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Error recognition in the Cantor cube

Abstract. Based on the notion of thin sets introduced recently by T. Banach, Sz. Głąb, E. Jabłońska and J. Swaczyna we deliver a study of the infinite single-message transmission protocols. Such protocols are associated with a set of admissible messages (i.e. subsets of the Cantor cube \mathbb{Z}_2^ω).

Using Banach-Mazur games we prove that all protocols detecting errors are Baire spaces and generic (in particular maximal) ones are not neither Borel nor meager.

We also show that the Cantor cube can be decomposed to two thin sets which can be considered as the infinite counterpart of the parity bit. This result is related to so-called xor-sets defined by D. Niwiński and E. Kopczyński in 2014.

1. Introduction

We deliver the error-recognition and error-correction approach to single-message transmission protocols which allows to send infinite messages only (i.e. elements of the Cantor cube \mathbb{Z}_2^ω). The background of this note are thin sets introduced recently in Banach-Głąb-Jabłońska-Swaczyna [3].

During our consideration we use all four types of natural numbers (including and excluding both zero and infinity). Thus, in order to avoid misunderstandings, let us define formally

$$\begin{aligned}\omega &:= \{0, 1, \dots\}, & \bar{\omega} &:= \omega \cup \{\omega\}, \\ \mathbb{N} &:= \{1, 2, \dots\}, & \bar{\mathbb{N}} &:= \mathbb{N} \cup \{\omega\}.\end{aligned}$$

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Now we recall that the Hamming distance [6] is the function which measures the number of bits where two vectors are different. More precisely, we define $\text{hd}: \bigcup_{n \in \bar{\mathbb{N}}} \mathbb{Z}_2^n \times \mathbb{Z}_2^n \rightarrow \bar{\omega}$ by

$$\text{hd}(x, y) := |\{k: x(k) \neq y(k)\}| \quad x, y \in \mathbb{Z}_2^n, n \in \bar{\mathbb{N}}.$$

One can easily show that hd is a metric on each \mathbb{Z}_2^n ($n \in \mathbb{N}$) and an extended metric on \mathbb{Z}_2^ω . Therefore we can define the equivalence relation \sim on \mathbb{Z}_2^ω by

$$x \sim y : \iff \text{hd}(x, y) < +\infty.$$

Obviously hd is a metric on every element of $\mathbb{Z}_2^\omega / \sim$.

This notion is deeply connected with the errors recognition, correction and checksums. Indeed, once we create a transmission protocol we shall to allow some errors during transmission.

The aim of this paper is to deliver an ideas and preliminary results to the scope of the error recognition on the Cantor set \mathbb{Z}_2^ω . Following the simple observation that thin sets can be considered separately on each class of abstraction of the relation \sim , we show some of their topological properties. In some sense we consider this paper as a preliminary study of thin sets which could leads to its potential applications in the various scopes of mathematics and computer science.

Protocol description

This part is based on [1, section 4] (with a natural extension to the case $n = \omega$). In the simplest model there are two nodes and a single one-sided signal transmission, i.e. Alice sends a single message to Bob. Furthermore we assume that number of bits of all admissible messages are the same (say $n \in \bar{\mathbb{N}}$). In such a trivial setup whole protocol can be described by a set of all admissible messages which Alice may send to Bob, that is the set $T \subseteq \mathbb{Z}_2^n$.

For the transmission Alice sends a message $x \in T$ and Bob receives some element $\bar{x} \in \mathbb{Z}_2^n$, but due to some distortions in the transmission we do not claim $x = \bar{x}$. Obviously, if $\bar{x} \notin T$ then Bob (or code) *detects* an error. In such a setting he tries to recover a message, i.e. assume that the Alice submits the element which is closest to \bar{x} (in the Hamming distance; so-called likelihood decoding) – say y (if there are more than one such elements Bob fails to recover). In general, we say that Bob (or code) *corrects* the original signal if $y = x$.

As in most of protocols we assume that \bar{x} is close to x , it is reasonable to define an d -neighbourhood of x , that is the set

$$B_d(x) := \{y \in \mathbb{Z}_2^n : \text{hd}(x, y) \leq d\}.$$

In such a setup Bob can detect up to $d \in \mathbb{N}$ errors in a code word if and only if $B_d(x) \cap T = \{x\}$ for all $x \in T$. Moreover, he can correct up to $d \in \mathbb{N}$ errors (i.e. recover all messages with at most d errors) if and only if $B_d(x) \cap B_d(y) = \emptyset$ for all $x, y \in T$ with $x \neq y$.

