}^{n} \hat{b}_{nv} \sum_{r=0}^{v} \hat{a}'_{vr} \bar{\Delta} I_r$.

Since $\hat{b}_{n0} = \bar{b}_{n0} - \bar{b}_{n-1,0} = 0$, we have

$$\begin{aligned}
\bar{\Delta} U_n &= \sum_{v=1}^{n} \hat{b}_{nv} \sum_{r=0}^{v} \hat{a}'_{vr} \bar{\Delta} I_r \\
&= \sum_{v=1}^{n} \hat{b}_{nv} \hat{a}'_{vv} \bar{\Delta} I_v + \sum_{v=1}^{n} \hat{b}_{nv} \hat{a}'_{v,v-1} \bar{\Delta} I_{v-1} + \sum_{v=1}^{n} \hat{b}_{nv} \sum_{r=0}^{v-2} \hat{a}'_{vr} \bar{\Delta} I_r \\
&= \hat{b}_{nn} \hat{a}'_{nn} \bar{\Delta} I_n + \sum_{v=1}^{n-1} (\hat{b}_{nv} \hat{a}'_{vv} + \hat{b}_{n,v+1} \hat{a}'_{v+1,v}) \bar{\Delta} I_v \\
&\quad + \sum_{r=0}^{n-2} \bar{\Delta} I_r \sum_{v=r+2}^{n} \hat{b}_{nv} \hat{a}'_{vr}.
\end{aligned}$$

For δ_{nv}(Kronecker delta), by using equality $\sum_{k=v}^{n} \hat{a}'_{nk} \hat{a}_{kv} = \delta_{nv}$, we get

$$\begin{aligned}
\hat{b}_{nv} \hat{a}'_{vv} + \hat{b}_{n,v+1} \hat{a}'_{v+1,v} &= \frac{\hat{b}_{nv}}{\hat{a}_{vv}} + \hat{b}_{n,v+1} \Big(- \frac{\hat{a}_{v+1,v}}{\hat{a}_{vv} \hat{a}_{v+1,v+1}} \Big) \\
&= \frac{\hat{b}_{nv}}{a_{vv}} - \hat{b}_{n,v+1} \frac{(\bar{a}_{v+1,v} - \bar{a}_{vv})}{a_{vv} a_{v+1,v+1}} \\
&= \frac{\hat{b}_{nv}}{a_{vv}} - \hat{b}_{n,v+1} \frac{(a_{v+1,v+1} + a_{v+1,v} - a_{vv})}{a_{vv} a_{v+1,v+1}} \\
&= \frac{\Delta_v(\hat{b}_{nv})}{a_{vv}} + \hat{b}_{n,v+1} \frac{(a_{vv} - a_{v+1,v})}{a_{vv} a_{v+1,v+1}}.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
\bar{\Delta} U_n &= \frac{b_{nn}}{a_{nn}} \bar{\Delta} I_n + \sum_{v=1}^{n-1} \frac{\Delta_v(\hat{b}_{nv})}{a_{vv}} \bar{\Delta} I_v + \sum_{v=1}^{n-1} \hat{b}_{n,v+1} \frac{(a_{vv} - a_{v+1,v})}{a_{vv} a_{v+1,v+1}} \bar{\Delta} I_v \\
&\quad + \sum_{r=0}^{n-2} \bar{\Delta} I_r \sum_{v=r+2}^{n} \hat{b}_{nv} \hat{a}'_{vr} \\
&= U_{n,1} + U_{n,2} + U_{n,3} + U_{n,4}.
\end{aligned}$$

Now, we write down the matrix transforming $\big(\varphi_n^{\frac{\beta(\delta k+k-1)}{k}} \bar{\Delta} I_n\big)$ into $\big(\varphi_n^{\frac{\beta(\delta k+k-1)}{k}} \bar{\Delta} U_n\big)$ by (13). The assertion (14) is equivalent to the assertion that this matrix $\in \big(l^k; l^k\big)$. Hence, by Lemma 1, in order that $\varphi - |A, \beta; \delta|_k \Rightarrow \varphi - |B, \beta; \delta|_k$, the condition (5) is necessary.

Sufficiency. Let the conditions be satisfied. We will prove that $\varphi - |A, \beta; \delta|_k \Rightarrow \varphi - |B, \beta; \delta|_k$. For this we need to show

$$\sum_{n=1}^{\infty} \varphi_n^{\beta(\delta k+k-1)} |U_{n,r}|^k < \infty \qquad \text{for } r = 1, 2, 3, 4.$$

First, we have

$$\sum_{n=1}^{m} \varphi_n^{\beta(\delta k+k-1)} |U_{n,1}|^k = \sum_{n=1}^{m} \varphi_n^{\beta(\delta k+k-1)} \frac{b_{nn}^k}{a_{nn}^k} |\bar{\Delta} I_n|^k.$$

Now, by (5) and using the fact that $\sum a_n$ is summable $\varphi - |A, \beta; \delta|_k$, we get

$$\sum_{n=1}^{m} \varphi_n^{\beta(\delta k+k-1)} |U_{n,1}|^k = O(1) \sum_{n=1}^{m} \varphi_n^{\beta(\delta k+k-1)} |\bar{\Delta} I_n|^k = O(1) \qquad \text{as } m \to \infty.$$

