Ann. Univ. Paedagog. Crac. Stud. Math. 22 (2023), 33-47
DOI: 10.2478/aupcsm-2023-0004

FOLIA 385

Annales Universitatis Paedagogicae Cracoviensis
Studia Mathematica 22 (2023)

Bharat Bhushan, Gurninder S. Sandhu and Deepak Kumar

Centrally-extended generalized Jordan derivations in rings

Abstract. In this article, we introduce the notion of centrally-extended generalized Jordan derivations and characterize the structure of a prime ring (resp. $*$-prime ring) R that admits a centrally-extended generalized Jordan derivation F satisfying $[F(x), x] \in Z(R)$ (resp. $[F(x), x^*] \in Z(R)$) for all $x \in R$.

1. Introduction

Throughout this study R is an associative ring with center $Z(R)$. Let $Q_{ml}(R)$ be the maximal left ring quotients of R, the center of $Q_{ml}(R)$ is denoted by C which is known as the extended centroid of R. Recall that C is a field in case R is prime ring. For any x, y in R, the commutator (resp. anti-commutator) of x, y is defined as $[x, y] = xy - yx$ (resp. $x \circ y = xy + yx$). In a *prime ring*, if there exist a, b in R such that $aRb = (0)$, then either $a = 0$ or $b = 0$, whereas in *semiprime ring*, if $aRa = (0)$, then $a = 0$. Clearly, every prime ring is semiprime ring but the converse need not be true, for instance $\mathbb{Z} \times \mathbb{Z}$ where $\mathbb{Z}$ is a ring of integers.

For any n in $\mathbb{Z}^+$, R is called n-torsion free if $nx = 0$ for all $x \in R$, implies $x = 0$. A mapping $\varphi\colon R \to R$ is said to be *centralizing* on a subset S of R, if $[\varphi(x), x] \in Z(R)$ for all $x \in S$. In particular, φ is called *commuting* on S if $[\varphi(x), x] = 0$ for all $x \in S$. An anti-automorphism '$*$' of R is called *involution* if $(x^*)^* = x$ for all $x \in R$. If R is a prime ring with involution '$*$' then '$*$' can be extended to central closure of R, that is $RC + C$, [16].

AMS (2020) Subject Classification: 16W10, 16N60, 16W25.
Keywords and phrases: Associative rings, involution, generalized Jordan derivation, centrally extended generalized Jordan derivation.
ISSN: 2081-545X, e-ISSN: 2300-133X.

An element x of a ring with involution '$*$' is *symmetric* if $x^* = x$ and is *skew symmetric* if $x^* = -x$. The set of symmetric elements in R is denoted by $H(R)$ whereas the set of skew symmetric elements denoted by $S(R)$. Note that, if R is 2-torsion free ring, then for each x in R, we have a unique representation $2x = h + k$, where $h \in H(R)$ and $k \in S(R)$.

Motivated by the definition of centralizing (resp. commuting) mapping, Ali and Dar [1] introduced $*$-centralizing (resp. $*$-commuting) mapping, which is defined as follows: A mapping φ is called $*$-centralizing (resp. $*$-commuting) on a set S if $[\varphi(x), x^*] \in Z(R)$ (resp. $[\varphi(x), x^*] = 0$) for all $x \in S$.

Recall that an additive self-mapping d of R is known as a *derivation* if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$ and is known as *Jordan derivation* if $d(x^2) = d(x)x + xd(x)$ for all $x \in R$. It is straightforward that every derivation is a Jordan derivation but the converse is not always true.

Example 1.1 ([2, Example 3.2.1])
Let R be a ring and $a \in R$ such that $xax = 0$ for all $x \in R$ and $xay \neq 0$ for some $y \neq x$ in R. Define a map $d\colon R \to R$ by $d(x) = ax$. Then, it is very easy to see that d is a Jordan derivation on R but not a derivation on R.

It can be seen that δ is a Jordan derivation but not a derivation. Moreover, the question "when Jordan derivation is a derivation?" raised by Herstein [12] caused significant work existed in the literature of Jordan mappings in rings (see [6], [10], [12], [13]). In 1991, Brešar [7] introduced the notion of *generalized derivation*. Accordingly, a generalized derivation $F\colon R \to R$ is an additive mapping which is uniquely determined by a derivation d such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$. In 2003, Jing and Lu [13] introduced the notion of *generalized Jordan derivation*, which is an additive mapping $F\colon R \to R$ with associated Jordan derivation $d\colon R \to R$ such that $F(x^2) = F(x)x + xd(x)$ for all $x \in R$, and proved that in a 2-torsion free prime ring every generalized Jordan derivation is a generalized derivation.