Motivated by this fact, for a subset $T \subset \mathbb{Z}_2^n$ ($n \in \bar{\mathbb{N}}$) having at least two elements, following [1, section 4.5] we define the *minimum distance* (of T) by

$$d(T) := \inf \{ \text{hd}(x, y) : x, y \in T, x \neq y \}.$$

Then, using some elementary geometrical argumentation from [1] we can establish the following folk result.

LEMMA 1

Let $T \subset \mathbb{Z}_2^n$ ($n \in \bar{\mathbb{N}}$). Then, for all $k \in \mathbb{N}$,

- a) a code T detects k errors if and only if $d(T) \geq k + 1$;
- b) a code T corrects k errors if and only if $d(T) \geq 2k + 1$.

Let us emphasize that in the case $n = \omega$ it could happen that $d(T) = +\infty$. In this setting the latter lemma states that code detects and corrects any (finite) numbers of errors. Furthermore, it shows that the problem of detection, correction, and the minimum distance are in some sense equivalent to each other.

In the present paper we are going to study properties of thin sets introduced by Banach et. al. [3] and their generalizations to k -thin sets. We show the Banach-Mazur game is undetermined for generic thin sets which implies few topological properties. On the other hand, we prove that thin sets (as well as their generalizations) naturally appears in the study of correction errors on the infinite bit streams. The key aim of these results is to bind the topological and the information-theory approaches to the Cantor cube.

2. Auxiliary results

2.1. Thin sets and their properties

A subset T of the Cantor cube \mathbb{Z}_2^ω is called *thin* if for every number $n \in \omega$ the restriction $\text{pr}_n \upharpoonright T$ of the projection $\text{pr}_n : \mathbb{Z}_2^\omega \rightarrow 2^{\omega \setminus \{n\}}$ given by $\text{pr}_n : x \mapsto x \upharpoonright_{\omega \setminus \{n\}}$ is injective. Equivalently, the minimal distance of T equals at least two – see Lemma 2 for the precise wording of this statement. In view of Lemma 1, this is a necessary and sufficient condition for a family of infinite streams of bits which allows to detect a single error. In this sense thin sets are the infinite counterpart of a parity bit.

Some properties of this family has been already given in [3]. In particular, it is known [3, Proposition 9.3] that each Borel thin subset of the Cantor cube is meager and has Haar measure zero. We deliver some further properties of thin sets. In turns out that they are deeply connected with Banach-Mazur games. The key results are obtained using the folk “capture-the-strategy” idea.

We also study some special subtype of this family, so-called xor-sets introduced by Niwiński–Kopczyński [8]. This allows us to prove that Cantor cube can be partitioned into two thin sets (see section 4 for details).

First, as a straightforward implication of the definition, we can prove that a subset of a thin set is also thin. Furthermore this family is closed under union of chains. These properties follow from analogous asserts of injective mappings (understood as sets of pairs with suitable assumptions).

LEMMA 2

Let $T \subset \mathbb{Z}_2^\omega$. The following statements are equivalent:

- (i) T is thin;
- (ii) every class of T/\sim is thin;
- (iii) $d(T) \geq 2$.

Proof. Implication (i) \Rightarrow (ii) is obvious as every subset of a thin set is thin.

To prove (ii) \Rightarrow (iii) assume to the contrary that $\text{hd}(x, y) = 1$ for some $x, y \in T$. Then $x \sim y$ and $\{n \in \omega : x(n) \neq y(n)\} = \{n_0\}$ for some $n_0 \in \omega$. Thus we have $\text{pr}_{n_0}(x) = \text{pr}_{n_0}(y)$. By the definition of thin set this implies $x = y$, and consequently $\text{hd}(x, y) = 0$ contradicting the assumption.

To show (iii) \Rightarrow (i) assume that T is not a thin set. Then there exist $n_0 \in \omega$ and two distinct elements $x, y \in T$ such that $\text{pr}_{n_0}(x) = \text{pr}_{n_0}(y)$. Then

$$\{n \in \omega : x(n) \neq y(n)\} = \{n_0\},$$

i.e. $\text{hd}(x, y) = 1$.

