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Akbar Rezaei, Hee Sik Kim and Arsham Borumand Saeid

Independence concepts for groupoids

Abstract. In this paper, for getting more results in groupoids, we consider a set and introduce the notion of a right (left) independent subset of a groupoid, and it is studied in detail. As a corollary of these properties, the following important result is proved: for any groupoid, there is a maximal right (left) independent subset.

Moreover, the notion of strongly right (left) independent subset is considered. It is proved that there exists a groupoid having a strongly right independent 2-set. Finally, we discuss the notion of dynamic elements with independence.

1. Introduction

Bruck [2] published a book, *A survey of binary systems* discussed in the theory of groupoids, loops and quasigroups, and several algebraic structures. Borůvka [3] stated the theory of decompositions of sets and its application to binary systems. Nebeský [17] introduced the notion of a travel groupoid by adding two axioms to a groupoid, and he described an algebraic interpretation of the graph theory. Chajda and Länger [5] assigned to every directed relational system a groupoid [4] and it was shown that properties of the relational system can be characterized by properties of a corresponding groupoid.

Allen et al. [1] introduced the concept of several types of groupoids related to semigroups, viz., twisted semigroups for which twisted versions of the associative law hold. Kim et al. [13] showed that every selective groupoid induced by a fuzzy

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subset is a pogroupoid, and they discussed several properties in quasi ordered sets by introducing the notion of a framework. In 2017, Liu et al. [15] extended the theory of groupoids already developed for semigroups $(Bin(X), \square)$ in a growing number of research papers with X a set and $Bin(X)$ the set of groupoids defined on X to the generalizations: fuzzy (sub)groupoids and fuzzy hyper (sub)groupoids. Hwang et al. [10] discussed the notion of the locally finiteness and convolution products in groupoids. Fayoumi [6] introduced the notion of the center semigroup $(Bin(X), \square)$ of all binary systems on a set X.

Also, she introduced two methods of factorization for this binary system under the binary groupoid product in the semigroup $(Bin(X), \square)$ and showed that a strong non-idempotent groupoid can be represented as a product of its similar- and signature- derived factors. Beside, in [7] it was shown that a groupoid with the orientation property is a product of its orient- and skew- factors.

Feng et al. [8] discussed some relations among axioms in groupoids, and got some useful properties.

The motivation of this study came from the idea of the converse of "injective function", and then we introduce the notion of (strongly) right (left) independent subset of a groupoid, and obtain a groupoid having a strongly right (left) independent 2-set. Moreover, we discuss the notion of dynamic elements with independence.

2. Preliminaries

A *d-algebra* ([18]) is a non-empty set X with a constant 0 and a binary operation "$*$" satisfying the following axioms:

(I) $x * x = 0$,

(II) $0 * x = 0$,

(III) $x * y = 0$ and $y * x = 0$ imply $x = y$ for all $x, y \in X$.

A *BCK-algebra* [9, 11, 16] is a d-algebra X satisfying the following additional axioms:

(IV) $((x * y) * (x * z)) * (z * y) = 0$,

(V) $(x * (x * y)) * y = 0$ for all $x, y, z \in X$.

Given a BCK-algebra, we define a binary relation "$\leq$" by $x \leq y$ if and only if $x * y = 0$. It is known that $\leq$ is a partially ordered set (shortly, poset) in BCK-algebras. A poset $\leq$ is said to be an *antichain* if $x \leq y$ if and only if $x = y$ for all $x, y \in X$.

Let $(X, \leq)$ be a poset with the least element 0. If we define a binary operation "$*$" on X as follows:

$$x * y = \begin{cases} 0, & \text{if } x \leq y, \\ x, & \text{otherwise.} \end{cases}$$

Then $(X, *, 0)$ is a BCK-algebra. Such an algebra is said to be a *standard BCK-algebra* inherited from the poset $(X, \leq)$ or a *Tanaka-type algebra* (see [12]).