A mapping $\delta\colon R \to R$ is called *centrally extended derivation* if $\delta(x + y) - \delta(x) - \delta(y) \in Z(R)$ and $\delta(xy) - \delta(x)y - x\delta(y) \in Z(R)$ for all $x, y \in R$. Bell and Daif [4] extended the notion of derivation by introducing centrally extended derivations and discussed their existence in rings. Very recently, we [5] introduced a more general map than CE-derivation, called *CE-Jordan derivation*, defined as $\delta(x+y) - \delta(x) - \delta(y) \in Z(R)$ and $\delta(x \circ y) - \delta(x) \circ y - x \circ \delta(y) \in Z(R)$ for all $x, y \in R$. In this article, we extend CE-Jordan derivations to CE-generalized Jordan derivations in rings as follow: A mapping $F\colon R \to R$ is called *CE-generalized Jordan derivation* constrained with CE-Jordan derivation, if

$$F(x + y) - F(x) - F(y) \in Z(R), \tag{A}$$

$$F(x \circ y) - F(x)y - F(y)x - x\delta(y) - y\delta(x) \in Z(R) \tag{B}$$

for all $x, y \in R$.

The main objective of this paper is to investigate the structure of a non-commutative prime ring (resp. $*$-prime ring) R involving CE-generalized Jordan derivation F and satisfying $[F(x), x] \in Z(R)$ (resp. $[F(x), x^*] \in Z(R)$). More specifically, we prove the following results:

Theorem 1.2
Let R be a 2-torsion free noncommutative prime ring and $F\colon R \to R$ a CE-generalized Jordan derivation constrained with CE-Jordan derivation δ. If F is centralizing on R, then R is an order in a central simple algebra of dimension at most 4 over its center or $F(x) = \lambda x$, where $\lambda \in C$.

Theorem 1.3
Let R be a 2-torsion free noncommutative prime ring and $F\colon R \to R$ a CE-generalized Jordan derivation constrained with a CE-Jordan derivation δ. If F is $*$-centralizing on R, then R is an order in a central simple algebra of dimension at most 4 over its center or $F = 0$.

2. Preliminaries

We shall denote the standard identity in four non commuting variables x_1, x_2, x_3, x_4 by s_4, which is defined as follows

$$s_4(x_1, x_2, x_3, x_4) = \sum_{\sigma \in S_4} (-1)^\sigma x_{\sigma(1)} x_{\sigma(2)} x_{\sigma(3)} x_{\sigma(4)},$$

where S_4 is the symmetric group of degree 4 and $(-1)^\sigma$ is the sign of permutation $\sigma \in S_4$.

Now we give some results from the literature that shall be used in order to develop the main results.

Lemma 2.1 ([1, Lemma 2.2])
Let R be a 2-torsion free semiprime ring with involution '$*$'. If an additive mapping f of R into itself such that $[f(x), x^*] \in Z(R)$ for all $x \in R$, then $[f(x), x^*] = 0$ for all $x \in R$.

Lemma 2.2 ([3, Proposition 2.1.7])
Let R be a prime ring, $Q_{mr}(R)$ be the maximal right ring of quotients of R and D be the set of all right dense ideals of R. Then for all $q \in Q_{mr}(R)$, there exists $J \in D$ such that $qJ \subseteq R$.

Lemma 2.3 ([5, Lemma 4])
Let R be a 2-torsion free ring with no nonzero central ideal. If δ is a CE-Jordan derivation of R, then δ is additive.

Lemma 2.4 ([5, Theorem 3.6])
Let R be a 2-torsion free noncommutative prime ring with involution '$*$' that admits a CE-Jordan derivation $\delta\colon R \to R$ such that $[\delta(x), x] \in Z(R)$ for all $x \in R$. Then either $\delta = 0$ or R is an order in a central simple algebra of dimension at most 4 over its center.

Lemma 2.5 ([8, Lemma 1])
Let R be a prime ring with C its extended centroid. Then the following assertions are equivalent:

(i) $dim_C(RC) \leq 4$.

(ii) *R satisfies s_4.*

(iii) *R is commutative or R embeds into $M_2(F)$, for a field F.*

(iv) *R is algebraic of bounded degree 2 over C.*

(v) *R satisfies $[[x^2, y], [x, y]]$.*

Lemma 2.6 ([9, Proposition 3.1])
Let R be a 2-torsion free semiprime ring and U be a Jordan subring of R. If an additive mapping $f\colon R \to R$ is centralizing on U, then f is commuting on U.

Lemma 2.7 ([9, Theorem 3.2])
Let R be a prime ring. If an additive mapping $f\colon R \to R$ is commuting on R, then there exists $\lambda \in C$ and an additive $\sigma\colon R \to C$, such that $F(x) = \lambda x + \sigma(x)$ for all $x \in R$.

Lemma 2.8 ([11, Theorem])
Let R be a prime ring of characteristic $\neq 2$ with right quotient ring U and extended centroid C, $F \neq 0$ a generalized derivation of R, L a non-central Lie ideal of R and $n \geq 1$. If $[F(u), u]^n = 0$, for all $u \in L$, then there exists an element a $\lambda \in C$ such that $F(x) = \lambda x$, for all $x \in R$, unless when R satisfies s_4 and there exists an element $b \in U$ such that $F(x) = bx + xb$, for all $x \in R$.