By the above results and Zorn Lemma we obtain the next proposition.

PROPOSITION 1

Let $T \subset \mathbb{Z}_2^\omega$ be a thin set. Then there exists a maximal thin set $T_0 \subset \mathbb{Z}_2^\omega$ such that $T \subseteq T_0$. Moreover, for all $Q \in \mathbb{Z}_2^\omega/\sim$ we have that $T_0 \cap Q$ is a maximal thin subset of Q .

Proof. As thin sets are closed under union of chains, the first part is an immediate implication of Zorn lemma. To show the moreover part assume that there exists $Q \in \mathbb{Z}_2^\omega/\sim$ such that $T_0 \cap Q$ is not a maximal thin subset of Q . Then there exists $q \in Q \setminus T_0$ such that $(T_0 \cap Q) \cup \{q\}$ is thin. Therefore applying the implication (ii) \Rightarrow (i) in Lemma 2 we obtain that $T_0 \cup \{q\}$ is also thin contradicting the maximality.

2.2. Banach-Mazur game

Following Berwanger-Grädel-Kreutzer [4] consider a special type of Banach-Mazur game parametrized by a set $F \subset \mathbb{Z}_2^\omega$ (with the product, i.e. Tychonoff topology). Let $\mathcal{G}(F)$ be an infinite two-player game with a complete information, where moves of players consist of selecting and extending finite path through a complete binary tree \mathbb{Z}_2^ω by an element in $\mathbb{Z}_2^+ := \bigcup_{n=1}^\infty \mathbb{Z}_2^n$. The players will be called Ego and Alter. The two players alternate turns, and each player is aware of all moves before making the next one; Ego begins. All plays are infinite and the result outcome of each play is an element of $x \in \mathbb{Z}_2^\omega$. Ego wins if $x \in F$, otherwise Alter wins. For detailed history of this games we refer the reader to Oxtoby [9] and Telgársky [10].

Using some unraveling techniques it is possible to embed $\mathcal{G}(F)$ to the classical Banach-Mazur game on a tree \mathbb{Z}_2^ω (see [4] for details). Thus we can reformulate the original Banach-Mazur theorem [2] in the flavour of Berwanger-Grädel-Kreutzer. Prior to this we need to recall the notion of strategy.

Ego's and Alter's strategy are the functions

$$e: \bigcup_{n=0}^{\infty} (\mathbb{Z}_2^+)^n \rightarrow \mathbb{Z}_2^+ \quad \text{and} \quad a: \bigcup_{n=1}^{\infty} (\mathbb{Z}_2^+)^n \rightarrow \mathbb{Z}_2^+,$$

respectively. Denote sets of all Alter's and Ego's strategies by \mathcal{A} and \mathcal{E} . For $a \in \mathcal{A}$ and $e \in \mathcal{E}$ one can consider a sequence of moves

$$\begin{aligned} \epsilon_0 &:= e(\emptyset), \\ \alpha_i &:= a(\epsilon_0, \dots, \epsilon_{i-1}), \quad i \in \mathbb{N}, \\ \epsilon_i &:= e(\alpha_1, \dots, \alpha_i), \quad i \in \mathbb{N}. \end{aligned}$$

Now let us define the concatenation operator

$$\text{Concat}: \bigcup_{n=1}^{\infty} (\mathbb{Z}_2^+)^n \cup \bigcup_{n=1}^{\infty} ((\mathbb{Z}_2^+)^n \times \mathbb{Z}_2^\omega) \cup (\mathbb{Z}_2^+)^\omega \rightarrow \mathbb{Z}_2^+ \cup \mathbb{Z}_2^\omega$$

