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Existence results of self-similar solutions of the space-fractional diffusion equation involving the generalized Riesz-Caputo fractional derivative

Abstract. In this paper, we have discussed the problem of existence and uniqueness of solutions under the self-similar form to the space-fractional diffusion equation. The space-fractional derivative which will be used is the generalized Riesz-Caputo fractional derivative. Based on the similarity variable η, we have introduced the equation satisfied by the self-similar solutions for the aforementioned problem. To study the existence and uniqueness of self-similar solutions for this problem, we have applied some known fixed point theorems (i.e. Banach's contraction principle, Schauder's fixed point theorem and the nonlinear alternative of Leray-Schauder type).

1. Introduction

Fractional calculus (FC) is a mathematical analysis subject which deals with different possible approaches of defining fractional order derivatives (FODs) and integrals (FOIs). The theory of classical (integer order) differential equations (IODEs) has been then generalized to the broader theory of fractional order differential equations (FDEs). For more details on the subject, the reader may refer to [7, 26, 31].

To define fractional integrals and derivatives, many approaches have been proposed in the literature, including Riemann-Liouville's (RL), Hadamard's, Caputo's, Riesz's, Erdelyi-Kober's approaches, etc. The development of each one of approches should go through a series of stages ranging from exponential functions

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to special functions. Later, in [23, 24], a new fractional operator which generalized both the RL and Hadamard operators was introduced by Katugampola. Not long ago, in [3], a Caputo-type modification of this operator was proposed. This later is the Caputo type of generalized fractional derivative (CGFD). It represents a generalization of the Caputo and Caputo-Hadamard FDs. Aleem et al. in [1], presented a generalisation of the Riesz fractional operator, where this operator covers as particular cases the classical Riesz fractional derivative. In the same paper, the authors have also proposed a Caputo-type modification of this operator. This new fractional derivative (FD) was named as the generalized Riesz-Caputo fractional derivative (or the Riesz-Caputo generalized fractional derivative (R-CGFD)). In the same paper, some fundamental results have been introduced and proved.

Different fixed-point theorems have been used by researchers to develop solutions and their existence for non-linear initial value problems (IVPs) and boundary value problems (BVPs) of fractional differential equations (see [4, 5, 34, 6, 8, 9])

Fractional partial DEs or simply FPDEs can be used for the modelling and study of many important phenomena in many different fields of science and engineering, such as diffusion processes, damping law, etc. One can find a variety of applications in [19, 22, 25, 29, 33].

The existence and uniqueness of solutions of non-linear FPDEs have been studied in many papers including [11, 12, 13, 21, 30].

Generally, for PDEs, we can search for special type solutions known as group-invariant solutions. As in [14], by solving a reduced system of equations (which has fewer independent variables compared to the original problem), the group-invariant solutions can be found. These solutions are also known as self-similar or scale-invariant solutions which are used to model many processes in mathematics and engineering's mechanics. The FPDEs which have self-similar solutions can easily be reduced to ordinary differential equations (ODEs). This latter process helps to simplify one's work on FPDEs.

The idea behind solutions' self-similarity along with Lie group analysis have also been applied in FDEs. For example, Luchko and Gorenflo in [27] and Buckwar in [14] have been discussed the application of Lie group analysis for the equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} = d\frac{\partial^\beta u}{\partial x^\beta}, \qquad x > 0,\ t > 0,\ d > 0,\ \alpha, \beta \geq 0.$$

They have found a scale-invariant solutions for the fractional ordinary differential equation (FODE) with a new independent variable $\eta = xt^{\frac{-\alpha}{\beta}}$. The left and right sided Erdélyi-Kober derivatives which depend on α, β of this equation and on the parameter γ of its scaled group are considered. They have derived a general solution in terms of the generalized Wright function.

Across the literature, one may easily be aware of the existence of plenty of research works on fractional (space, time and space-time) diffusion equations by using the similarity method. For more details, the reader may check [10, 17, 28].

In this paper, we discuss the existence, uniqueness and main properties of the solution of the following space-fractional diffusion equation, which is

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^{\alpha,\rho} u(x,t)}{\partial |x|^\alpha}, \qquad (x,t) \in [0, X] \times [t_0, \infty[,\ 1 < \alpha \leq 2, \tag{1}$$

under the following self-similar form

$$u(x,t) = t^{\beta} f\left(\frac{x}{t^{\frac{1}{\alpha\rho}}}\right) \qquad \text{with } (x,t) \in [0,X] \times [t_0, \infty[, \tag{2}$$

where $u(x,t)$ is a scalar function of space and time variables $(x,t) \in [0,X] \times [t_0, \infty[$ with $X, t_0 > 0$, $\frac{\partial^{\alpha,\rho}}{\partial|x|^{\alpha}}$ is the R-CGFD of order α with $\rho > 0$ and which is the main motivation of the present research, the "basic profile" f in (2) is not known in advance and is to be identified and $\beta \in \mathbb{R}$ is a constant chosen so that the solutions exist.

The rest of this paper is structured as follows. In the next section, we recall preliminaries related to some definitions of fractional integrals and derivatives, theorems and lemmas of FC. The main results are given in section 3. Finally, this paper is ended with a conclusion.

2. Preliminaries and definitions

In this section, we give the necessary definitions, notations, lemmas and theorems from FC theory which will be used through the whole of this work. Let $J = [0, \mu]$ be a finite interval of $\mathbb{R}$ with $\mu > 0$. We denote by $C(J, \mathbb{R})$ the Banach space of all continuous functions $g\colon [0,\mu] \to \mathbb{R}$ with the norm

$$\|g\|_{\infty} = \sup_{\eta \in [0,\mu]} |g(\eta)|.$$

We denote also $C^n(J, \mathbb{R})$ with $n \in \mathbb{N}_0$ the set of mappings having n times continuously differentiable on J.

As in [26], for $1 \le p \le \infty$ and $c \in \mathbb{R}$, consider the space $X_c^p[a,b]$ as follows

$$X_c^p[a,b] = \left\{ g\colon [a,b] \to \mathbb{R} :\ \|g\|_{X_c^p} = \left(\int_a^b |s^c g(s)|^p \frac{ds}{s} \right)^{\frac{1}{p}} < \infty \right\},$$

for $1 \le p < \infty$, c $\in \mathbb{R}$. For the case $p = \infty$,

$$\|g\|_{X_c^{\infty}} = \operatorname*{ess\,sup}_{a \le \eta \le b} [\eta^c |g(\eta)|], \qquad c \in \mathbb{R}.$$

Definition 2.1 (Generalized fractional integrals. (see [2, 23]))
The left-sided and right-sided of the generalized fractional integrals of order $\alpha > 0$ and parameter $\rho > 0$ of an integrable function $g\colon [0,\mu] \to \mathbb{R}$ with $\mu > 0$ are defined respectively by

$$(I_{0^+}^{\alpha,\rho} g)(\eta) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^{\eta} s^{\rho-1} (\eta^{\rho} - s^{\rho})^{\alpha-1} g(s) ds, \qquad \eta > 0 \tag{3}$$

and

$$(I_{\mu^-}^{\alpha,\rho} g)(\eta) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_{\eta}^{\mu} s^{\rho-1} (s^{\rho} - \eta^{\rho})^{\alpha-1} g(s) ds, \qquad \eta < \mu, \tag{4}$$

where $\Gamma(.)$ is Euler's gamma function defined as

$$\Gamma(\alpha) = \int_0^{+\infty} x^{\alpha-1} e^{-x} dx, \qquad \alpha \in \mathbb{C},\ \operatorname{Re}(\alpha) > 0.$$

Definition 2.2 (CGFDs. (see [2]))
Let $\mu > 0$, ρ be a positive real number, $\alpha \in \mathbb{R}^+$ and $n \in \mathbb{N}$ be such that $\alpha \in (n-1, n)$, and $g\colon [0,\mu] \to \mathbb{R}$ a function of class C^n. The left-sided and right-sided of CGFDs of order α and parameter ρ are defined respectively by

$$ {}^{C}D^{\alpha,\rho}_{0^+} g(\eta) = \frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)} \int_0^{\eta} s^{\rho-1}(\eta^\rho - s^\rho)^{n-\alpha-1}\Big(s^{1-\rho}\frac{d}{ds}\Big)^n g(s)ds \tag{5} $$

and

$$ {}^{C}D^{\alpha,\rho}_{\mu^-} g(\eta) = \frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)} \int_\eta^{\mu} s^{\rho-1}(s^\rho - \eta^\rho)^{n-\alpha-1}\Big(-s^{1-\rho}\frac{d}{ds}\Big)^n g(s)ds. \tag{6} $$

The next two results justify the definition 2.2, since the Caputo-type of the generalized fractional derivative is an inverse operation of the generalized fractional integral.

Theorem 2.3 (see [2])
Let $\alpha > 0$ be such that $\alpha \in (n-1, n)$, $n \in \mathbb{N}$ and $\rho > 0$. Given a function $g \in C^n[0,\mu]$, we have

$$ (I^{\alpha,\rho}_{0^+}\, {}^{C}D^{\alpha,\rho}_{0^+} g)(\eta) = g(\eta) - \sum_{k=0}^{n-1} \frac{g^{(k)}(0)}{k!}\Big(\frac{\eta^\rho}{\rho}\Big)^k, $$

and

$$ (I^{\alpha,\rho}_{\mu^-}\, {}^{C}D^{\alpha,\rho}_{\mu^-} g)(\eta) = g(\eta) - \sum_{k=0}^{n-1} \frac{(-1)^k g^{(k)}(\mu)}{k!}\Big(\frac{\mu^\rho - \eta^\rho}{\rho}\Big)^k. $$

The following theorem yields the compositions of the fractional integral operators $I^{\alpha,\rho}_{0^+}$ and $I^{\alpha,\rho}_{\mu^-}$ with the fractional differential operators ${}^{C}D^{\alpha,\rho}_{0^+}$ and ${}^{C}D^{\alpha,\rho}_{\mu^-}$, respectively.