A groupoid $(X, *)$, i.e. a set X together with a binary operation "$*$" on the set X, is said to be a *right zero semigroup* if $x*y = y$ for any $x, y \in X$, and a groupoid $(X, *)$ is said to be a *left zero semigroup* if $x * y = x$ for any $x, y \in X$. A groupoid $(X, *)$ is said to be a *rightoid* for $f\colon X \to X$ if $x * y = f(y)$ for any $x, y \in X$. Similarly, a groupoid $(X, *)$ is said to be a *leftoid* for $f\colon X \to X$ if $x * y = f(x)$ for any $x, y \in X$. Note that a right (left) zero semigroup is a special case of a rightoid (leftoid) (see [14]). A groupoid $(X, *)$ is said to be *right cancellative* (*left cancellative*) if $y * x = z * x$ $(x * y = x * z)$ implies $y = z$.

3. Right (left) independence in groupoids

Given a groupoid $(X, *)$, a non-empty subset E of X is said to be *right independent* if $x \neq y \in E$, then $x * u \neq y * u$ for all $u \in X$. Also E is said to be *left independent* if $x \neq y \in E$, then $u * x \neq u * y$ for all $u \in X$. E is said to be *independent* if it is both right and left independent.

Notice that a groupoid $(X, *)$ is right independent if the set X is right independent. In other words, $(X, *)$ is right independent if and only if every subset E of X is right independent.

Example 1

(a) Let $X := \{0, 1, 2, 3, 4\}$ with the following table:

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	2	0
3	3	1	3	0	1
4	4	4	4	4	0

Then $(X, *, 0)$ is a BCK-algebra (see [16]), but it is not right independent, since $1 \neq 2$ in X, but $1 * 4 = 0 = 2 * 4$. It is easy to check that $E_1 := \{2, 3\}$, $E_2 := \{3, 4\}$ are right independent, but $E_3 := \{2, 4\}$ neither right independent, since $2 \neq 4$ in E_3, but $2 * 4 = 0 = 4 * 4$, nor left independent, since $2 \neq 4$ in E_3, but $0 * 2 = 0 = 0 * 4$.

(b) Let $X := \{0, 1\}$ with the following table:

$*$	0	1
0	0	1
1	1	0

Then $(X, *)$ is a groupoid and X is independent.

(c) Let K be a field. We define a binary operation "$*$" on K by $x*y := x(x-y)$ for all $x, y \in K$. Then $(K, *, 0)$ is a d-algebra, but not a BCK-algebra. We claim that $(K, *)$ is not right independent. In fact, if $x \neq y$ in K, then $x * (x + y) = x(x - (x + y)) = -xy$ and $y * (x + y) = y(y - (x + y)) = -xy$.

Example 2
Every group is (right, left) independent, since cancellation laws hold in a group.

Note that every right (left) independent subset may not be closed. Consider the right independent subset E_2 in Example 1. It is not closed, since $3*4 = 1 \notin E_2$.

Proposition 1

(a) *Let $(X, *)$ be a leftoid for φ, i.e. $x * y = \varphi(x)$ for all $x, y \in X$, and let E be a right independent subset of X. Then φ is injective on E.*

(b) *Let $(X, *)$ is a rightoid for φ, i.e. $x * y = \varphi(y)$ for all $x, y \in X$, and let E be a right independent subset of X. Then E is a singleton.*

Proof. (a). If $x, y \in E$ with $x \neq y$, then $\varphi(x) = x * u \neq y * u = \varphi(y)$ for all $u \in X$, since E is right independent.

(b). Assume E is not a singleton. Then there exist $x, y \in E$ such that $x \neq y$. Since E is right independent, we have $x*u \neq y*u$ for all $u \in X$. On the other hand, since $(X, *)$ is a rightoid for φ, we have $x * u = \varphi(u) = y * u$, a contradiction.

Lemma 1
Let $(X, *)$ be a groupoid, and let $x \in X$. Then $\{x\}$ is left (right) independent.

Proof. The singleton set $\{x\}$ has no element y in $\{x\}$ such that $x \neq y$. It follows that the independence criteria are fulfilled vacuously.

Theorem 1
Let $(X, *)$ be a groupoid. Then there exists a maximal right (left) independent subset M in X.