as follows

$$\begin{aligned} \text{Concat}(\alpha_1, \dots, \alpha_n) &= (\alpha_{1,1}, \dots, \alpha_{1,k_1}, \dots, \alpha_{n,1}, \dots, \alpha_{n,k_n}) \in \mathbb{Z}_2^+ \\ &\text{for } n \in \mathbb{N}, k_1, \dots, k_n \in \mathbb{N} \text{ and } (\alpha_1, \dots, \alpha_n) \in \prod_{j=1}^n \mathbb{Z}_2^{k_j}; \\ \text{Concat}(\alpha_1, \dots, \alpha_n, \beta) &= (\alpha_{1,1}, \dots, \alpha_{1,k_1}, \dots, \alpha_{n,1}, \dots, \alpha_{n,k_n}, \beta_1, \beta_2, \dots) \in \mathbb{Z}_2^\omega \\ &\text{for } n \in \mathbb{N}, k_1, \dots, k_n \in \mathbb{N}, (\alpha_1, \dots, \alpha_n) \in \prod_{j=1}^n \mathbb{Z}_2^{k_j} \text{ and } \beta \in \mathbb{Z}_2^\omega; \\ \text{Concat}(\alpha_1, \alpha_2, \dots) &= (\alpha_{1,1}, \dots, \alpha_{1,k_1}, \alpha_{2,1}, \dots, \alpha_{2,k_2}, \dots) \in \mathbb{Z}_2^\omega \\ &\text{for } k_1, \dots, k_n, \dots \in \mathbb{N}, (\alpha_1, \dots, \alpha_n) \in \prod_{j=1}^{\infty} \mathbb{Z}_2^{k_j}. \end{aligned}$$

Then we define a *play*

$$\text{Play}: \mathcal{E} \times \mathcal{A} \ni (e, a) \mapsto \text{Concat}(\epsilon_0, \alpha_1, \epsilon_1, \alpha_2, \epsilon_2, \alpha_3, \dots) \in \mathbb{Z}_2^\omega.$$

This is very usual notion in game theory – instead of sequence of moves players show whole strategy at the beginning. We also use the time-lapse approach to a strategy. We treat it as a sequence of replies for the opponent's moves and write it in terms of pseudocode – it is a classical approach in game theory which is equivalent to the one above.

We say that $a_0 \in \mathcal{A}$ is an *Alter's winning strategy* (in $\mathcal{G}(F)$) if $\text{Play}(e, a_0) \notin F$ for all $e \in \mathcal{E}$. Analogously $e_0 \in \mathcal{E}$ is an *Ego's winning strategy* (in $\mathcal{G}(F)$) if $\text{Play}(e_0, a) \in F$ for all $a \in \mathcal{A}$. If one of players has a winning strategy then the game $\mathcal{G}(F)$ is *determined*. Now we can recall celebrated Banach-Mazur theorem.

THEOREM 1 (Banach-Mazur)

Let $F \subset \mathbb{Z}_2^\omega$.

- (a) Alter has a winning strategy for the game $\mathcal{G}(F)$ if and only if F is meager.
- (b) Ego has a winning strategy for the game $\mathcal{G}(F)$ if and only if there exists a finite word $x \in \mathbb{Z}_2^+$ such that $(x \cdot \mathbb{Z}_2^\omega) \setminus F$ is meager.

As a result we have the following corollary.

COROLLARY 1

Games $\mathcal{G}(F)$ are determined for all Borel sets F .

In the following two propositions we present necessary conditions for Ego and Alter to have a winning strategy. The first result essentially follows the idea of Niwiński and Kopczyński from [8].

PROPOSITION 2

Let $F \subseteq \mathbb{Z}_2^\omega$ be a thin set. Then Ego has no winning strategy in a game $\mathcal{G}(F)$.

Proof. Assume to the contrary that Ego has a winning strategy in $\mathcal{G}(F)$. We play this game two times simultaneously – we call them “initial” and “mirror” play. Denote the Ego’s moves in the initial and mirror plays as $(\alpha_i)_{i=0}^\infty$ and $(\beta_i)_{i=0}^\infty$, respectively. Obviously $\alpha_0 = \beta_0$ as the first Ego’s move is fixed.

First Alter’s reply in the initial play is 0. In the mirror play it is $\text{Concat}((1), \alpha_1)$. From now on Alter captures the Ego’s strategy in the following way:

- each time Ego moves α_k ($k \geq 2$) in the initial play, Alter copies this move to the mirror play as his $(k - 1)$ -th move;
- each time Ego moves β_k ($k \geq 1$) in the mirror play, Alter copies this move to the initial play as his k -th move.