Theorem 2.4 (see [32])
Let $\alpha, \beta, \rho \in \mathbb{R}$ be such that $\alpha > \beta$ and $\beta > 0$. If $g \in X^p_c[0,\mu]$, then for $\rho > 0$,

$$ ({}^{C}D^{\beta,\rho}_{0^+} I^{\alpha,\rho}_{0^+} g)(\eta) = (I^{\alpha-\beta,\rho}_{0^+} g)(\eta), $$
$$ ({}^{C}D^{\beta,\rho}_{\mu^-} I^{\alpha,\rho}_{\mu^-} g)(\eta) = (I^{\alpha-\beta,\rho}_{\mu^-} g)(\eta). $$

Definition 2.5 (Riesz-generalized fractional integral. (see [1]))
Let $g(\eta) \in X^p_c(0,\mu)$ and $\alpha, \rho > 0$. Then, for $0 \le \eta \le \mu$, the generalized Riesz type integral is defined as

$$ \begin{aligned} ({}^{RG}_{\ 0}I^{\alpha,\rho}_{\mu} g)(\eta) &= \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^{\mu} s^{\rho-1}|(s^\rho - \eta^\rho)|^{\alpha-1} g(s)ds \\ &= (I^{\alpha,\rho}_{0^+} g)(\eta) + (I^{\alpha,\rho}_{\mu^-} g)(\eta), \end{aligned} \tag{7} $$

where $I^{\alpha,\rho}_{0^+}$ and $I^{\alpha,\rho}_{\mu^-}$ are left and right sided generalized fractional integrals, defined in (3) and (4), respectively.

Now, we define the R-CGFD.

Definition 2.6 (Riesz-CGFD (see [1]))
Let $\alpha,\rho\in\mathbb{C}$ with $\mathrm{Re}(\alpha),\mathrm{Re}(\rho)>0$ and $g(\eta)\in X_c^p(0,\mu)$ for $0\le\eta\le\mu$. Then the R-CGFD is defined as

$$\begin{aligned}{}^{RC}_{\;0}D^{\alpha,\rho}_{\mu}g(\eta)&=\frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)}\int_0^{\mu}s^{\rho-1}|(\eta^\rho-s^\rho)|^{n-\alpha-1}\Big(s^{1-\rho}\frac{d}{ds}\Big)^n g(s)ds\\&=\frac{1}{2}({}^{C}D^{\alpha,\rho}_{0^+}+(-1)^n\,{}^{C}D^{\alpha,\rho}_{\mu^-})g(\eta),\end{aligned}$$

where ${}^{C}D^{\alpha,\rho}_{0^+}$ and ${}^{C}D^{\alpha,\rho}_{\mu^-}$ are left and right sided of CGFDs which defined in (5) and (6), respectively.

Lemma 2.7 (see [1])
Let $g\in AC^n_\delta[0,\mu]$ with $0\le\eta\le\mu$. Then the following relation is true

$${}^{RG}_{\;0}I^{\alpha,\rho}_{\mu}\,{}^{RC}_{\;0}D^{\alpha,\rho}_{\mu}g(\eta)=\frac{1}{2}(I^{\alpha,\rho}_{0^+}\,{}^{C}D^{\alpha,\rho}_{0^+}+(-1)^nI^{\alpha,\rho}_{\mu^-}\,{}^{C}D^{\alpha,\rho}_{\mu^-})g(\eta). \tag{8}$$

Remark 2.8
If $1<\alpha\le 2$ and $\rho>0$, then for $g(\eta)\in C[0,\mu]$, the relation illustrated in (8) becomes

$${}^{RG}_{\;0}I^{\alpha,\rho}_{\mu}\,{}^{RC}_{\;0}D^{\alpha,\rho}_{\mu}g(\eta)=g(\eta)-\frac{1}{2}[g(0)+g(\mu)]-\frac{\eta^\rho}{2\rho}[g'(0)+g'(\mu)]+\frac{\mu^\rho}{2\rho}g'(\mu).$$

Remark 2.9
If $1<\alpha\le 2$, then, for all $g\in C[0,\mu]$, and by Remark 2.8, Theorem 2.4, we have

$$\begin{aligned}&{}^{RG}_{\;0}I^{\alpha-1,\rho}_{\mu}\,{}^{RC}_{\;0}D^{\alpha,\rho}_{\mu}g(\eta)\\&\quad=\frac{d}{d\eta}\,{}^{RG}_{\;0}I^{\alpha,\rho}_{\mu}\,{}^{RC}_{\;0}D^{\alpha,\rho}_{\mu}g(\eta)\\&\quad=\frac{d}{d\eta}\Big[g(\eta)-\frac{1}{2}[g(0)+g(\mu)]+\frac{\mu^\rho}{2\rho}g'(\mu)-\frac{\eta^\rho}{2\rho}[g'(0)+g'(\mu)]\Big]\\&\quad=g'(\eta)-\frac{[g'(0)+g'(\mu)]}{2}\eta^{\rho-1}.\end{aligned}\tag{9}$$

Furthermore, if $g'(0)+g'(\mu)=0$, then

$${}^{RG}_{\;0}I^{\alpha-1,\rho}_{\mu}\,{}^{RC}_{\;0}D^{\alpha,\rho}_{\mu}g(\eta)=g'(\eta). \tag{10}$$

In addition, for each $\eta\in[0,\mu]$, using the fact that $\Gamma(\alpha+1)=\alpha\Gamma(\alpha)$, we obtain

$$\begin{aligned}|g'(\eta)|&=|{}^{RG}_{\;0}I^{\alpha-1,\rho}_{\mu}\,{}^{RC}_{\;0}D^{\alpha,\rho}_{\mu}g(\eta)|\\&=\left|\frac{\rho^{2-\alpha}}{\Gamma(\alpha-1)}\int_0^\mu\zeta^{\rho-1}|(\zeta^\rho-\eta^\rho)|^{\alpha-2}\,{}^{RC}_{\;0}D^{\alpha,\rho}_{\mu}g(\zeta)d\zeta\right|\\&\le\frac{\rho^{2-\alpha}}{\Gamma(\alpha-1)}\int_0^\mu\zeta^{\rho-1}|(\zeta^\rho-\eta^\rho)|^{\alpha-2}\,|{}^{RC}_{\;0}D^{\alpha,\rho}_{\mu}g(\zeta)|d\zeta\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{\rho^{2-\alpha}}{\Gamma(\alpha-1)}\int_0^\eta \zeta^{\rho-1}(\eta^\rho-\zeta^\rho)^{\alpha-2}\,|{}^{RC}_{0}D^{\alpha,\rho}_{\mu} g(\zeta)|d\zeta \\
&\quad + \frac{\rho^{2-\alpha}}{\Gamma(\alpha-1)}\int_\eta^\mu \zeta^{\rho-1}(\zeta^\rho-\eta^\rho)^{\alpha-2}\,|{}^{RC}_{0}D^{\alpha,\rho}_{\mu} g(\zeta)|d\zeta \\
&\leq \sup\nolimits_{0\leq\eta\leq\mu}|{}^{RC}_{0}D^{\alpha,\rho}_{\mu} g(\eta)|\Big[\frac{\rho^{2-\alpha}}{\Gamma(\alpha-1)}\int_0^\eta \zeta^{\rho-1}(\eta^\rho-\zeta^\rho)^{\alpha-2}d\zeta \\
&\quad + \frac{\rho^{2-\alpha}}{\Gamma(\alpha-1)}\int_\eta^\mu \zeta^{\rho-1}(\zeta^\rho-\eta^\rho)^{\alpha-2}d\zeta\Big] \qquad (11)\\
&= \|{}^{RC}_{0}D^{\alpha,\rho}_{\mu} g\|_\infty\Big[\frac{-\rho^{2-\alpha}}{\rho(\alpha-1)\Gamma(\alpha-1)}\int_0^\eta \frac{d}{d\zeta}((\eta^\rho-\zeta^\rho)^{\alpha-1}) \\
&\quad + \frac{\rho^{2-\alpha}}{\rho(\alpha-1)\Gamma(\alpha-1)}\int_\eta^\mu \frac{d}{d\zeta}((\zeta^\rho-\eta^\rho)^{\alpha-1})\Big] \\
&= \|{}^{RC}_{0}D^{\alpha,\rho}_{\mu} g\|_\infty\Big\{\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\big[-[(\eta^\rho-\zeta^\rho)^{\alpha-1}]_0^\eta+[(\zeta^\rho-\eta^\rho)^{\alpha-1}]_\eta^\mu\big]\Big\} \\
&= \frac{(\eta^{\rho(\alpha-1)}+(\mu^\rho-\eta^\rho)^{\alpha-1})}{\rho^{\alpha-1}\Gamma(\alpha)}\|{}^{RC}_{0}D^{\alpha,\rho}_{\mu} g\|_\infty \\
&\leq \frac{2\mu^{\rho(\alpha-1)}}{\rho^{\alpha-1}\Gamma(\alpha)}\|{}^{RC}_{0}D^{\alpha,\rho}_{\mu} g\|_\infty .
\end{aligned}
$$

Definition 2.10 (Equicontinuity)
Let E be a Banach space. We call a part P in $C(E)$ is equicontinuous if

$$\forall\varepsilon>0\ \exists\delta>0\ \forall u,v\in E\ \forall A\in P\ \|u-v\|<\delta\Rightarrow\|A(u)-A(v)\|<\varepsilon.$$

Definition 2.11 (Complete continuity (see [16]))
We say $A\colon E\to E$ is completely continuous if for any bounded subset P of E, the set $A(P)$ is relatively compact.

Theorem 2.12 (Arzelà-Ascoli's Theorem (see [18]))
Let $B\subset C(E,\mathbb{R}^n)$, where $E=[a,b]\subset\mathbb{R}$. B is relatively compact (i.e. $\overline{B}$ is compact) if and only if

1. *B is uniformly bounded,*
2. *B is equicontinuous.*

Recall that a function f is uniformly bounded in B if there exists a constant $M>0$ such that

$$\|f\|=\sup_{x\in E}|f(x)|\leq M \qquad \text{for all } f\in B.$$

Lemma 2.13 (The generalized Gronwall inequality (see [1]))
Let $\alpha>0$, $0<\eta<\mu$ and assume that $g(\eta),u_1(\eta)$ and $u_2(\eta)$ are locally integrable, nonnegative and non-decreasing functions. Also, assume that $v_1(\eta)$ and $v_2(\eta)$ are a non-decreasing continuous functions such that $0\leq v_1(\eta),v_2(\eta)\leq L$, where L is a constant.

Furthermore, if $g(\eta)$ satisfies the inequality

$$g(\eta) \leq u_1(\eta) + \rho^{1-\alpha} v_1(\eta) \int_0^{\eta} \zeta^{\rho-1} (\eta^{\rho} - \zeta^{\rho})^{\alpha-1} g(\zeta) d\zeta + u_2(\eta) + \rho^{1-\alpha} v_2(\eta) \int_{\eta}^{\mu} \zeta^{\rho-1} (\zeta^{\rho} - \eta^{\rho})^{\alpha-1} g(\zeta) d\zeta,$$

then the following inequality holds true

$$g(\eta) \leq (u_1(\eta) + u_2(\eta)) E_{\alpha,1}(\rho^{-\alpha} v_2(\eta) \Gamma(\alpha)(\mu^{\rho} - \eta^{\rho})^{\alpha}) \times E_{\alpha,1}(\rho^{-\alpha} v_1(\eta) \Gamma(\alpha) \eta^{\rho\alpha}),$$

where $E_{\alpha,1}(.)$ is a Mittag-Leffler function.