By using Hölder's inequality, we have

$$\begin{aligned} &\sum_{n=2}^{m+1} \varphi_n^{\beta(\delta k+k-1)} |U_{n,2}|^k \\ &\qquad \le \sum_{n=2}^{m+1} \varphi_n^{\beta(\delta k+k-1)} \sum_{v=1}^{n-1} \frac{|\Delta_v(\hat{b}_{nv})|}{a_{vv}^k} |\bar{\Delta} I_v|^k \left(\sum_{v=1}^{n-1} |\Delta_v(\hat{b}_{nv})| \right)^{k-1}. \end{aligned}$$

Here (1), (2), (4) and (7) imply $\sum_{v=1}^{n-1} |\Delta_v(\hat{b}_{nv})| = \sum_{v=1}^{n-1} (b_{n-1,v} - b_{nv}) \le b_{nn}$. Then, by (16) and (5), we have

$$\begin{aligned} \sum_{n=2}^{m+1} \varphi_n^{\beta(\delta k+k-1)} |U_{n,2}|^k &\le \sum_{n=2}^{m+1} \varphi_n^{\beta(\delta k+k-1)} b_{nn}^{k-1} \sum_{v=1}^{n-1} \frac{|\Delta_v(\hat{b}_{nv})|}{a_{vv}^k} |\bar{\Delta} I_v|^k \\ &\le \sum_{v=1}^{m} \frac{|\bar{\Delta} I_v|^k}{a_{vv}^k} \sum_{n=v+1}^{m+1} \varphi_n^{\beta(\delta k+k-1)} b_{nn}^{k-1} |\Delta_v(\hat{b}_{nv})| \\ &= O(1) \sum_{v=1}^{m} \varphi_v^{\beta(\delta k+k-1)} |\bar{\Delta} I_v|^k \\ &= O(1) \qquad \text{as } m \to \infty. \end{aligned}$$

Now, by (8), we get

$$\begin{aligned} \sum_{n=2}^{m+1} \varphi_n^{\beta(\delta k+k-1)} |U_{n,3}|^k &= \sum_{n=2}^{m+1} \varphi_n^{\beta(\delta k+k-1)} \left| \sum_{v=1}^{n-1} \hat{b}_{n,v+1} \frac{(a_{vv} - a_{v+1,v})}{a_{vv} a_{v+1,v+1}} \bar{\Delta} I_v \right|^k \\ &= O(1) \sum_{n=2}^{m+1} \varphi_n^{\beta(\delta k+k-1)} \left(\sum_{v=1}^{n-1} \hat{b}_{n,v+1} |\bar{\Delta} I_v| \right)^k. \end{aligned}$$

Then by using Hölder's inequality, and the conditions (9), (15), we obtain

$$\begin{aligned} &\sum_{n=2}^{m+1} \varphi_n^{\beta(\delta k+k-1)} |U_{n,3}|^k \\ &\qquad = O(1) \sum_{n=2}^{m+1} \varphi_n^{\beta(\delta k+k-1)} \sum_{v=1}^{n-1} \hat{b}_{n,v+1} \frac{b_{vv}}{b_{vv}^k} |\bar{\Delta} I_v|^k \left(\sum_{v=1}^{n-1} \hat{b}_{n,v+1} b_{vv} \right)^{k-1} \end{aligned}$$

$$
\begin{aligned}
&= O(1) \sum_{n=2}^{m+1} \varphi_n^{\beta(\delta k+k-1)} b_{nn}^{k-1} \sum_{v=1}^{n-1} \hat{b}_{n,v+1} |\bar{\Delta} I_v|^k \frac{1}{b_{vv}^{k-1}} \\
&= O(1) \sum_{v=1}^{m} \frac{1}{b_{vv}^{k-1}} |\bar{\Delta} I_v|^k \sum_{n=v+1}^{m+1} \varphi_n^{\beta(\delta k+k-1)} b_{nn}^{k-1} \hat{b}_{n,v+1} \\
&= O(1) \sum_{v=1}^{m} \varphi_v^{\beta(\delta k+k-1)} |\bar{\Delta} I_v|^k \\
&= O(1) \qquad \text{as } m \to \infty.
\end{aligned}
$$

Finally, by using (12), we get

$$
\begin{aligned}
\sum_{n=2}^{m+1} \varphi_n^{\beta(\delta k+k-1)} |U_{n,4}|^k &= O(1) \sum_{n=2}^{m+1} \varphi_n^{\beta(\delta k+k-1)} \left(\sum_{r=0}^{n-2} |\bar{\Delta} I_r| \hat{b}_{n,r+1} \right)^k \\
&= O(1) \qquad \text{as } m \to \infty,
\end{aligned}
$$

as in $U_{n,3}$. This completes the proof of Theorem 2.

4. Conclusion

In this paper, a theorem on the relative strength of two absolute matrix summability methods is generalized. In case of $\beta = 1$ and $\delta = 0$, the conditions (15), (16) reduce to the conditions (10), (11) respectively and so Theorem 2 reduces to Theorem 1. Also, in case of $\beta = 1$, Theorem 2 reduces to the result in [12].

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Received: August 22, 2022; final version: August 22, 2022;
available online: March 28, 2023.