These two plays can be illustrated in the table-like form (Simulation 1)

		Initial play						
Ego	α_0	α_1	α_2	\dots	α_k	\dots		
Alter	0		β_1	β_2	\dots	β_k	\dots	
		Mirror play						
Ego	α_0		β_1	β_2	\dots	β_k	\dots	
Alter	$\text{Concat}(1, \alpha_1)$		α_2	\dots	α_k		\dots	

Fig. 1: Capturing Ego’s strategy

The final outcome of the initial and mirror plays are

$$a := \text{Concat}(\alpha_0, (0), \alpha_1, \beta_1, \alpha_2, \beta_2, \dots)$$

and

$$b := \text{Concat}(\alpha_0, (1), \alpha_1, \beta_1, \alpha_2, \beta_2, \dots),$$

respectively. As Ego has a winning strategy in $\mathcal{G}(F)$ then we obtain $a, b \in F$. However in this case we have $\text{hd}(a, b) = 1$. This implies that F is not a thin set, contradicting the assumptions.

Krom [7] proved that Ego has no winning strategy in $\mathcal{G}(F)$ if and only if $F \subset \mathbb{Z}_2^\omega$ is a Baire space. Thus applying Proposition 2 we immediately obtain that thin sets are Baire spaces.

Now we are heading toward the necessary condition to Alter's winning strategy, however we need to introduce few notions first. For $k \in \omega$ define the function $\text{bit}_k: \omega \rightarrow \{0, 1\}$ such that $\text{bit}_k(x)$ is the k -th bit from the right in the binary notation of x (counting from zero). More precisely, for $k, n \in \omega$ we have

$$\text{bit}_k(n) = \begin{cases} 0, & \text{if } (n \bmod 2^{k+1}) \in \{0, \dots, 2^k - 1\}, \\ 1, & \text{if } (n \bmod 2^{k+1}) \in \{2^k, \dots, 2^{k+1} - 1\}. \end{cases}$$

For $n \in \mathbb{N}$, $x \in \mathbb{Z}_2^n$ and $m \in \{0, 1, \dots, 2^n - 1\}$ define $\Theta(x, m) \in \mathbb{Z}_2^n$ as follows (\oplus stands for a bitwise xor)

$$\Theta(x, m) := (x_k \oplus \text{bit}_k(m))_{k \in \{0, \dots, n-1\}}.$$

For an infinite sequence $x \in \mathbb{Z}_2^\omega$ and $m \in \omega$ define $\Theta(x, m) \in \mathbb{Z}_2^\omega$ by

$$\Theta(x, m) := (x_k \oplus \text{bit}_k(m))_{k \in \omega}.$$

We can now proceed to formulate and proof the most technical proposition of this paper.

PROPOSITION 3

Let $F \subseteq \mathbb{Z}_2^\omega$ be a set such that Alter has a winning strategy in a game $\mathcal{G}(F)$. Then there exists an element $X \in \mathbb{Z}_2^\omega / \sim$ such that $X \cap F = \emptyset$.

Proof. Fix an Alter strategy. Now we consider infinitely many plays of $\mathcal{G}(F)$ and show that their outputs cover whole class of abstraction of \sim .

Let $(v_i)_{i=1}^\infty$ of elements in \mathbb{Z}_2^+ be a sequence of Alter replies (in all plays), enumerated by the order of moves. Ego spreads Alters replies among all plays in a way which are described by the algorithm below. There are two types of Ego's moves: $\text{START}_i(\alpha)$ and $\text{MOVE}_i(\alpha)$ for $i \in \omega$ and $\alpha \in \mathbb{Z}_2^+$:

1. $\text{START}_i(\alpha)$ – Ego starts Play i with the initial move α ;
2. $\text{MOVE}_i(\alpha)$ – Ego makes a subsequent move α in Play i .

We now present an algorithmic description of the Ego's strategy (in infinitely many plays). It depends on the Alter's strategy which is emphasized as an argument (this is a sort of an input stream to this procedure). For the sake of brevity, let us denote $v_0 = (0)$. Moreover, for all $p, q \in \omega$ with $p \leq q$ the sequence which is obtained by the concatenation of the elements $(v_i)_{i=p}^q$ by V_p^q , that is $V_p^q := \text{Concat}(v_p, \dots, v_q)$.