Theorem 2.14 (Banach's Fixed Point Theorem (see [15]))
Let E be a Banach space and $Q \colon E \to E$ is a contraction mapping. Then Q has a fixed point, i.e.

$$\exists ! x \in E : \; Qx = x.$$

Theorem 2.15 (Schauder's Fixed Point Theorem (see [20]))
Let E be a Banach space, and let P be a closed, convex and non-empty subset of E. Let $T \colon PtoP$ be a continuous mapping such that $T(P)$ is a relatively compact subset of E. Then T has at least one fixed point in P.

Theorem 2.16 (Nonlinear alternative of Leray-Schauder type (see [20]))
Let E be a Banach space with $P \subset E$ be a closed and convex. U be an open subset of P with $0 \in U$. Assume that $A \colon \overline{U} \to P$ is a continuous, compact (that is, $A(\overline{U})$ is a relatively compact subset of P) map. Then either;

(i) *A has a fixed point in $\overline{U}$; or*

(ii) *there is a point $u \in \partial U$ and $\sigma \in (0,1)$ with $u = \sigma A(u)$.*

3. The main results

3.1. Statement of the problem

In this subsection, we consider the following problem of the space-fractional diffusion equation

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = \dfrac{\partial^{\alpha,\rho} u(x,t)}{\partial |x|^{\alpha}}, & (x,t) \in [0,X] \times [t_0, \infty[, \\ u(0,t) + u(X,t) = t^{\beta} M, & t \in [t_0, \infty[, \\ \dfrac{\partial u(0,t)}{\partial x} + \dfrac{\partial u(X,t)}{\partial x} = 0, & t \in [t_0, \infty[, \end{cases}$$

where $\frac{\partial^{\alpha,\rho} u}{\partial |x|^{\alpha}}$ the R-CGFD of order α $(1 < \alpha \leq 2)$, and $\beta, M \in \mathbb{R}$. Under the self-similar solution form

$$u(x,t) = t^{\beta} f\Big(\frac{x}{t^{\frac{1}{\alpha\rho}}}\Big), \qquad \beta \in \mathbb{R}. \tag{12}$$

We should first deduce the equation satisfied by the function f in (12).

Theorem 3.1
Let $\alpha, \beta, \rho \in \mathbb{R}$ provided that $1 < \alpha \leq 2$, $\rho > 0$ and $(x,t) \in [0,X] \times [t_0, \infty[$ for some $X, t_0 > 0$. Then the transformation

$$u(x,t) = t^{\beta} f(\eta) \qquad \text{with } \eta = \frac{x}{t^{\frac{1}{\alpha\rho}}},$$

reduces the partial fractional differential equation (1) to the ordinary differential equation of fractional order of the form

$${}^{RC}_{\ 0}D^{\alpha,\rho}_{\mu} f(\eta) = \beta f(\eta) - \tfrac{1}{\alpha\rho}\eta f'(\eta), \qquad \eta \in [0,\mu],$$

where $\mu = X t_0^{\frac{-1}{\alpha\rho}}$.

Proof. Let $\eta = \frac{x}{t^{\frac{1}{\alpha\rho}}}$. From (12), we obtain

$$\begin{aligned} \frac{\partial u(x,t)}{\partial t} &= \beta t^{\beta-1} f(\eta) + t^{\beta}\Big[-\frac{1}{\alpha\rho} t^{-\frac{1}{\alpha\rho}-1} x f'(\eta)\Big] \\ &= t^{\beta-1}\Big[\beta f(\eta) - \frac{1}{\alpha\rho}\eta f'(\eta)\Big]. \end{aligned} \tag{13}$$

Furtheremore, for $1 < \alpha \leq 2$, $\rho > 0$, by the definition 2.6 of the R-CGFD, equation (12) and by putting $\zeta = \frac{s}{t^{\frac{1}{\alpha\rho}}}$, we get

$$\begin{aligned} &\frac{\partial^{\alpha,\rho} u(x,t)}{\partial |x|^{\alpha}} \\ &= \frac{t^{\beta}\rho^{\alpha-n+1}}{\Gamma(n-\alpha)}\int_0^X |(x^{\rho}-s^{\rho})|^{n-\alpha-1} s^{\rho-1}\Big(s^{1-\rho}\frac{d}{ds}\Big)^n f\Big(\frac{s}{t^{\frac{1}{\alpha\rho}}}\Big) ds \\ &= \frac{t^{\beta}}{2}({}^{C}D^{\alpha,\rho}_{0^+} + {}^{C}D^{\alpha,\rho}_{X^-}) f\Big(\frac{x}{t^{\frac{1}{\alpha\rho}}}\Big) \\ &= \frac{t^{\beta}\rho^{\alpha-n+1}}{2\Gamma(n-\alpha)}\int_0^x (x^{\rho}-s^{\rho})^{n-\alpha-1} s^{\rho-1}\Big(s^{1-\rho}\frac{d}{ds}\Big)^n f\Big(\frac{s}{t^{\frac{1}{\alpha\rho}}}\Big) ds \\ &\quad + \frac{t^{\beta}\rho^{\alpha-n+1}}{2\Gamma(n-\alpha)}\int_x^X (s^{\rho}-x^{\rho})^{n-\alpha-1} s^{\rho-1}\Big(s^{1-\rho}\frac{d}{ds}\Big)^n f\Big(\frac{s}{t^{\frac{1}{\alpha\rho}}}\Big) ds \\ &= \frac{t^{\beta+\frac{1}{\alpha\rho}}\rho^{\alpha-n+1}}{2\Gamma(n-\alpha)} \\ &\quad \times \int_0^{\eta} (x^{\rho} - (\zeta t^{\frac{1}{\alpha\rho}})^{\rho})^{n-\alpha-1} (\zeta t^{\frac{1}{\alpha\rho}})^{\rho-1}\Big((\zeta t^{\frac{1}{\alpha\rho}})^{1-\rho}\frac{d}{t^{\frac{1}{\alpha\rho}} d\zeta}\Big)^n f(\zeta) d\zeta \\ &\quad + \frac{t^{\beta+\frac{1}{\alpha\rho}}\rho^{\alpha-n+1}}{2\Gamma(n-\alpha)} \\ &\quad \times \int_{\eta}^{\mu} ((\zeta t^{\frac{1}{\alpha\rho}})^{\rho} - x^{\rho})^{n-\alpha-1} (\zeta t^{\frac{1}{\alpha\rho}})^{\rho-1}\Big((\zeta t^{\frac{1}{\alpha\rho}})^{1-\rho}\frac{d}{t^{\frac{1}{\alpha\rho}} d\zeta}\Big)^n f(\zeta) d\zeta \end{aligned} \tag{14}$$

$$
\begin{aligned}
&= \frac{t^{\beta+\frac{1}{\alpha\rho}}\rho^{\alpha-n+1}}{2\Gamma(n-\alpha)} \\
&\quad \times \int_0^\eta t^{\frac{1}{\alpha}[n-\alpha-1]+\frac{1}{\alpha\rho}[\rho-1+n(1-\rho)-n]}(\eta^\rho-\zeta^\rho)^{n-\alpha-1}\zeta^{\rho-1}\Big(\zeta^{1-\rho}\frac{d}{d\zeta}\Big)^n f(\zeta)d\zeta \\
&\quad + \frac{t^{\beta+\frac{1}{\alpha\rho}}\rho^{\alpha-n+1}}{2\Gamma(n-\alpha)} \\
&\quad \times \int_\eta^\mu t^{\frac{1}{\alpha}[n-\alpha-1]+\frac{1}{\alpha\rho}[\rho-1+n(1-\rho)-n]}(\zeta^\rho-\eta^\rho)^{n-\alpha-1}\zeta^{\rho-1}\Big(\zeta^{1-\rho}\frac{d}{d\zeta}\Big)^n f(\zeta)d\zeta \\
&= t^{\beta-1}\frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)}\int_0^\mu |(\eta^\rho-\zeta^\rho)|^{n-\alpha-1}\zeta^{\rho-1}\Big(\zeta^{1-\rho}\frac{d}{d\zeta}\Big)^n f(\zeta)d\zeta \\
&= t^{\beta-1}\, {}^{RC}_{0}D^{\alpha,\rho}_{\mu} f(\eta).
\end{aligned}
$$

By substituting (13) and (14) in (1), we get the following equation

$$
{}^{RC}_{0}D^{\alpha,\rho}_{\mu} f(\eta) = \beta f(\eta) - \frac{1}{\alpha\rho}\eta f'(\eta), \qquad \eta \in [0,\mu],
$$

where $\mu = X t_0^{\frac{-1}{\alpha\rho}}$.

3.2. Existence and uniqueness results of the basic profile

In this subsection, to study the following problem, we will need the results in subsection 3.1 along with Theorem 3.1,

$$
{}^{RC}_{0}D^{\alpha,\rho}_{\mu} f(\eta) = \beta f(\eta) - \frac{1}{\alpha\rho}\eta f'(\eta), \qquad 1<\alpha\le 2,\ \eta \in [0,\mu], \tag{15}
$$

with the conditions

$$
f(0)+f(\mu)=M, \quad f'(0)+f'(\mu)=0 \tag{16}
$$

where $\beta \in \mathbb{R}$ and $\rho, \mu > 0$.

In what follows, to derive the principal theorems, we will need the following lemmas.

Lemma 3.2
Let $\mu > 0$, we define

$$
E = \{f \in C[0,\mu] :\ f'(0)+f'(\mu)=0\}. \tag{17}
$$

Then $(E, \|.\|_\infty)$ is a Banach space.

Proof. Let μ be a positive parameter. It is obvious that the space E with the norm $\|.\|_\infty$ is a subspace of the Banach space $C[0,\mu]$. So, to show that E is a Banach space, it is enough to demonstrate that this later is closed in $C[0,\mu]$.