Lemma 2.9 ([13, Theorem 2.5])
Let R be a 2-torsion free prime ring, then every generalized Jordan derivation on R is a generalized derivation.

Lemma 2.10 ([14, Theorem 3])
Every generalized derivation g on a dense right ideal of R can be extended to $Q_{ml}(R)$ and assumes the form $g(x) = ax + \delta(x)$ for some $a \in Q_{ml}(R)$ and a derivation δ on $Q_{ml}(R)$.

Lemma 2.11 ([15, Theorem 1])
Let R be a prime ring with involution '$*$' and center $Z(R)$. If d is a nonzero derivation such that $[d(x), x] \in Z(R)$ for all $x \in H(R)$, then R satisfies s_4.

Lemma 2.12 ([15, Theorem 3])
Let R be a prime ring with involution '$*$' and center $Z(R)$. If n is a fixed natural number such that $x^n \in Z(R)$ for all $x \in H(R)$, then R satisfies s_4.

Lemma 2.13 ([15, Theorem 6])
Let R be a prime ring with involution '$*$' and center $Z(R)$. If d is a nonzero derivation on R such that $d(x)x + xd(x) \in Z(R)$ for all $x \in H(R)$, then R satisfies s_4.

Lemma 2.14 ([15, Theorem 7])
Let R be a prime ring with involution '$*$' and center $Z(R)$. If d is a nonzero derivation on R such that $d(x)x + xd(x) \in Z(R)$ for all $x \in S(R)$, then R satisfies s_4.

3. Proofs of the Main Results

Proposition 3.1

Let R be a 2-torsion free ring with no nonzero central ideal. If F is a CE-generalized Jordan derivation constrained with a CE-Jordan derivation of R, then F is additive.

Proof. Let F be a CE-generalized Jordan derivation. In view of condition (A), for any elements $x, y, z \in R$, it follows that

$$F(x+y) = F(x) + F(y) + c_{F(x,y,+)}, \tag{1}$$

where $c_{F(x,y,+)} \in Z(R)$, and there exists some $c_{F(z,x+y,\circ)} \in Z(R)$ such that

$$F(z \circ (x+y)) = F(z)(x+y) + z\delta(x+y) + F(x+y)z + (x+y)\delta(z) + c_{F(z,x+y,\circ)}.$$

By Lemma 2.3, δ is additive, and hence we find

$$\begin{aligned} F(z \circ (x+y)) = F(z)x + F(z)y + F(x)z + F(y)z + c_{F(x,y,+)}z \\ + z\delta(x) + z\delta(y) + x\delta(z) + y\delta(z) + c_{F(z,x+y,\circ)}. \end{aligned} \tag{2}$$

On the other hand, we compute

$$\begin{aligned} F(z \circ (x+y)) &= F(z \circ x + z \circ y) \\ &= F(z \circ x) + F(z \circ y) + c_{F(z\circ x, z\circ y,+)} \\ &= F(z)x + z\delta(z) + F(x)z + x\delta(z) + F(z)y + z\delta(y) \\ &\quad + F(y)z + y\delta(z) + c_{F(z\circ x, z\circ y,+)} + c_{F(z,x,\circ)} + c_{F(z,y,\circ)}, \end{aligned} \tag{3}$$

where $c_{F(z\circ x, z\circ y,+)}, c_{F(z,x,\circ)}$ and $c_{F(z,y,\circ)}$ are the corresponding central elements.

Comparing expressions (2) and (3), we find

$$zc_{F(x,y,+)} + c_{F(z,x+y,\circ)} = c_{F(z\circ x, z\circ y,+)} + c_{F(z,x,\circ)} + c_{F(z,y,\circ)} \in Z(R)$$

for all $z \in R$. It forces that $Rc_{F(x,y,+)} \subseteq Z(R)$, where $c_{F(x,y,+)}$ is a fixed central element in R, but R has no nonzero central ideal, therefore $Rc_{F(x,y,+)} = \{0\}$. Likewise, we get $c_{F(x,y,+)}R = \{0\}$. It implies that $c_{F(x,y,+)} \in A(R)$, the annihilator of R. But $A(R)$ is always a central ideal in R, hence our hypothesis forces $A(R) = (0)$ and so $c_{F(x,y,+)} = 0$. From (1), we find $F(x+y) = F(x)+F(y)$ for all $x, y \in R$, as desired.

Corollary 3.2

Let R be a 2-torsion free noncommutative prime ring. If F is a CE- generalized Jordan derivation of R, constrained with CE-Jordan derivation δ, then F is additive.

Lemma 3.3

Let R be a 2-torsion free prime ring such that $[h,k] = 0$ for all $h \in H(R)$, $k \in S(R)$, then R satisfies s_4.