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procedure CAPTURE(Alter Strategy  $\sigma$ )
  START0(0) Alter replies: v1
  for  $i = 1$  to  $+\infty$  do
    START $i$ ( $\Theta(V_0^{\frac{i(i+3)}{2}-1}, i)$ ) Alter replies: v\frac{i(i+3)}{2}
    for  $j = 0$  to  $i$  do
      MOVE $j$ ( $V^{\frac{i(i+1)}{2}+i+j}_{\frac{i(i+1)}{2}+1+j}$ ) Alter replies: v\frac{i(i+3)}{2}+j+1
    end for
  end for
end procedure

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Similarly to the previous proof let us illustrate several first moves in a tabular form.

		Play 0					
Ego	0	v_2	V_4^5	V_7^9			
Alter	v_1	v_3	v_6	v_{10}			
		Play 1					
Ego	Concat(1, v_1)	v_3	V_5^6	V_8^{10}			
Alter	v_2	v_4	v_7	\dots			
		Play 2					
Ego	$\Theta(V_0^4, 2)$		V_6^7	\dots			
Alter		v_5	v_8	\dots			
		Play 3					
Ego	$\Theta(V_0^8, 3)$			\dots			
Alter			v_9	\dots			
		\dots					

Fig. 2: Capturing Alter's strategy

Obviously both players make infinitely many moves in each of plays. Furthermore, in order to show that this algorithm is correct, we need to show that Alter's replies are properly enumerated (i.e. id of the element coincide with the replies number). This proof is a straightforward application of the "loop invariant" method; for the details we refer the reader to the classical book [5].

Now let $r_i \in \mathbb{Z}_2^\omega$ be the output of Play i ($i \in \omega$). As Ego rewrites all Alter answers except the initial move we obtain

$$r_i = \Theta(r_0, i) \quad \text{for all } i \in \omega.$$

Therefore if Alter has a winning strategy we get $\{r_i : i \in \omega\} \cap F = \emptyset$, and thus $[r_0]_\sim \cap F = \emptyset$.

3. Main result

In this brief section we present three results. First of them is presented in the game setting approach, while second and third one are topological properties of thin sets.

PROPOSITION 4

Let $T \subset \mathbb{Z}_2^\omega$ be a thin set such that $T \cap Q \neq \emptyset$ for all $Q \in \mathbb{Z}_2^\omega / \sim$. Then the game $\mathcal{G}(T)$ is undetermined.

Proof. Since T is thin, Proposition 2 implies that Ego has no winning strategy in $\mathcal{G}(T)$. On the other hand, as $T \cap Q \neq \emptyset$ for all $Q \in \mathbb{Z}_2^\omega / \sim$, by Proposition 3 we get that Alter has no winning strategy in $\mathcal{G}(T)$, too.

Now, applying Corollary 1, we can formulate the main result of this paper.

THEOREM 2

If $T \subset \mathbb{Z}_2^\omega$ is a thin set such that $T \cap Q \neq \emptyset$ for all $Q \in \mathbb{Z}_2^\omega / \sim$ then T is not Borel.

As a singleton is a thin set, by Proposition 1 we easily obtain

COROLLARY 2

Maximal thin sets are neither Borel nor meager.

Proof. Let $T \subset \mathbb{Z}_2^\omega$ be a maximal thin set. Then, by Proposition 1, we know that $T \cap Q \neq \emptyset$ for all $Q \in \mathbb{Z}_2^\omega / \sim$. Whence, by Theorem 2, T is not Borel. Moreover, by Proposition 4, the game $\mathcal{G}(T)$ is undetermined. Thus, by the Banach-Mazur theorem, we obtain that T is not meager.

4. Applications to Xor-sets

We show that xor-sets introduced by Niwiński–Kopczyński [8] are maximal thin sets. Prior to this, let us introduce some sort of conjugacy. For $x \in \mathbb{Z}_2^\omega$ and $n \in \omega$ we define $x^{\#n} \in \mathbb{Z}_2^\omega$ by

$$x^{\#n}(k) := \begin{cases} x(k), & k \in \omega \setminus \{n\}, \\ 1 - x(n), & k = n. \end{cases}$$

For fixed $n \in \omega$ the operator $(\cdot)^{\#n}$ is a symmetry, i.e. $(x^{\#n})^{\#n} = x$ for all $x \in \mathbb{Z}_2^\omega$. We are now in the position to present the main definition of this section.