Let $(f_n)_{n\in\mathbb{N}} \in E$ be a real sequence such that $\lim_{n\to\infty} f_n = f$ in $C[0,\mu]$. Then we demonstrate that $f \in E$. Let $\eta, \upsilon \in [0,\mu]\times[0,\mu]$. We have

$$\begin{cases} \dfrac{d}{d\eta}[f_n(\eta) - f(\eta)] = \dfrac{d}{d\eta} f_n(\eta) - \dfrac{d}{d\eta} f(\eta), \\ \dfrac{d}{d\upsilon}[f_n(\upsilon) - f(\upsilon)] = \dfrac{d}{d\upsilon} f_n(\upsilon) - \dfrac{d}{d\upsilon} f(\upsilon). \end{cases}$$

Since f_n is continuous, we get

$$\begin{cases} \lim\limits_{n\to\infty} \dfrac{d}{d\eta} f_n(\eta) = \dfrac{d}{d\eta} f(\eta), \\ \lim\limits_{n\to\infty} \dfrac{d}{d\upsilon} f_n(\upsilon) = \dfrac{d}{d\upsilon} f(\upsilon) \end{cases} \qquad \text{for all } \eta, \upsilon \in [0,\mu]\times[0,\mu].$$

Then

$$\sup_{\eta} \lim_{n\to\infty} \left| \frac{d}{d\eta} f_n(\eta) - \frac{d}{d\eta} f(\eta) \right| = \lim_{n\to\infty} \sup_{\eta} \left| \frac{d}{d\eta} f_n(\eta) - \frac{d}{d\eta} f(\eta) \right| = 0,$$

and

$$\sup_{\upsilon} \lim_{n\to\infty} \left| \frac{d}{d\upsilon} f_n(\upsilon) - \frac{d}{d\upsilon} f(\upsilon) \right| = \lim_{n\to\infty} \sup_{\upsilon} \left| \frac{d}{d\upsilon} f_n(\upsilon) - \frac{d}{d\upsilon} f(\upsilon) \right| = 0.$$

This implies that

$$\begin{cases} \lim\limits_{n\to\infty} \left\| \dfrac{d}{d\eta} f_n(\eta) - \dfrac{d}{d\eta} f(\eta) \right\|_\infty = 0, \\ \lim\limits_{n\to\infty} \left\| \dfrac{d}{d\upsilon} f_n(\upsilon) - \dfrac{d}{d\upsilon} f(\upsilon) \right\|_\infty = 0. \end{cases}$$

Thus,

$$\begin{aligned} &\lim_{n\to\infty} \left\| \frac{d}{d\eta} f_n(\eta) - \frac{d}{d\eta} f(\eta) + \frac{d}{d\upsilon} f_n(\upsilon) - \frac{d}{d\upsilon} f(\upsilon) \right\|_\infty \\ &\qquad \leq \lim_{n\to\infty} \left\| \frac{d}{d\eta} f_n(\eta) - \frac{d}{d\eta} f(\eta) \right\|_\infty + \lim_{n\to\infty} \left\| \frac{d}{d\upsilon} f_n(\upsilon) - \frac{d}{d\upsilon} f(\upsilon) \right\|_\infty \\ &\qquad \leq 0. \end{aligned}$$

Therefore

$$\lim_{n\to\infty} \left\| \frac{d}{d\eta} f_n(\eta) - \frac{d}{d\eta} f(\eta) + \frac{d}{d\upsilon} f_n(\upsilon) - \frac{d}{d\upsilon} f(\upsilon) \right\|_\infty = 0.$$

Then, for $\eta = 0$ and $\upsilon = \mu$, we have also

$$\lim_{n\to\infty} \left(\frac{d}{d\eta} f_n \right)(0) + \lim_{n\to\infty} \left(\frac{d}{d\upsilon} f_n \right)(\upsilon) = f'(0) + f'(\mu) = 0,$$

then $f \in E$. Consequently, the subspace E is closed in $C[0,\mu]$. Hence $(E, \|.\|_\infty)$ is a Banach space.

In the next lemma, we will give the solution of problem (15)–(16).

Lemma 3.3
Let $\alpha, \beta, \rho, \mu \in \mathbb{R}$ provided that $1 < \alpha \leq 2$ and $\rho, \mu > 0$. For a given f, f', ${}^{RC}_{0}D^{\alpha,\rho}_{\mu} f \in C[0,\mu]$. Then the problem (15)–(16) is equivalent to the following integral equation

$$f(\eta) = w + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^{\eta} (\eta^\rho - \zeta^\rho)^{\alpha-1} \zeta^{\rho-1} \Big(\beta f(\zeta) - \frac{1}{\alpha\rho} \zeta f'(\zeta)\Big) d\zeta$$
$$+ \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_{\eta}^{\mu} (\zeta^\rho - \eta^\rho)^{\alpha-1} \zeta^{\rho-1} \Big(\beta f(\zeta) - \frac{1}{\alpha\rho} \zeta f'(\zeta)\Big) d\zeta$$

for all $\eta \in [0,\mu]$, where

$$w = \frac{M}{2} - \frac{\mu^\rho}{2\rho} f'(\mu). \tag{18}$$

Proof. First, by applying the Riesz-generalized fractional integral ${}^{RG}_{0}I^{\alpha,\rho}_{\mu}$ defined in (7) to both sides of equation (15), we obtain

$${}^{RG}_{0}I^{\alpha,\rho}_{\mu}\, {}^{RC}_{0}D^{\alpha,\rho}_{\mu} = {}^{RG}_{0}I^{\alpha,\rho}_{\mu} \Big(\beta f(\eta) - \frac{1}{\alpha\rho} \eta f'(\eta)\Big). \tag{19}$$

From Lemma 2.7 and Remark 2.8, we get

$${}^{RG}_{0}I^{\alpha,\rho}_{\mu}\, {}^{RC}_{0}D^{\alpha,\rho}_{\mu} f(\eta) = f(\eta) - \frac{1}{2}[f(0) + f(\mu)] - \frac{\eta^\rho}{2\rho}[f'(0) + f'(\mu)] + \frac{\mu^\rho}{2\rho} f'(\mu).$$

Then the fractional integral equation (19), can be re-written as follows

$$\begin{aligned} f(\eta) = {}^{RG}_{0}I^{\alpha,\rho}_{\mu} \Big(\beta f(\eta) - \frac{1}{\alpha\rho} \eta f'(\eta)\Big) + \frac{1}{2}[f(0) + f(\mu)] \\ + \frac{\eta^\rho}{2\rho}[f'(0) + f'(\mu)] - \frac{\mu^\rho}{2\rho} f'(\mu). \end{aligned} \tag{20}$$

Applying (16) to (20) yields

$$f(\eta) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^{\mu} |(\zeta^\rho - \eta^\rho)|^{\alpha-1} \zeta^{\rho-1} \Big(\beta f(\zeta) - \frac{1}{\alpha\rho} \zeta f'(\zeta)\Big) + \frac{M}{2} - \frac{\mu^\rho}{2\rho} f'(\mu).$$

Then, according to (18), the problem (15)–(16) is equivalent to

$$f(\eta) = w + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^{\eta} (\eta^\rho - \zeta^\rho)^{\alpha-1} \zeta^{\rho-1} \Big(\beta f(\zeta) - \frac{1}{\alpha\rho} \zeta f'(\zeta)\Big) d\zeta$$
$$+ \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_{\eta}^{\mu} (\zeta^\rho - \eta^\rho)^{\alpha-1} \zeta^{\rho-1} \Big(\beta f(\zeta) - \frac{1}{\alpha\rho} \zeta f'(\zeta)\Big) d\zeta.$$

Lemma 3.4
Let T be an integral operator defined by

$$\begin{aligned} Tf(\eta) = w &+ \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^{\eta} (\eta^\rho - \zeta^\rho)^{\alpha-1} \zeta^{\rho-1} \Big(\beta f(\zeta) - \frac{1}{\alpha\rho} \zeta f'(\zeta)\Big) d\zeta \\ &+ \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_\eta^{\mu} (\zeta^\rho - \eta^\rho)^{\alpha-1} \zeta^{\rho-1} \Big(\beta f(\zeta) - \frac{1}{\alpha\rho} \zeta f'(\zeta)\Big) d\zeta. \end{aligned} \tag{21}$$

provided that the supremum norm is

$$\|Tf\|_\infty = \sup_{0 \leq \eta \leq \mu} |Tf(\eta)|.$$

Then, T maps E into itself ($T\colon E \to E$).

Proof. Let $1 < \alpha \leq 2$ and $f \in E$ satisfy

$${}^{RC}_{\ 0}D^{\alpha,\rho}_{\mu} f(\eta) = \beta f(\eta) - \frac{1}{\alpha\rho} \eta f'(\eta),$$

where E is the Banach space defined by (17). Then, from (21), we have

$$\begin{aligned} \frac{d}{d\eta} Tf(\eta) &= \frac{d}{d\eta} \Big[w + {}^{RG}_{\ 0}I^{\alpha,\rho}_{\mu} (\beta f(\eta) - \frac{1}{\alpha\rho} \eta f'(\eta)) \Big] \\ &= \frac{d}{d\eta} \Big[{}^{RG}_{\ 0}I^{\alpha,\rho}_{\mu} (\beta f(\eta) - \frac{1}{\alpha\rho} \eta f'(\eta)) \Big] = {}^{RG}_{\ 0}I^{\alpha-1,\rho}_{\mu} \Big(\beta f(\eta) - \frac{1}{\alpha\rho} \eta f'(\eta)\Big) \\ &= {}^{RG}_{\ 0}I^{\alpha-1,\rho}_{\mu}\, {}^{RG}_{\ 0}D^{\alpha,\rho}_{\mu} f(\eta). \end{aligned}$$

It follows from (9) and (10) in Remark 2.9 that

$$\frac{d}{d\eta} Tf(\eta) = {}^{RG}_{\ 0}I^{\alpha-1,\rho}_{\mu}\, {}^{RC}_{\ 0}D^{\alpha,\rho}_{\mu} f(\eta) = f'(\eta).$$

Therefore

$$\frac{d}{d\eta} Tf(0) + \frac{d}{d\eta} Tf(\mu) = f'(0) + f'(\mu) = 0.$$

Hence, $T(E) \subset E$.

Next, we will deal with the existence and uniqueness of solution for (15)–(16).

Firstly, using Banach's Fixed Point Theorem, we will derive the conditions of the solutions' existence.

Theorem 3.5
Let $\alpha, \beta, \rho, \mu \in \mathbb{R}$, provided that $1 < \alpha \leq 2$, $\rho > 0$ and $\mu \in \big(0, (\frac{\rho^\alpha \Gamma(\alpha+1)}{2})^{\frac{1}{\rho(\alpha-1)+1}}\big)$. If

$$\frac{2\mu^{\rho\alpha}|\beta|}{\rho^\alpha \Gamma(\alpha+1) - 2\mu^{\rho(\alpha-1)+1}} < 1. \tag{22}$$

Then, the problem (15)–(16) has a unique solution on $[0, \mu]$.