Proof. In the given condition, replace h by h^2 to obtain $[h,k]h + h[h,k] = 0$. For any fixed k in $S(R)$, we have $d(h)h + hd(h) = 0$ for all $h \in H(R)$, where $d(x) = [x,k]$ for all $x \in R$. For nonzero d, we have the desired result by Lemma 2.13. In case, $d = 0$, we conclude $S(R) \subseteq Z(R)$. It gives that $[u,r] = 0$ for all $u \in S(R)$ and $r \in R$. Since for each x in R, $x - x^*$ in $S(R)$, we have

$$[x - x^*, r] = 0 \qquad \text{for all } x, r \in R. \tag{4}$$

Replacing x by xk in (4), where $k \in S(R) \subseteq Z(R)$, we find $[x + x^*, r]k = 0$ for all $x, r \in R$ and $k \in S(R)$. Right multiply (4) by k and then compare with the last expression in order to get $2[x,r]k = 0$ for all $x, r \in R$ and $k \in S(R)$. It forces that either R is commutative or $S(R) = \{0\}$. In case $S(R) = \{0\}$, we see $xy = (xy)^* = y^*x^* = yx$ for all $x, y \in R$, i.e. R is commutative. Hence in each case R is commutative, and we are done.

Proposition 3.4
Let R be a 2-torsion free noncommutative prime ring with involution '$*$' and $F\colon R \to R$ a generalized derivation constrained with derivation δ. If $[F(x), x^*] = 0$ for all $x \in R$, then R is an order in a central simple algebra of dimension at most 4 over its center or $F = 0$.

Proof. By the given hypothesis, we have $[F(x), x^*] = 0$ for all $x \in R$. It follows that

$$[F(x)^*, x] = 0 \qquad \text{for all } x \in R.$$

Since F^* is additive and commuting function, thereby using Lemma 2.7, there exists $\lambda \in C$ and a mapping $\sigma\colon R \to C$ such that

$$F(x)^* = \lambda x + \sigma(x) \qquad \text{for all } x \in R.$$

It implies

$$F(x) = \lambda^* x^* + \sigma(x)^* \qquad \text{for all } x \in R. \tag{5}$$

Using Lemma 2.10, we have a in $Q_{ml}(R)$ such that

$$F(x) = ax + \delta(x) \qquad \text{for all } x \in R. \tag{6}$$

Compare (5) and (6) to obtain

$$\lambda^* x^* + \sigma(x)^* = ax + \delta(x) \qquad \text{for all } x \in R. \tag{7}$$

For any c in C, replace x by c in (7) to conclude $ac \in C$. Using primeness of R, and $C \neq \{0\}$, we conclude $a \in C$. Using this fact and taking h instead of x in (7), where $h \in H(R)$, we find $[\delta(h), h] = 0$ for all $h \in H(R)$. For nonzero δ, R satisfies s_4 identity by Lemma 2.11, but as R is assumed to be noncommutative, invoking Lemma 2.5 R is an order in a central simple algebra of dimension at most 4 over its center, as desired.

Now, If $\delta = 0$ from (6), we have $F(x) = ax$, where $a \in C$ for all $x \in R$. Replace x by h, where $h \in H(R)$ in (7) to obtain $(\lambda^* - a)h = -\sigma(h)^* \in C$ for all $h \in H(R)$. It implies either $\lambda^* = a$ or $H(R) \subseteq Z(R)$. In case $H(R) \subseteq Z(R)$, by Lemma 2.12, R satisfies s_4 identity by Lemma 2.11, but as R is assumed to be noncommutative, invoking Lemma 2.5 R is an order in a central simple algebra of dimension at most 4 over its center, as desired. Now if $\lambda^* = a$, then replacement of x by k in $S(R)$ in (7) gives $\lambda^* k$ in C for all $k \in S(R)$. Using primeness of R, if $0 \neq \lambda$, we have $S(R) \subseteq Z(R)$, which further implies R is commutative as we have already seen in the proof Lemma 3.3.

On the other hand, if $\lambda = 0$, then we have $F = 0$ as desired.

Remark 3.5

(a) Let R is 2-torsion free noncommutative prime ring. We now show that an additive map F is a CE-generalized Jordan derivation if and only if $F(x^2) - F(x)x - x\delta(x) \in Z(R)$.

$\Rightarrow$ Let F be an additive CE-gneralized Jordan derivation, i.e.

$$F(x \circ y) - F(x)y - F(y)x - x\delta(y) - y\delta(x) \in Z(R) \qquad \text{for all } x, y \in R.$$

Taking $x = y$ in this relation, we get

$$F(2x^2) - 2F(x)x - 2x\delta(x) \in Z(R).$$

Since F is additive and R is 2-torsion free, we have

$$F(x^2) - F(x)x - x\delta(x) \in Z(R) \qquad \text{for all } x \in R,$$

as desired.