DEFINITION 3

A set $\mathcal{X} \subset \mathbb{Z}_2^\omega$ is called an *xor-set* if for every $n \in \omega$ and $x \in \mathbb{Z}_2^\omega$ we have $x \in \mathcal{X} \iff x^{\#n} \notin \mathcal{X}$.

Since $(\cdot)^{\#n}$ is a symmetry we can easily check that for every xor-set \mathcal{X} the set $\mathbb{Z}_2^\omega \setminus \mathcal{X}$ is an xor-set, too. We prove that xor-sets are maximal thin sets. Now we are going to introduce few technical notions.

First, let us define the relation \approx on \mathbb{Z}_2^ω by

$$x \approx y : \iff \text{hd}(x, y) \text{ is finite and even.}$$

Observe that $x \approx y$ implies $x \sim y$. Moreover, each element of $\mathbb{Z}_2^\omega / \sim$ split into two elements of $\mathbb{Z}_2^\omega / \approx$. Therefore, as $\mathbb{Z}_2^\omega / \sim$ has a cardinality continuum, one consider a partition of \mathbb{Z}_2^ω to a family of disjoint sets $\mathcal{U} := \{U_{\eta, j} : \eta \in \mathbb{R} \text{ and } j \in \{0, 1\}\}$ such that $U_{\eta, 0}, U_{\eta, 1} \in \mathbb{Z}_2^\omega / \approx$ and $U_{\eta, 0} \cup U_{\eta, 1} \in \mathbb{Z}_2^\omega / \sim$ ($\eta \in \mathbb{R}$). These sets play an essential role in the theory of xor-sets.

LEMMA 3

A set $\mathcal{X} \subset \mathbb{Z}_2^\omega$ is an xor-set if and only if there exists a selector \mathcal{S} of $\{\{U_{\eta,0}, U_{\eta,1}\} : \eta \in \mathbb{R}\}$ such that $\mathcal{X} = \bigcup \mathcal{S}$.

Proof. Let $\eta \in \mathbb{R}$. Observe that for all $x \in U_{\eta,0} \cup U_{\eta,1}$ and $n \in \omega$ we have $x^{\#n} \sim x$ and $x^{\#n} \not\approx x$. Therefore $x \in U_{\eta,0} \iff x^{\#n} \in U_{\eta,1}$.

Furthermore, for all $y \in \mathbb{Z}_2^\omega$ such that $y \approx x$ there exist $k \in \mathbb{N}$ and a set $\{i_1, \dots, i_{2k}\}$ such that $y = (\dots((x^{\#i_1})^{\#i_2})\dots)^{\#i_{2k}}$. Whence $x \in \mathcal{X}$ implies $[x]_\approx \subset \mathcal{X}$.

Moreover, $x \in U_{\eta,0} \iff y \in U_{\eta,0}$ and $x \in U_{\eta,1} \iff y \in U_{\eta,1}$. Thus $[x]_\approx \in \{U_{\eta,0}, U_{\eta,1}\}$. As $[x]_\sim$ contains two classes of abstraction of \approx we get

$$\{U_{\eta,0}, U_{\eta,1}\} = \{[x]_\approx, ([x]_\sim \setminus [x]_\approx)\}.$$

By the definition every xor-set contains exactly one element of each pair, which implies our assertion.

PROPOSITION 5

Every xor-set $\mathcal{X} \subset \mathbb{Z}_2^\omega$ is a maximal thin set. In particular, \mathcal{X} is not Borel.

Proof. As \mathcal{X} an xor-set we have that for all $x, y \in \mathcal{X}$ the distance $\text{hd}(x, y)$ is either infinite or even. Then the implication (iii) \Rightarrow (i) in Lemma 2 yields that \mathcal{X} is a thin set.

To show the maximality assume to the contrary that there exist an xor-set \mathcal{X} and an element $x \in \mathbb{Z}_2^\omega \setminus \mathcal{X}$ such that $\mathcal{X} \cup \{x\}$ is thin. By the definition of an xor-set we have $x^{\#1} \in \mathcal{X}$. However, $\pi_1(x) = \pi_1(x^{\#1})$ which leads to a contradiction, as $\mathcal{X} \cup \{x\}$ was supposed to be thin.