Proof. First, we will transform the problem (15)–(16) into a fixed point problem. By Lemma 3.3, we define the operator $T\colon E \to E$ as follows

$$\begin{aligned} Tf(\eta) = w &+ \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^\eta (\eta^\rho - \zeta^\rho)^{\alpha-1} \zeta^{\rho-1} \Big(\beta f(\zeta) - \frac{1}{\alpha\rho} \zeta f'(\zeta) \Big) d\zeta \\ &+ \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_\eta^\mu (\zeta^\rho - \eta^\rho)^{\alpha-1} \zeta^{\rho-1} \Big(\beta f(\zeta) - \frac{1}{\alpha\rho} \zeta f'(\zeta) \Big) d\zeta. \end{aligned} \tag{23}$$

Since the problem (15)–(16) can be written in the form of the fractional integral equation (23), the fixed point of T is to be considered as a solution for (15)–(16).

Let $f, G \in E$, provided that

$$\begin{aligned} {}^{RC}_{\ 0}D^{\alpha,\rho}_\mu f(\eta) &= \beta f(\eta) - \frac{1}{\alpha\rho} \eta f'(\eta), \\ {}^{RC}_{\ 0}D^{\alpha,\rho}_\mu G(\eta) &= \beta G(\eta) - \frac{1}{\alpha\rho} \eta G'(\eta). \end{aligned}$$

Then

$$\begin{aligned} &Tf(\eta) - TG(\eta) \\ &\quad = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^\eta (\eta^\rho - \zeta^\rho)^{\alpha-1} \zeta^{\rho-1} \Big(\beta(f(\zeta) - G(\zeta)) - \frac{\zeta}{\alpha\rho} (f'(\zeta) - G'(\zeta)) \Big) d\zeta \\ &\qquad + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_\eta^\mu (\zeta^\rho - \eta^\rho)^{\alpha-1} \zeta^{\rho-1} \Big(\beta(f(\zeta) - G(\zeta)) - \frac{\zeta}{\alpha\rho} (f'(\zeta) - G'(\zeta)) \Big) d\zeta. \end{aligned}$$

Therefore

$$\begin{aligned} &|Tf(\eta) - TG(\eta)| \\ &\quad \le \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^\eta (\eta^\rho - \zeta^\rho)^{\alpha-1} \zeta^{\rho-1} |{}^{RC}_{\ 0}D^{\alpha,\rho}_\mu f(\zeta) - {}^{RC}_{\ 0}D^{\alpha,\rho}_\mu G(\zeta)| d\zeta \\ &\qquad + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_\eta^\mu (\zeta^\rho - \eta^\rho)^{\alpha-1} \zeta^{\rho-1} |{}^{RC}_{\ 0}D^{\alpha,\rho}_\mu f(\zeta) - {}^{RC}_{\ 0}D^{\alpha,\rho}_\mu G(\zeta)| d\zeta. \end{aligned} \tag{24}$$

Moreover, for each $\eta \in [0, \mu]$, we have

$$\begin{aligned} |{}^{RC}_{\ 0}D^{\alpha,\rho}_\mu f(\eta) - {}^{RC}_{\ 0}D^{\alpha,\rho}_\mu G(\eta)| &= \Big| \beta(f(\eta) - G(\eta)) - \frac{\eta}{\alpha\rho} (f'(\eta) - G'(\eta)) \Big| \\ &\le |\beta| |f(\eta) - G(\eta)| + \frac{\mu}{\alpha\rho} |f'(\eta) - G'(\eta)|. \end{aligned}$$

Using (11) in Remark 2.9, we find

$$\|{}^{RC}_{\ 0}D^{\alpha,\rho}_\mu f - {}^{RC}_{\ 0}D^{\alpha,\rho}_\mu G\|_\infty \le |\beta| \|f - G\|_\infty + \frac{2\mu^{\rho(\alpha-1)+1}}{\rho^\alpha \Gamma(\alpha+1)} \|{}^{RC}_{\ 0}D^{\alpha,\rho}_\mu f - {}^{RC}_{\ 0}D^{\alpha,\rho}_\mu G\|_\infty,$$

which yields

$$\Big[1 - \frac{2\mu^{\rho(\alpha-1)+1}}{\rho^\alpha \Gamma(\alpha+1)} \Big] \|{}^{RC}_{\ 0}D^{\alpha,\rho}_\mu f - {}^{RC}_{\ 0}D^{\alpha,\rho}_\mu G\|_\infty \le |\beta| \|f - G\|_\infty.$$

As $\rho^{\alpha}\Gamma(\alpha+1)-2\mu^{\rho(\alpha-1)+1}>0$, we get

$$\|{}^{RC}_{\ 0}D^{\alpha,\rho}_{\mu}f-{}^{RC}_{\ 0}D^{\alpha,\rho}_{\mu}G\|_{\infty}\leq\frac{|\beta|\rho^{\alpha}\Gamma(\alpha+1)}{\rho^{\alpha}\Gamma(\alpha+1)-2\mu^{\rho(\alpha-1)+1}}\|f-G\|_{\infty}.$$

Thus, (24) can be re-written as

$$\begin{aligned}\|Tf-TG\|_{\infty}&\\ &\leq\frac{|\beta|\rho^{\alpha}\Gamma(\alpha+1)}{\rho^{\alpha}\Gamma(\alpha+1)-2\mu^{\rho(\alpha-1)+1}}\|f-G\|_{\infty}\\ &\quad\times\left[\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^{\eta}(\eta^{\rho}-\zeta^{\rho})^{\alpha-1}\zeta^{\rho-1}d\zeta+\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{\eta}^{\mu}(\zeta^{\rho}-\eta^{\rho})^{\alpha-1}\zeta^{\rho-1}d\zeta\right]\\ &\leq\Big(\frac{2\mu^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}\Big)\Big(\frac{|\beta|\rho^{\alpha}\Gamma(\alpha+1)}{\rho^{\alpha}\Gamma(\alpha+1)-2\mu^{\rho(\alpha-1)+1}}\Big)\|f-G\|_{\infty}\\ &\leq\frac{2\mu^{\rho\alpha}|\beta|}{\rho^{\alpha}\Gamma(\alpha+1)-2\mu^{\rho(\alpha-1)+1}}\|f-G\|_{\infty}.\end{aligned}$$

By (22), T is a contraction mapping. Using the principle of Banach's Fixed Point Theorem 2.14, we deduce that T admits a unique fixed point which is a unique solution of the problem (15)–(16) on $[0,\mu]$.

Secondly, using the fixed point theorem of Schauder, we will derive the conditions of the solutions' existence.

Theorem 3.6
Let $\rho,\mu>0$, $\beta\in\mathbb{R}$ and $1<\alpha\leq 2$. If

$$\frac{\mu^{\rho(\alpha-1)+1}+\mu^{\rho\alpha}|\beta|}{\rho^{\alpha}\Gamma(\alpha+1)}<\frac{1}{2},\tag{25}$$

then, the problem (15)–(16) *has at least one solution on* $[0,\mu]$.

Proof. Let the operator T be defined in (23). We have already transformed the problem (15)–(16) into a fixed point problem

$$\begin{aligned}Tf(\eta)=w&+\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^{\eta}(\eta^{\rho}-\zeta^{\rho})^{\alpha-1}\zeta^{\rho-1}\Big(\beta f(\zeta)-\frac{1}{\alpha\rho}\zeta f'(\zeta)\Big)d\zeta\\ &+\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{\eta}^{\mu}(\zeta^{\rho}-\eta^{\rho})^{\alpha-1}\zeta^{\rho-1}\Big(\beta f(\zeta)-\frac{1}{\alpha\rho}\zeta f'(\zeta)\Big)d\zeta.\end{aligned}$$

We shall show that T satisfies the assumption of Schauder's Fixed Point Theorem 2.15. The proof will be given in three claims.

Claim 1: T is a continuous operator.
Let $(f_n)_{n\in\mathbb{N}}$ be a sequence provided that $\lim_{n\to\infty}f_n=f$ in E. Then, for each $\eta\in[0,\mu]$, we have

$$
\begin{aligned}
&|Tf_n(\eta) - Tf(\eta)| \\
&\leq \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^\eta (\eta^\rho - \zeta^\rho)^{\alpha-1} \zeta^{\rho-1} \Big| \beta(f_n(\zeta) - f(\zeta)) - \frac{\zeta}{\alpha\rho}(f_n'(\zeta) - f'(\zeta)) \Big| d\zeta \qquad (26)\\
&\quad + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_\eta^\mu (\zeta^\rho - \eta^\rho)^{\alpha-1} \zeta^{\rho-1} \Big| \beta(f_n(\zeta) - f(\zeta)) - \frac{\zeta}{\alpha\rho}(f_n'(\zeta) - f'(\zeta)) \Big| d\zeta,
\end{aligned}
$$

where

$$
{}^{RC}_{\ 0}D^{\alpha,\rho}_{\mu} f_n(\eta) = \beta f_n(\eta) - \frac{\eta}{\alpha\rho} f_n'(\eta) \quad \text{and} \quad {}^{RC}_{\ 0}D^{\alpha,\rho}_{\mu} f(\eta) = \beta f(\eta) - \frac{\eta}{\alpha\rho} f'(\eta).
$$

We have

$$
\begin{aligned}
|{}^{RC}_{\ 0}D^{\alpha,\rho}_{\mu} f_n(\eta) - {}^{RC}_{\ 0}D^{\alpha,\rho}_{\mu} f(\eta)| &= \Big| \beta(f_n(\eta) - f(\eta)) - \frac{\eta}{\alpha\rho}(f_n'(\eta) - f'(\eta)) \Big| \\
&\leq |\beta||f_n(\eta) - f(\eta)| + \frac{\mu}{\alpha\rho}|f_n'(\eta) - f'(\eta)|.
\end{aligned}
$$

Using (11) in Remark 2.9, we find

$$
\|{}^{RC}_{\ 0}D^{\alpha,\rho}_{\mu} f_n - {}^{RC}_{\ 0}D^{\alpha,\rho}_{\mu} f\|_\infty \leq |\beta|\|f_n - f\|_\infty + \frac{2\mu^{\rho(\alpha-1)+1}}{\rho^\alpha \Gamma(\alpha+1)} \|{}^{RC}_{\ 0}D^{\alpha,\rho}_{\mu} f_n - {}^{RC}_{\ 0}D^{\alpha,\rho}_{\mu} f\|_\infty,
$$

which yields

$$
\Big[1 - \frac{2\mu^{\rho(\alpha-1)+1}}{\rho^\alpha \Gamma(\alpha+1)}\Big] \|{}^{RC}_{\ 0}D^{\alpha,\rho}_{\mu} f_n - {}^{RC}_{\ 0}D^{\alpha,\rho}_{\mu} f\|_\infty \leq |\beta|\|f_n - f\|_\infty.
$$

As $\rho^\alpha \Gamma(\alpha+1) - 2\mu^{\rho(\alpha-1)+1} > 2\mu^{\rho\alpha}|\beta| > 0$, we get

$$
\|{}^{RC}_{\ 0}D^{\alpha,\rho}_{\mu} f_n - {}^{RC}_{\ 0}D^{\alpha,\rho}_{\mu} f\|_\infty \leq \frac{|\beta|\rho^\alpha \Gamma(\alpha+1)}{\rho^\alpha \Gamma(\alpha+1) - 2\mu^{\rho(\alpha-1)+1}} \|f_n - f\|_\infty.
$$

Because $f_n \to f$ as $n \to \infty$, then we get ${}^{RC}_{\ 0}D^{\alpha,\rho}_{\mu} f_n \to {}^{RC}_{\ 0}D^{\alpha,\rho}_{\mu} f$ as $n \to \infty$ for every $\eta \in [0, \mu]$.