$\Leftarrow$ On the other hand, let us suppose that F is an additive map satisfying

$$F(x^2) - F(x)x - x\delta(x) \in Z(R) \qquad \text{for all } x \in R.$$

Linearizing this relation, we find

$$\begin{aligned} &F(x^2 + x\circ y + y^2) - F(x)x - F(x)y - yF(x) - F(y)y \\ &\qquad - x\delta(x) - x\delta(y) - y\delta(x) - y\delta(y) \in Z(R) \qquad \text{for all } x, y \in R. \end{aligned}$$

Since F is additive, it yields

$$\begin{aligned} &(F(x^2) - F(x)x - x\delta(x)) + (F(x \circ y) - F(x)y - yF(x) - x\delta(y) - y\delta(x)) \\ &\qquad + (F(y^2) - (F(y)y - y\delta(y)) \in Z(R) \qquad \text{for all } x, y \in R. \end{aligned}$$

The given hypothesis reduces it to

$$F(x \circ y) - F(x)y - yF(x) - x\delta(y) - y\delta(x) \in Z(R) \qquad \text{for all } x, y \in R,$$

hence F is an additive CE-generalized Jordan derivation.

(b) In case R is a noncommutative prime ring, we have the following example of CE-generalized Jordan derivation. Let $\mathbb{Z}$ be the ring of integers and

$$R = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \ a, b, c, d \in \mathbb{Z} \right\},$$

a noncommutative prime ring. Then the mapping $F\colon R \to R$ such that

$$F\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & 2b \\ b+c & d \end{pmatrix}$$

with associated mapping $\delta\colon R \to R$ defined as

$$\delta\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix}.$$

Then one can notice that F is a CE-generalized Jordan derivation of R, which is not necessarily a generalized Jordan derivation or CE-generalized derivation.

3.1. Proof of Theorem 1.2

If $Z(R) = \{0\}$, then CE-generalized Jordan derivation is clearly a generalized Jordan derivation and by Lemma 2.9, every generalized Jordan derivation is a generalized derivation. Thus by the hypothesis, we have the situation $[F(x), x] = 0$ for all $x \in R$, where F is a generalized derivation of R. By a direct consequence of Lemma 2.8 there exists λ in C such that $F(x) = \lambda x$ for all $x \in R$.

For non-trivial implication, we assume $Z(R) \neq \{0\}$. By the given hypothesis, we have $[F(x), x] \in Z(R)$ for all $x \in R$. In view of Corollary 3.2, F is additive and hence by Lemma 2.6, it follows that

$$[F(x), x] = 0 \qquad \text{for all } x \in R.$$

Since F is additive and commuting map, thereby using Lemma 2.7 there exist $\lambda \in C$ and a mapping $\sigma\colon R \to C$ such that

$$F(x) = \lambda x + \sigma(x) \qquad \text{for all } x \in R. \tag{8}$$

It is obvious to see from (B) that

$$F(x^2) - F(x)x - x\delta(x) \in Z(R) \qquad \text{for all } x \in R, \tag{9}$$

In view of (9) and (8), it follows that

$$\lambda x^2 + \sigma(x^2) - \lambda x^2 - \sigma(x)x - x\delta(x) \in C \qquad \text{for all } x \in R.$$

It implies

$$\sigma(x)x + x\delta(x) \in C \qquad \text{for all } x \in R. \tag{10}$$

In particular, if $x \in Z(R)$ we obtain

$$x[\delta(x), y] = 0 \qquad \text{for all } y \in R,\ x \in Z(R),$$

which gives $xR[\delta(x), R] = \{0\}$. Therefore, for each $x \in Z(R)$ either $x = 0$ or $\delta(x) \in Z(R)$. As $Z(R)$ is an additive subgroup of R, by applying Brauer's trick, we have either $Z(R) = \{0\}$ or $\delta(Z(R)) \subseteq Z(R)$. In view of our assumption $Z(R) \neq \{0\}$, therefore we are left with $\delta(Z(R)) \subseteq Z(R)$. Polarizing (10), we have

$$\sigma(x)y + \sigma(y)x + y\delta(x) + x\delta(y) \in C \qquad \text{for all } x \in R. \tag{11}$$

In particular, take $0 \neq y \in Z(R)$ in (11) to conclude

$$[\delta(x), x] = 0 \qquad \text{for all } x \in R.$$

By Lemma 2.4, R is an order in a central simple algebra of dimension at most 4 over its center or $\delta = 0$. In case $\delta = 0$, from (10) we obtain $\sigma(x)x \in C$. Using the primeness of R and Brauer's trick, we conclude that either $\sigma = 0$ or R is commutative. Clearly, R can not be commutative, therefore we have from (8), $F(x) = \lambda x$ for all $x \in R$. This completes the proof.

3.2. Proof of Theorem 1.3

If $Z(R) = \{0\}$, then the CE-generalized Jordan derivation F is just an ordinary generalized Jordan derivation and hence a generalized derivation by Lemma 2.9. For a generalized derivation, we get the conclusion by Proposition 3.4.