The remaining part is a straightforward implication of Corollary 2.

As a complementary of an xor-set is an xor-set we obtain the following interesting property.

COROLLARY 3

There exist two non-Borel, thin and disjoint sets $T_0, T_1 \subset \mathbb{Z}_2^\omega$ such that

$$T_0 \cup T_1 = \mathbb{Z}_2^\omega.$$

In fact we can also prove a sort of the reverse statement

PROPOSITION 6

Let T_0, T_1 be two thin sets such that $T_0 \cup T_1 = \mathbb{Z}_2^\omega$. Then T_0 and T_1 are disjoint xor-sets.

Proof. Indeed, as T_0 is thin and $T_0 \cup T_1 = \mathbb{Z}_2^\omega$ we have

$$x \in T_0 \Rightarrow x^{\#n} \notin T_0 \Rightarrow x^{\#n} \in T_1 \quad \text{for all } x \in \mathbb{Z}_2^\omega \text{ and } n \in \omega. \quad (1)$$

Similarly $x \in T_1 \Rightarrow x^{\#n} \in T_0$, which yields $x \in T_0 \iff x^{\#n} \in T_1$.

If there existed $x \in T_0 \cap T_1$ then by (1) we would obtain $x^{\#n} \in T_1$, contradicting the fact that T_1 is a thin set. Therefore $T_0 \cap T_1 = \emptyset$. Then we have

$$x \in T_i \iff x^{\#n} \notin T_i \quad \text{for all } x \in \mathbb{Z}_2^\omega, n \in \omega \text{ and } i \in \{0, 1\},$$

which shows that both T_0 and T_1 are xor-sets.

REMARK 1

Applying above results we can easily show that $\mathcal{X} \subset \mathbb{Z}_2^\omega$ is an xor-set if and only if both \mathcal{X} and $\mathbb{Z}_2^\omega \setminus \mathcal{X}$ are thin.

5. Generalization to k -thin sets

At the very end of this note let us just mention the natural generalization of thin sets to k -thin sets. Namely, the set $T \subset \mathbb{Z}_2^n$ ($n \in \bar{\mathbb{N}}$) is called k -thin if its minimum distance equals at least k . Then 2-thin sets are precisely thin sets.

There appears the natural question, if there exists a partition of the Cantor cube to finitely many k -thin sets. For $k = 2$ the answer easily follows from Proposition 6. We show that it cannot be generalized to k -thin sets.

For $k \in \mathbb{N}$ and $n \in \bar{\mathbb{N}}$ with $2 \leq k \leq n$ we define $Q(n, k)$ as the smallest extended natural number s such that there exists a partition of \mathbb{Z}_2^n into s sets which are k -thin.

Then we obviously have $Q(n, 2) = 2$ for all $n \geq 2$. This case turns out to be very special.

PROPOSITION 7

Let $k \in \mathbb{N}$, $n \in \bar{\mathbb{N}}$ with $3 \leq k \leq n$. Then

- (a) for finite n we have $Q(n, k) \geq \lfloor \frac{n}{k-1} \rfloor$;
- (b) for $n = \omega$ we have $Q(\omega, k) = +\infty$.

Proof. Consider an arbitrary partition (T_1, \dots, T_q) of \mathbb{Z}_2^n to k -thin sets, where $q \in \bar{\mathbb{N}}$. Furthermore, set

$$S := \{x \in \mathbb{Z}_2^n : x_0 + \dots + x_{n-1} \leq \lfloor \frac{k-1}{2} \rfloor\}.$$

Then for all $x, y \in S$ we have

$$\text{hd}(x, y) \leq \text{hd}(x, 0) + \text{hd}(0, y) \leq 2 \lfloor \frac{k-1}{2} \rfloor \leq k-1.$$

In particular, each T_i contains at most one element of S . Thus in the case where n is finite we obtain $q \geq |S| \geq \lfloor \frac{n}{k-1} \rfloor$. If $n = \omega$ we simply get $q \geq |S| = +\infty$.

Finally, observe that we can reapply Theorem 2 and Proposition 1 to generalize Corollary 2 as follows

PROPOSITION 8

Maximal k -thin sets are neither Borel nor meager.

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