Now let $S_0 > 0$, such that for every $\eta \in [0, \mu]$, we have

$$
|{}^{RC}_{\ 0}D^{\alpha,\rho}_{\mu} f_n| \leq S_0 \quad \text{and} \quad |{}^{RC}_{\ 0}D^{\alpha,\rho}_{\mu} f| \leq S_0.
$$

Then, we have

$$
\begin{aligned}
&|Tf_n(\eta) - Tf(\eta)| \\
&\leq \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^\eta (\eta^\rho - \zeta^\rho)^{\alpha-1} \zeta^{\rho-1} \Big| \beta(f_n(\zeta) - f(\zeta)) - \frac{\zeta}{\alpha\rho}(f_n'(\zeta) - f'(\zeta)) \Big| d\zeta \\
&\quad + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_\eta^\mu (\zeta^\rho - \eta^\rho)^{\alpha-1} \zeta^{\rho-1} \Big| \beta(f_n(\zeta) - f(\zeta)) - \frac{\zeta}{\alpha\rho}(f_n'(\zeta) - f'(\zeta)) \Big| d\zeta \\
&\leq \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^\eta (\eta^\rho - \zeta^\rho)^{\alpha-1} \zeta^{\rho-1} |{}^{RC}_{\ 0}D^{\alpha,\rho}_{\mu} f_n(\zeta) - {}^{RC}_{\ 0}D^{\alpha,\rho}_{\mu} f(\zeta)| d\zeta
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{\eta}^{\mu}(\zeta^{\rho}-\eta^{\rho})^{\alpha-1}\zeta^{\rho-1}|{}^{RC}_{\ 0}D^{\alpha,\rho}_{\mu}f_n(\zeta)-{}^{RC}_{\ 0}D^{\alpha,\rho}_{\mu}f(\zeta)|d\zeta\\
\leq&\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{0}^{\eta}(\eta^{\rho}-\zeta^{\rho})^{\alpha-1}\zeta^{\rho-1}[|{}^{RC}_{\ 0}D^{\alpha,\rho}_{\mu}f_n(\zeta)|+|{}^{RC}_{\ 0}D^{\alpha,\rho}_{\mu}f(\zeta)|]d\zeta\\
&+\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{\eta}^{\mu}(\zeta^{\rho}-\eta^{\rho})^{\alpha-1}\zeta^{\rho-1}[|{}^{RC}_{\ 0}D^{\alpha,\rho}_{\mu}f_n(\zeta)|+|{}^{RC}_{\ 0}D^{\alpha,\rho}_{\mu}f(\zeta)|]d\zeta\\
\leq&\frac{2S_0\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{0}^{\eta}(\eta^{\rho}-\zeta^{\rho})^{\alpha-1}\zeta^{\rho-1}d\zeta+\frac{2S_0\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{\eta}^{\mu}(\zeta^{\rho}-\eta^{\rho})^{\alpha-1}\zeta^{\rho-1}d\zeta.
\end{aligned}
$$

Since the functions

$$
\zeta\to\frac{2S_0\rho^{1-\alpha}}{\Gamma(\alpha)}[(\eta^{\rho}-\zeta^{\rho})^{\alpha-1}\zeta^{\rho-1}]\quad\text{and}\quad\zeta\to\frac{2S_0\rho^{1-\alpha}}{\Gamma(\alpha)}[(\zeta^{\rho}-\eta^{\rho})^{\alpha-1}\zeta^{\rho-1}]
$$

are integrable on $[0,\eta]$ and $[\eta,\mu]$ respectively for each $\eta\in[0,\mu]$, then the Lebesgue dominated convergence theorem and (26) implies that$|Tf_n(\eta)-Tf(\eta)|\to 0$ as $n\to\infty$. Thus

$$
\lim_{n\to\infty}\|Tf_n-Tf\|_{\infty}=0.
$$

Consequently, T is continuous.

Claim 2: According to (25), let

$$
R\geq\Big(1+\frac{2\mu^{\rho\alpha}|\beta|}{\rho^{\alpha}\Gamma(\alpha+1)-2(\mu^{\rho(\alpha-1)+1}+\mu^{\rho\alpha}|\beta|)}\Big)|w|,
$$

and define a subset

$$
E_R=\{f\in E:\ \|f\|_{\infty}\leq R,\ R>0\}.
$$

Thus, E_R is a closed, bounded and convex subset of E.

Let $f\in E_R$ and T be the integral operator defined in (23). Then, we prove that $T(E_R)\subset E_R$. In fact, by (11) in Remark 2.9, we have

$$
|{}^{RC}_{\ 0}D^{\alpha,\rho}_{\mu}f(\eta)|=\Big|\beta f(\eta)-\frac{\eta}{\alpha\rho}f'(\eta)\Big|\leq|\beta||f(\eta)|+\frac{\mu}{\alpha\rho}|f'(\eta)|.
$$

This implies that

$$
\|{}^{RC}_{\ 0}D^{\alpha,\rho}_{\mu}f\|_{\infty}\leq\frac{|\beta|\rho^{\alpha}\Gamma(\alpha+1)}{\rho^{\alpha}\Gamma(\alpha+1)-2\mu^{\rho(\alpha-1)+1}}R. \tag{27}
$$

Therefore

$$\begin{aligned}
|Tf(\eta)| &\leq |w| + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^{\eta}(\eta^\rho-\zeta^\rho)^{\alpha-1}\zeta^{\rho-1}\Big|\beta f(\zeta)-\frac{\zeta}{\alpha\rho}f'(\zeta)\Big|d\zeta \\
&\quad + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_\eta^{\mu}(\zeta^\rho-\eta^\rho)^{\alpha-1}\zeta^{\rho-1}\Big|\beta f(\zeta)-\frac{\zeta}{\alpha\rho}f'(\zeta)\Big|d\zeta \\
&\leq |w| + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^{\eta}(\eta^\rho-\zeta^\rho)^{\alpha-1}\zeta^{\rho-1}|{}^{RC}_{\,0}D^{\alpha,\rho}_{\mu}f(\zeta)|d\zeta \\
&\quad + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_\eta^{\mu}(\zeta^\rho-\eta^\rho)^{\alpha-1}\zeta^{\rho-1}|{}^{RC}_{\,0}D^{\alpha,\rho}_{\mu}f(\zeta)|d\zeta \\
&\leq |w| + \frac{|\beta|\rho^\alpha\Gamma(\alpha+1)}{\rho^\alpha\Gamma(\alpha+1)-2\mu^{\rho(\alpha-1)+1}}R\Big[\frac{2\mu^{\rho\alpha}}{\rho^\alpha\Gamma(\alpha+1)}\Big] \\
&\leq |w| + \frac{2\mu^{\rho\alpha}|\beta|R}{\rho^\alpha\Gamma(\alpha+1)-2\mu^{\rho(\alpha-1)+1}} \\
&\leq \frac{|w|\Big(1+\dfrac{2\mu^{\rho\alpha}|\beta|}{\rho^\alpha\Gamma(\alpha+1)-2(\mu^{\rho(\alpha-1)+1}+\mu^{\rho\alpha}|\beta|)}\Big)}{\Big(1+\dfrac{2\mu^{\rho\alpha}|\beta|}{\rho^\alpha\Gamma(\alpha+1)-2(\mu^{\rho(\alpha-1)+1}+\mu^{\rho\alpha}|\beta|)}\Big)} \\
&\quad + \frac{2\mu^{\rho\alpha}|\beta|R}{\rho^\alpha\Gamma(\alpha+1)-2\mu^{\rho(\alpha-1)+1}} \\
&\leq \frac{R(\rho^\alpha\Gamma(\alpha+1)-2(\mu^{\rho(\alpha-1)+1}+\mu^{\rho\alpha}|\beta|))}{\rho^\alpha\Gamma(\alpha+1)-2\mu^{\rho(\alpha-1)+1}} \\
&\quad + \frac{2\mu^{\rho\alpha}|\beta|R}{\rho^\alpha\Gamma(\alpha+1)-2\mu^{\rho(\alpha-1)+1}} \\
&\leq R.
\end{aligned}$$

Thus $T(E_R)\subset E_R$, hence $T(E_R)$ is bounded.

Claim 3: $T(E_R)$ is relatively compact.