Now we suppose that $Z(R) \neq \{0\}$. By the given hypothesis, we have

$$[F(x), x^*] \in Z(R) \qquad \text{for all } x \in R.$$

With the aid of Lemma 2.1 and Corollary 3.1 we get

$$[F(x), x^*] = 0 \qquad \text{for all } x \in R. \tag{12}$$

Applying involution on both sides in (12) we find

$$[F(x)^*, x] = 0 \qquad \text{for all } x \in R.$$

Using Lemma 2.7, there exist λ in C and a mapping $\sigma\colon R \to C$ such that

$$F(x)^* = \lambda x + \sigma(x) \qquad \text{for all } x \in R. \tag{13}$$

It implies that

$$F(x) = \lambda^* x^* + \sigma(x)^* \qquad \text{for all } x \in R. \tag{14}$$

Using (B), we find

$$F(x \circ h_c) - F(x)(h_c) - F(h_c)x - x\delta(h_c) - h_c\delta(x) \in Z(R) \qquad \text{for all } x \in R. \tag{15}$$

By (14), we have

$$\lambda^*(x \circ h_c)^* - \lambda^*(x)^* h_c - \sigma(h_c)^* x - \lambda^* h_c x - x\delta(h_c) - h_c\delta(x) \in C \quad \text{for all } x \in R. \tag{16}$$

Replace x by $0 \neq h_c$, where $h_c \in H(R) \cap Z(R)$ in (16) to obtain

$$\delta(h_c) \in C. \tag{17}$$

Expanding (16), we find

$$\lambda^*(x^* - x)h_c - \sigma(h_c)^* x - x\delta(h_c) - h_c\delta(x) \in C \qquad \text{for all } x \in R. \tag{18}$$

We now split the proof into two parts.

Case 1. Suppose that the involution induced on C is not identity. Then there exists c in C such that $c^* \neq c$. Let $c^* - c = z_c$. Clearly $z_c^* = -z_c \neq 0$ and z_c in C. By Lemma 2.2, there exists a nonzero ideal J of R such that $z_c J \subseteq R$. Replace x by jz_c, where j in J in (18) to obtain

$$\lambda^*(-j^* - j)z_c h_c - \sigma(h_c)^* j z_c - j\delta(h_c)z_c - h_c\delta(jz_c) \in C \qquad \text{for all } j \in J. \tag{19}$$

In particular, put $x = j$, where j in J in (18) to conclude

$$\lambda^*(j^* - j)h_c - \sigma(h_c)^* j - j\delta(h_c) - h_c\delta(j) \in C \qquad \text{for all } j \in J. \tag{20}$$

Multiply (20) with z_c and then compare with (19) to get

$$-2\lambda^* j^* z_c h_c - h_c\delta(jz_c) + h_c\delta(j)z_c \in C \qquad \text{for all } j \in J.$$

As $0 \neq h_c$, it gives

$$-2\lambda^* j^* z_c - \delta(jz_c) + \delta(j)z_c \in C \qquad \text{for all } j \in J. \tag{21}$$

Replacing j by $j \circ y$ in (21), we may infer that

$$(-2\lambda^* j^* z_c) \circ y^* - \delta(jz_c) \circ y - (jz_c) \circ \delta(y) + (\delta(j) \circ y + j \circ \delta(y))z_c \in C \qquad \text{for all } y \in R, j \in J.$$

It can also be written as

$$(-2\lambda^* j^* z_c) \circ y^* - \delta(jz_c) \circ y + (\delta(j) \circ y)z_c \in C \text{ for all } j \in J,\ y \in R. \tag{22}$$

Replace y by $j_1 z_c$ in (22) to obtain

$$2(\lambda^* j^* z_c) \circ j_1^* z_c - (\delta(jz_c) \circ j_1)z_c + (\delta(j) \circ j_1)z_c^2 \in C \qquad \text{for all } j, j_1 \in J,\ y \in R. \tag{23}$$

Replace y by j_1 in (22) to get

$$(-2\lambda^* j^* z_c) \circ j_1^* - (\delta(jz_c)) \circ j_1 + (\delta(j) \circ j_1)z_c \in C \qquad \text{for all } j, j_1 \in J.$$

Right multiplying the above expression by z_c, we have

$$(-2\lambda^* j^* z_c) \circ j_1^* z_c - ((\delta(jz_c)) \circ j_1)z_c + (\delta(j) \circ j_1)z_c^2 \in C \qquad \text{for all } j, j_1 \in J. \tag{24}$$

Compare (23) and (24) to obtain $4(\lambda^* j^* z_c) \circ (j_1^* z_c) \in C$.

Since $z_c \neq 0$, by using primness of R, we find either $4\lambda = 0$ or $j \circ j_1 \in Z(R)$ for all $j, j_1 \in J$. If $J^2 \subseteq Z(R)$, then it is not difficult to get R is commutative, which is a contradiction. Therefore we have $\lambda = 0$; using it in (13) to conclude $[\delta(x), x] = 0$ for all $x \in R$. In view of Lemma 2.4, we are done.

Case 2. If the involution induced on C is identity, then $c^* = c$ for all $c \in C$. Replacing x by h in (18), where $h \in H(R)$, we have

$$-\sigma(h_c)h - h\delta(h_c) - h_c\delta(h) \in C. \qquad (25)$$

Commuting with x and using (17) give

$$\delta(h_c)[h,x] + \sigma(h_c)[h,x] + h_c[\delta(h),x] = 0$$

Substituting h by h^2 in the last relation and then simplify it, we conclude

$$\delta(h)[h,x] + [h,x]\delta(h) = 0. \qquad (26)$$

Polarizing the variable h in (26), we find

$$\delta(h_1)[h,x] + \delta(h)[h_1,x] + [h,x]\delta(h_1) + [h_1,x]\delta(h) = 0 \qquad \text{for all } h, h_1 \in H(R).$$

In particular, replace h_1 by h_c to obtain

$$2\delta(h_c)[h,x] = 0 \qquad \text{for all } h \in H(R).$$

Using primeness of R, if $\delta(h_c) \neq 0$, then $H(R) \subseteq Z(R)$ and hence R is an order in a central simple algebra of dimension at most 4 over its center by Lemma 2.12. In case $\delta(h_c) = 0$, replacing x by k in (18), where k in $S(R)$, we obtain

$$-2\lambda k h_c - \sigma(h_c)k - h_c\delta(k) \in C \qquad \text{for all } k \in S(R). \qquad (27)$$

It implies

$$(-2\lambda k h_c - \sigma(h_c)k - h_c\delta(k))^* \in C \qquad \text{for all } k \in S(R).$$

It can also be written as

$$2\lambda k h_c + \sigma(h_c)k - h_c\delta(k)^* \in C \qquad \text{for all } k \in S(R). \qquad (28)$$

Adding (27) and (28) yields

$$\delta(k) + \delta(k)^* \in C \qquad \text{for all } k \in S(R). \qquad (29)$$

From (25), we also have

$$-\sigma(h_c)h - h_c\delta(h) \in C \qquad \text{for all } h \in H(R).$$

In view of our assumption, it follows that

$$(-\sigma(h_c)h - h_c\delta(h))^* = -\sigma(h_c)h - h_c\delta(h)$$

Since h_c is nonzero, it implies that $\delta(h)^* = \delta(h)$ for all $h \in H(R)$. From (27), we also have

$$[\delta(k), k] = 0 \qquad \text{for all } k \in S(R).$$

Polarize the above equation to obtain

$$[\delta(k), k_1] + [\delta(k_1), k] = 0 \qquad \text{for all } k, k_1 \in S(R).$$

Replace k_1 by $h \circ k$, where h in $H(R)$, k in $S(R)$ to get

$$[\delta(h) \circ k, k] + [h \circ \delta(k), k] + [\delta(k), h \circ k] = 0. \tag{30}$$

Taking involution on both sides in (30) and using the fact that $\delta(h)^* = \delta(h)$ for all h in $H(R)$, we find

$$-[\delta(h) \circ k, k] + [h \circ \delta(k)^*, k] + [\delta(k)^*, h \circ k] = 0 \qquad \text{for all } k \in S(R),\ h \in H(R). \tag{31}$$

Adding (30) and (31) yields

$$[h \circ (\delta(k) + \delta(k)^*), k] + [\delta(k) + \delta(k)^*, h \circ k] = 0 \qquad \text{for all } k \in S(R),\ h \in H(R).$$

Using (29), we have

$$(\delta(k) + \delta(k)^*)2[h, k] = 0 \qquad \text{for all } k \in S(R),\ h \in H(R).$$

It forces that for each k in $S(R)$ either $[h, k] = 0$ for all $h \in H(R)$ or $\delta(k)+\delta(k)^* = 0$. Invoking Brauer's trick, we have either $[H(R), S(R)] = \{0\}$ or $\delta(k)^* = -\delta(k)$ for all $k \in S(R)$. In the former case, we get our conclusion from Lemma 3.3.