Let $f\in E_R$, $\eta_1,\eta_2\in[0,\mu]$ with $\eta_1<\eta_2$, by (27), we get

$$\begin{aligned}
&|Tf(\eta_1)-Tf(\eta_2)| \\
&\quad = \Big|\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^{\eta_1}(\eta_1^\rho-\zeta^\rho)^{\alpha-1}\zeta^{\rho-1}\Big(\beta f(\zeta)-\frac{\zeta}{\alpha\rho}f'(\zeta)\Big)d\zeta \\
&\qquad + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{\eta_1}^{\mu}(\zeta^\rho-\eta_1^\rho)^{\alpha-1}\zeta^{\rho-1}\Big(\beta f(\zeta)-\frac{\zeta}{\alpha\rho}f'(\zeta)\Big)d\zeta \\
&\qquad - \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^{\eta_2}(\eta_2^\rho-\zeta^\rho)^{\alpha-1}\zeta^{\rho-1}\Big(\beta f(\zeta)-\frac{\zeta}{\alpha\rho}f'(\zeta)\Big)d\zeta \\
&\qquad - \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{\eta_2}^{\mu}(\zeta^\rho-\eta_2^\rho)^{\alpha-1}\zeta^{\rho-1}\Big(\beta f(\zeta)-\frac{\zeta}{\alpha\rho}f'(\zeta)\Big)d\zeta\Big|
\end{aligned}$$

$$\begin{aligned}
&\le \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^{\eta_1}|[\zeta^{\rho-1}(\eta_1^\rho-\zeta^\rho)^{\alpha-1}-\zeta^{\rho-1}(\eta_2^\rho-\zeta^\rho)^{\alpha-1}]|\,|{}^{RC}_{\ 0}D^{\alpha,\rho}_{\mu}f(\zeta)|d\zeta\\
&\quad+\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{\eta_2}^{\mu}|[\zeta^{\rho-1}(\zeta^\rho-\eta_1^\rho)^{\alpha-1}-\zeta^{\rho-1}(\zeta^\rho-\eta_2^\rho)^{\alpha-1}]|\,|{}^{RC}_{\ 0}D^{\alpha,\rho}_{\mu}f(\zeta)|d\zeta\\
&\quad+\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{\eta_1}^{\eta_2}\zeta^{\rho-1}(\eta_2^\rho-\zeta^\rho)^{\alpha-1}|{}^{RC}_{\ 0}D^{\alpha,\rho}_{\mu}f(\zeta)|d\zeta\\
&\quad+\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{\eta_1}^{\eta_2}\zeta^{\rho-1}(\zeta^\rho-\eta_1^\rho)^{\alpha-1}|{}^{RC}_{\ 0}D^{\alpha,\rho}_{\mu}f(\zeta)|d\zeta \qquad (28)\\
&\le \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\frac{|\beta|\rho^\alpha\Gamma(\alpha+1)}{\rho^\alpha\Gamma(\alpha+1)-2\mu^{\rho(\alpha-1)+1}}R\\
&\quad\times\Bigg(\int_0^{\eta_1}|[\zeta^{\rho-1}(\eta_1^\rho-\zeta^\rho)^{\alpha-1}-\zeta^{\rho-1}(\eta_2^\rho-\zeta^\rho)^{\alpha-1}]|d\zeta\\
&\quad+\int_{\eta_2}^{\mu}|[\zeta^{\rho-1}(\zeta^\rho-\eta_1^\rho)^{\alpha-1}-\zeta^{\rho-1}(\zeta^\rho-\eta_2^\rho)^{\alpha-1}]|d\zeta\\
&\quad+\int_{\eta_1}^{\eta_2}\zeta^{\rho-1}(\eta_2^\rho-\zeta^\rho)^{\alpha-1}d\zeta+\int_{\eta_1}^{\eta_2}\zeta^{\rho-1}(\zeta^\rho-\eta_1^\rho)^{\alpha-1}d\zeta\Bigg).
\end{aligned}$$

We have

$$\zeta^{\rho-1}(\eta_1^\rho-\zeta^\rho)^{\alpha-1}-\zeta^{\rho-1}(\eta_2^\rho-\zeta^\rho)^{\alpha-1}=-\frac{1}{\alpha\rho}\frac{d}{d\zeta}[(\eta_1^\rho-\zeta^\rho)^\alpha-(\eta_2^\rho-\zeta^\rho)^\alpha]$$

and

$$\zeta^{\rho-1}(\zeta^\rho-\eta_1^\rho)^{\alpha-1}-\zeta^{\rho-1}(\zeta^\rho-\eta_2^\rho)^{\alpha-1}=\frac{1}{\alpha\rho}\frac{d}{d\zeta}[(\zeta^\rho-\eta_1^\rho)^\alpha-(\zeta^\rho-\eta_2^\rho)^\alpha],$$

then

$$\begin{aligned}\int_0^{\eta_1}&|[\zeta^{\rho-1}(\eta_1^\rho-\zeta^\rho)^{\alpha-1}-\zeta^{\rho-1}(\eta_2^\rho-\zeta^\rho)^{\alpha-1}]|d\zeta\\
&\le\frac{1}{\alpha\rho}[(\eta_2^\rho-\eta_1^\rho)^\alpha+(\eta_2^{\rho\alpha}-\eta_1^{\alpha\rho})]\end{aligned}\tag{29}$$

and

$$\begin{aligned}\int_{\eta_2}^{\mu}&|[\zeta^{\rho-1}(\zeta^\rho-\eta_1^\rho)^{\alpha-1}-\zeta^{\rho-1}(\zeta^\rho-\eta_2^\rho)^{\alpha-1}]|d\zeta\\
&\le\frac{1}{\alpha\rho}[(\mu^\rho-\eta_1^\rho)^\alpha-(\eta_2^\rho-\eta_1^\rho)^\alpha-(\mu^\rho-\eta_2^\rho)^\alpha],\end{aligned}\tag{30}$$

we have also

$$\int_{\eta_1}^{\eta_2}\zeta^{\rho-1}(\eta_2^\rho-\zeta^\rho)^{\alpha-1}d\zeta=-\frac{1}{\alpha\rho}[(\eta_2^\rho-\zeta^\rho)^\alpha]_{\eta_1}^{\eta_2}=\frac{1}{\alpha\rho}(\eta_2^\rho-\eta_1^\rho)^\alpha,\tag{31}$$

and

$$\int_{\eta_1}^{\eta_2}\zeta^{\rho-1}(\zeta^\rho-\eta_1^\rho)^{\alpha-1}d\zeta=\frac{1}{\alpha\rho}[(\zeta^\rho-\eta_1^\rho)^{\alpha-1}]_{\eta_1}^{\eta_2}=\frac{1}{\alpha\rho}(\eta_2^\rho-\eta_1^\rho)^\alpha.\tag{32}$$

By substituting (29), (30), (31) and (32) in (28), we obtain

$$|Tf(\eta_1) - Tf(\eta_2)| \leq \frac{|\beta|R}{\rho^{\alpha}\Gamma(\alpha+1) - 2\mu^{\rho(\alpha-1)+1}} \times [(\mu^{\rho} - \eta_1^{\rho})^{\alpha} + 2(\eta_2^{\rho} - \eta_1^{\rho})^{\alpha} + (\eta_2^{\rho\alpha} - \eta_1^{\alpha\rho}) - (\mu^{\rho} - \eta_2^{\rho})^{\alpha}].$$

So, the right-hand side of the above inequality tends to zero as $\eta_2 \to \eta_1$. Hence, we obtain that $T(E_R)$ is equicontinuous. Therefore, combining claims 1 to 3 and by the means of the Ascoli-Arzela Theorem 2.12, we get that $T\colon E_R \to E_R$ is continuous and relatively compact. As a consequence, Schauder's Fixed Point Theorem assures the existence of at least one fixed point of operator (23) which is the solution of the problem (15)–(16).

Finally, using the fixed point theorem of Leray-Schauder, we will derive the conditions of the solutions' existence.

Theorem 3.7
Let $\alpha, \beta, \rho, \mu \in \mathbb{R}$, provided that $1 < \alpha \leq 2$, $\rho > 0$ and $\mu \in \big(0, (\frac{\rho^{\alpha}\Gamma(\alpha+1)}{2})^{\frac{1}{\rho(\alpha-1)+1}}\big)$. Then, the problem (15)–(16) admits at least one solution on $[0,\mu]$.

Proof. Consider the operator T defined in (23). Then we shall show that all assumption of Leray-Schauder Fixed Point Theorem 2.16 are satisfied by the operator T. The proof will be divided to four claims.

Claim 1: It is clear that T is continuous.

Claim 2: T maps bounded sets into bounded sets in E.
Actually, it is enough to show that for any $\theta > 0$, there exists $N > 0$ such that for each $f \in D_{\theta} = \{f \in E: \|f\|_{\infty} \leq \theta\}$, we have $\|Tf\|_{\infty} \leq N$. Let $f \in D_{\theta}$ for each $\eta \in [0,\mu]$, we have

$$\begin{aligned} |Tf(\eta)| \leq |w| &+ \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^{\eta} (\eta^{\rho} - \zeta^{\rho})^{\alpha-1}\zeta^{\rho-1}\Big|\beta f(\zeta) - \frac{\zeta}{\alpha\rho}f'(\zeta)\Big|d\zeta \\ &+ \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_{\eta}^{\mu} (\zeta^{\rho} - \eta^{\rho})^{\alpha-1}\zeta^{\rho-1}\Big|\beta f(\zeta) - \frac{\zeta}{\alpha\rho}f'(\zeta)\Big|d\zeta \end{aligned} \tag{33}$$

As a similar way as in (27), we have

$$\Big|\beta f(\eta) - \frac{\eta}{\alpha\rho}f'(\eta)\Big| \leq \frac{|\beta|\rho^{\alpha}\Gamma(\alpha+1)}{\rho^{\alpha}\Gamma(\alpha+1) - 2\mu^{\rho(\alpha-1)+1}}\theta.$$

Therefore, (33) implies that

$$\|Tf\|_{\infty} \leq |w| + \frac{2\mu^{\rho\alpha}|\beta|}{\rho^{\alpha}\Gamma(\alpha+1) - 2\mu^{\rho(\alpha-1)+1}}\theta = N.$$

Claim 3: It is clear that T maps bounded sets into equicontinuous sets of E. From Claim1-Claim3, we conclude that $T\colon E \to E$ is continuous and completely continuous.

Claim 4: A priori bounds.
Now, we shaw that there exists an open set $H \subset E$ with $f \neq \lambda T(f)$ for some $\lambda \in (0,1)$ and $f \in \partial H$.