Therefore, we left with $\delta(k)^* = -\delta(k)$ for all $k \in S(R)$. From (27), we have

$$(-2\lambda k h_c - \sigma(h_c)k - h_c\delta(k))^* = -2\lambda k h_c - \sigma(h_c)k - h_c\delta(k) \qquad \text{for all } k \in S(R).$$

Since $c^* = c$ for all $c \in C$, it implies

$$2\lambda k h_c + \sigma(h_c)k + h_c\delta(k) = -2\lambda k h_c - \sigma(h_c)k - h_c\delta(k).$$

It can also be written as

$$4\lambda k h_c + 2\sigma(h_c)k + 2h_c\delta(k) = 0 \qquad \text{for all } k \in S(R). \tag{32}$$

Replace k by $k \circ h$, where h in $H(R)$, k in $S(R)$ to obtain

$$(4\lambda k h_c + 2\sigma(h_c)k + 2h_c\delta(k))h + h(4\lambda k h_c + 2\sigma(h_c)k + 2h_c\delta(k)) + 2h_c(k \circ \delta(h)) + 2h_c c_{\delta(k,h,\circ)} = 0,$$

where $c_{\delta(k,h,\circ)} \in Z(R)$. In view of (32), it follows that

$$k \circ \delta(h) \in Z(R) \qquad \text{for all } h \in H(R),\ k \in S(R).$$

Commuting with k, we get

$$[\delta(h), k]k + k[\delta(h), k] = 0 \qquad \text{for all } k \in S(R),\ h \in H(R).$$

For fixed h in $H(R)$, we have $d(k)k + kd(k) = 0$ for all $k \in S(R)$, where $d(x) = [\delta(h), x]$. Using Lemma 2.14, we have our conclusion or $\delta(h)$ in $Z(R)$ for all $h \in H(R)$. Using (26), we have $\delta(h)[h, r] = 0$. That gives $\delta(h)R[h, r] = 0$ for all $h \in H(R)$, $r \in R$. Using Brauer's trick, we have either $H(R) \subseteq Z(R)$ or $\delta(H(R)) = \{0\}$.

The former case gives the desired result by Lemma 2.12 and in the latter case, using (25), we have $\sigma(h_c)h \in C$ for all $h \in H(R)$. It implies $\sigma(h_c) = 0$ or $h \subseteq Z(R)$. In view of Lemma 2.12, h in $Z(R)$ for all $h \in H(R)$ gives the desired result.

Assume that $\sigma(h_c) = 0$ for all $h_c \in H(R) \cap Z(R)$ and using it in (32), we get $2h_c(2\lambda k + \delta(k)) = 0$. Since $h_c \neq 0$, it implies that $\delta(k) = -2\lambda k$ for all $k \in S(R)$. Now from (B), we have

$$F(k^2) - F(k)k - k\delta(k) \in Z(R) \qquad \text{for all } k \in S(R). \tag{33}$$

Using (14) in (33), we find

$$\lambda^*(k^2)^* + \sigma(k^2)^* - \lambda^* k^* k - \sigma(k)^* k - k(-2\lambda k) \in C.$$

It implies

$$4\lambda k^2 - \sigma(k)k \in C \qquad \text{for all } k \in S(R). \tag{34}$$

Since $c^* = c$ for all $c \in C$. So, we conclude that

$$(4\lambda k^2 - \sigma(k)k)^* = 4\lambda k^2 - \sigma(k)k \qquad \text{for all } k \in S(R),$$

$$4\lambda k^2 + \sigma(k)k = 4\lambda k^2 - \sigma(k)k \qquad \text{for all } k \in S(R).$$

It implies $\sigma(k)k = 0$. Using primeness of R and Brauer's trick, we obtain that either $\sigma(k) = 0$ for all $k \in S(R)$ or $S(R) = \{0\}$. The case $S(R) = \{0\}$ leads a contradiction, as it gives R commutative.

On the other hand, using (34), we have λk^2 in C. It implies either $\lambda = 0$ or $k^2 \in Z(R)$. Suppose that k^2 in $Z(R)$ for all $k \in S(R)$, we have $[k, x]k + k[k, x] = 0$. For any fixed x in R, we have $d(k)k + kd(k) = 0$ for all $k \in S(R)$, where $d(y) = [y, x]$ for all $y \in R$. Using Lemma 2.14, either R satisfy s_4 identity or $d = 0$, i.e. $[x, y] = 0$ for all x, y in R. Thus, we have the result.

If $\lambda = 0$, then using (18), we obtain $[\delta(x), x] = 0$ for all $x \in R$. With the aid of Lemma 2.4, we get the desired outcome. It completes the proof.

Acknowledgement. We would like to thank the anonymous referee for their expert suggestion and comments on the earlier draft of this paper. The research for the first author is supported by the University Grants Commission, New Delhi under the Junior Research Fellowship Award (Grant no. 402996).

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Bharat Bhushan
1 Chitkara University Institute
of Engineering and Technology
Chitkara University
Rajpura 140401 Punjab
India
E-mail: bharat_rs18@pbi.ac.in

Gurninder S. Sandhu
Department of Mathematics
Patel Memorial National College
Rajpura 140401, Punjab
India
E-mail: gurninder_rs@pbi.ac.in

Deepak Kumar
Department of Mathematics
Punjabi University
Patiala 147002, Punjab
India
E-mail: deep_math1@yahoo.com

Received: October 16, 2022; final version: March 31, 2023;
available online: May 26, 2023.