Let $f \in E$ and $f = \lambda T(f)$ for $0 < \lambda < 1$. Then, we have for each $\eta \in [0, \mu]$,

$$\begin{aligned} |f(\eta)| &= |\lambda T f(\eta)| \\ &= \left| \lambda w + \frac{\lambda \rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^\mu |(\eta^\rho - \zeta^\rho)|^{\alpha-1} \zeta^{\rho-1} \Big(\beta f(\zeta) - \frac{\zeta}{\alpha\rho} f'(\zeta) \Big) d\zeta \right| \\ &\leq |w| + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^\eta (\eta^\rho - \zeta^\rho)^{\alpha-1} \zeta^{\rho-1} \Big| \beta f(\zeta) - \frac{\zeta}{\alpha\rho} f'(\zeta) \Big| d\zeta \\ &\quad + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_\eta^\mu (\zeta^\rho - \eta^\rho)^{\alpha-1} \zeta^{\rho-1} \Big| \beta f(\zeta) - \frac{\zeta}{\alpha\rho} f'(\zeta) \Big| d\zeta \\ &\leq |w| + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^\eta (\eta^\rho - \zeta^\rho)^{\alpha-1} \zeta^{\rho-1} |{}^{RC}_{\ 0}D^{\alpha,\rho}_{\mu} f(\zeta)| d\zeta \\ &\quad + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_\eta^\mu (\zeta^\rho - \eta^\rho)^{\alpha-1} \zeta^{\rho-1} |{}^{RC}_{\ 0}D^{\alpha,\rho}_{\mu} f(\zeta)| d\zeta. \end{aligned} \tag{34}$$

We have

$$\begin{aligned} |{}^{RC}_{\ 0}D^{\alpha,\rho}_{\mu} f(\eta)| = \Big| \beta f(\zeta) - \frac{\zeta}{\alpha\rho} f'(\zeta) \Big| &\leq |\beta|\ |f(\eta)| + \frac{\mu}{\alpha\rho} |f'(\eta)| \\ &\leq |\beta|\ |f(\eta)| + \frac{2\mu^{\rho(\alpha-1)+1}}{\rho^\alpha \Gamma(\alpha+1)} \sup_{0\leq\eta\leq\mu} |{}^{RC}_{\ 0}D^{\alpha,\rho}_{\mu} f(\eta)|, \end{aligned}$$

which implies that

$$\sup_{0\leq\eta\leq\mu} |{}^{RC}_{\ 0}D^{\alpha,\rho}_{\mu} f(\eta)| \leq \frac{|\beta| \rho^\alpha \Gamma(\alpha+1)}{\rho^\alpha \Gamma(\alpha+1) - 2\mu^{\rho(\alpha-1)+1}} \sup_{0\leq\eta\leq\mu} |f(\eta)|.$$

Thus, (34) gives

$$\begin{aligned} \sup_{0\leq\eta\leq\mu} |f(\eta)| \leq |w| &+ \frac{|\beta| \rho^\alpha \Gamma(\alpha+1)}{\Gamma(\alpha)[\rho^\alpha \Gamma(\alpha+1) - 2\mu^{\rho(\alpha-1)+1}]} \\ &\times \left[\rho^{1-\alpha} \int_0^\eta \zeta^{\rho-1} (\eta^\rho - \zeta^\rho)^{\alpha-1} \sup_{0\leq\eta\leq\mu} |f(\zeta)| d\zeta \right. \\ &\left. + \rho^{1-\alpha} \int_\eta^\mu \zeta^{\rho-1} (\zeta^\rho - \eta^\rho)^{\alpha-1} \sup_{0\leq\eta\leq\mu} |f(\zeta)| d\zeta \right]. \end{aligned} \tag{35}$$

By using the generalized Gronwall Lemma 2.13, (35) can be re-written as

$$\begin{aligned} &\sup_{0\leq\eta\leq\mu} |f(\eta)| \\ &\quad \leq |w| E_{\alpha,1} \Big(\rho^{-\alpha} \Gamma(\alpha) (\mu^\rho - \eta^\rho)^\alpha \Big(\frac{|\beta| \rho^\alpha \Gamma(\alpha+1)}{\Gamma(\alpha)(\rho^\alpha \Gamma(\alpha+1) - 2\mu^{\rho(\alpha-1)+1})} \Big) \Big) \\ &\quad \times E_{\alpha,1} \Big(\rho^{-\alpha} \Gamma(\alpha) \eta^{\rho\alpha} \Big(\frac{|\beta| \rho^\alpha \Gamma(\alpha+1)}{\Gamma(\alpha)(\rho^\alpha \Gamma(\alpha+1) - 2\mu^{\rho(\alpha-1)+1})} \Big) \Big), \end{aligned}$$

which can be simplified to

$$\begin{aligned}\|f\|_\infty &\le |w| E_{\alpha,1}\Big(\frac{(\mu^\rho-\eta^\rho)^\alpha|\beta|\Gamma(\alpha+1)}{\rho^\alpha\Gamma(\alpha+1)-2\mu^{\rho(\alpha-1)+1}}\Big) E_{\alpha,1}\Big(\frac{\eta^{\rho\alpha}|\beta|\Gamma(\alpha+1)}{\rho^\alpha\Gamma(\alpha+1)-2\mu^{\rho(\alpha-1)+1}}\Big)\\ &= N_1.\end{aligned}$$

Let

$$H=\{f\in E:\ \|f\|_\infty<N_1+1\}.$$

By choosing of H, there is no $f\in\partial H$, such that $f=\lambda T(f)$ for some $\lambda\in(0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder's Fixed Point Theorem 2.16, the operator T has a fixed point f in H, which is a solution to the problem (15)–(16) on $[0,\mu]$.

Now, we prove the principal theorems.

3.3. Existence results to the original problem

In this subsection, we demonstrate the existence and uniqueness of solutions of the following space-fractional diffusion equation

$$\begin{cases}\dfrac{\partial u(x,t)}{\partial t}=\dfrac{\partial^{\alpha,\rho}u(x,t)}{\partial|x|^\alpha}, & (x,t)\in[0,X]\times[t_0,\infty[,\\ u(0,t)+u(X,t)=t^\beta M, & t\in[t_0,\infty[,\\ \dfrac{\partial u(0,t)}{\partial x}+\dfrac{\partial u(X,t)}{\partial x}=0, & t\in[t_0,\infty[.\end{cases}\tag{36}$$

where $1<\alpha\le 2$ and $\beta,M\in\mathbb{R}$. Under the self-similar solution form, which is

$$u(x,t)=t^\beta f(\eta)\qquad\text{with } \eta=\frac{x}{t^{\frac{1}{\alpha\rho}}}.\tag{37}$$

Theorem 3.8
Let $\alpha,\beta,\rho,X,t_0\in\mathbb{R}$, provided that $1<\alpha\le 2$, $\rho,t_0>0$ and

$$X\in\Bigg(0,\Big(\frac{t_0^{\frac{\rho(\alpha-1)+1}{\rho\alpha}}\rho^\alpha\Gamma(\alpha+1)}{2}\Big)^{\frac{1}{\rho(\alpha-1)+1}}\Bigg).$$

If

$$\frac{2X^{\rho\alpha}|\beta|}{t_0\rho^\alpha\Gamma(\alpha+1)-2X^{\rho(\alpha-1)+1}t_0^{\frac{\rho-1}{\rho\alpha}}}<1.\tag{38}$$

Then, for $f\in E$, (36) *has a unique solution in the self-similar form* (37).

Proof. The transformation (37) reduces the space-fractional diffusion equation (36) to the ordinary fractional differential equation of the following form

$${}^{RC}_{\ 0}D^{\alpha,\rho}_{\mu}f(\eta)=\beta f(\eta)-\frac{1}{\alpha\rho}\eta f'(\eta),\tag{39}$$

where

$$\mu = X t_0^{\frac{-1}{\alpha\rho}} \quad \text{and} \quad X \in \left(0, \left(\frac{t_0^{\frac{\rho(\alpha-1)+1}{\rho\alpha}} \rho^\alpha \Gamma(\alpha+1)}{2}\right)^{\frac{1}{\rho(\alpha-1)+1}}\right),$$

with the conditions

$$f(0) + f(\mu) = M, \quad f'(0) + f'(\mu) = 0. \tag{40}$$

Let $f \in E$ be a continuous function. By using (37), the condition (38) is equivalent to (22), which is

$$\frac{2\mu^{\rho\alpha}|\beta|}{\rho^\alpha\Gamma(\alpha+1) - 2\mu^{\rho(\alpha-1)+1}} < 1. \tag{41}$$

We already proved in Theorem 3.5, the existence and uniqueness of a solution of the problem (39)–(40) such that (41) is satisfied. As a consequence, there exists a unique solution of the problem (36) under the self-similar form (37) provided that (38) holds.

Remark 3.9
When $\rho \to 1$, (38) reduces to

$$\frac{2X^\alpha|\beta|}{t_0\Gamma(\alpha+1) - 2X^\alpha} < 1, \tag{42}$$

which represents the standard Riesz-Caputo derivative case. When we let $\alpha = 2$ in (42), we get

$$\frac{2X^2|\beta|}{t_0\Gamma(3) - 2X^2} < 1, \tag{43}$$

which gives the integer-order derivative case of the space-fractional diffusion equation in (1).

Theorem 3.10
Let $\alpha, \beta, \rho, X, t_0 \in \mathbb{R}$, provided that $1 < \alpha \leq 2$ and $\rho, t_0, X > 0$. If

$$\frac{X^{\rho(\alpha-1)+1} t_0^{\frac{\rho-1}{\rho\alpha}} + X^{\rho\alpha}|\beta|}{t_0\rho^\alpha\Gamma(\alpha+1)} < \frac{1}{2}. \tag{44}$$

Then, for $f \in E_R$, (36) *has at least one solution in the self-similar form* (37)*.*

Proof. By considering Theorem 3.6, and using the same steps followed in the proof of Theorem 3.8, we can prove that (36) has at least one solution in the self-similar form (37), if (44) is satisfied.

Remark 3.11
When $\rho \to 1$, (44) reduces to

$$\frac{X^\alpha(1+|\beta|)}{t_0\Gamma(\alpha+1)} < \frac{1}{2}, \tag{45}$$

which represents the standard Riesz-Caputo derivative case. When we let $\alpha = 2$ in (45), we get

$$\frac{X^2(1+|\beta|)}{t_0\Gamma(3)} < \frac{1}{2},$$

which gives the integer-order derivative case of the space-fractional diffusion equation in (1).

Theorem 3.12
Let $\alpha, \beta, \rho, X, t_0 \in \mathbb{R}$, provided that $1 < \alpha \leq 2$, $\rho, t_0 > 0$ and let

$$X \in \left(0, \left(\frac{t_0^{\frac{\rho(\alpha-1)+1}{\rho\alpha}} \rho^\alpha \Gamma(\alpha+1)}{2}\right)^{\frac{1}{\rho(\alpha-1)+1}}\right).$$

Then, for $f \in E$, (36) *has at least one solution in the self-similar form* (37)*.*

Proof. Based on Theorem 3.7, and using the same steps followed in the proof of Theorem 3.8, we can prove the existence of at least one solution of the problem (36) in the self-similar form (37).

4. Conclusion

In this paper, we have carried out a theoretical study concerning the space-fractional diffusion equation with anti-periodic boundary conditions. Using special transformation (the self-similar form (12), we first reduced the considered FPDE to the FODE (see Theorem 3.1) and then we have applied some fixed point theorems (Banach's contraction principle, Schauder's fixed point theorem and the nonlinear alternative of Leray-Schauder type) to this ODE to prove the existence and uniqueness results and thus, the existence and uniqueness results to the original problem. The differential operator we have considered is the Riesz-Caputo generalized fractional derivative, so the Riesz-Caputo and Riesz-Caputo Hadamard fractional derivatives can be considered as particular cases from our generalized problem. This study serves as a new way for the researchers to discuss interesting problems in fractional differential and integral calculus.

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