Proof. Suppose $\{E_\lambda\}_{\lambda \in \Lambda}$ is a chain of right (left) independent subsets of $(X, *)$, and let $E := \cup_{\lambda \in \Lambda} E_\lambda$. Then E is a right (left) independent subset of $(X, *)$. In fact, let $x, y \in E$ such that $x \neq y$. Then there exist $\alpha, \beta \in \Lambda$ such that $x \in E_\alpha$, $y \in E_\beta$. Without loss of generality, we let $E_\alpha \subseteq E_\beta$. Then $x, y \in E_\beta$. Since E_β is right (left) independent, we obtain $x * u \neq y * u$ $(u * x \neq u * y)$ for all $u \in X$. This shows that E is right (left) independent. By Zorn's Lemma, there exists a maximal right (left) independent subsets M in X.

Theorem 2
Let $(X, *)$ be a groupoid. Then $(X, *)$ is right (left) cancellative if and only if $(X, *)$ is right (left) independent.

Proof. Assume $\emptyset \neq E \subseteq X$, and $x \neq y$ in E. If $x * u = y * u$ (or $u * x = u * y$) for some $u \in X$, which leads to $x = y$, a contradiction, since $(X, *)$ is right (left) cancellative. Thus, $x*u \neq y*u$ (or $u*x \neq u*y$) for all $u \in X$, and so E is a right (left) independent subset in X.

The proof of converse is clear.

Corollary 1
Every group has no proper maximal right (left) independent subset.

Proof. Every group has cancellative laws.

Let us observe that, if $(X, *, e)$ is a group, then X is a maximal independent subset of X.

Proposition 2
Let $(X, *)$ be a groupoid satisfying the condition $x * y = (x * y) * y$ for all $x, y \in X$. If E is a right independent subset of X, then $(E, *)$ is a left zero subgroupoid.

Proof. Let $x \neq y$ in E such that $x * y \neq x$. If we take $u := y$, then $x * u = x * y = (x * y) * y = (x * y) * u$. This shows that E is not right independent, a contradiction.

The converse of Proposition 2 is trivial. Since if X is a left zero semigroup, then it is right independent.

Proposition 3
Let $(X, *, 0)$ be a standard BCK-algebra, i.e.

$$x * y = \begin{cases} 0, & \text{if } x \leq y, \\ x, & \text{otherwise.} \end{cases}$$

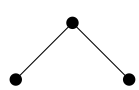
If M is a right independent subset of X, then M is an antichain.

Proof. Assume that there exist $x \neq y$ in M such that $x \leq y$. Then $x * y = 0 = y * y$, proving that M is not right independent, a contradiction. Hence M is an antichain.

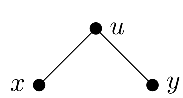
Proposition 4
Let $(X, \leq)$ be a poset with $|X| \geq 3$. Define a binary operation "$*$" on X by

$$x * y := \begin{cases} y, & \text{if } x \leq y, \\ x, & \text{otherwise.} \end{cases}$$

If $(X, *)$ is a right independent, then the poset $(X, \leq)$ has no sub-poset which is isomorphic to the following Hasse diagram:



Proof. Assume the poset $(X, \leq)$ has a sub-poset:



for some $u, x, y \in X$. It follows that $x * u = u = y * u$. Since $x \neq y$, $(X, *)$ is not right independent, a contradiction.

Note that in the Proposition 4 the condition $|X| \geq 3$ is necessary. For a counterexample, let $(X; *)$ be the two-element chain $X = \{0, 1\}$, $0 < 1$. The poset $(X; *)$ has no sub-poset isomorphic to the three-element poset described in the Proposition 4, so the hypotheses of the theorem are satisfied. We have $0 * 1 = 1$, but $0 * 1 = 1$ and $1 * 1 = 1$, so the set $X = \{0, 1\}$ is not right independent in $(X; *)$, which falsifies the claim.

Proposition 5
Let $\varphi\colon (X, *) \to (Y, \star)$ be a homomorphism of groupoids. If $\varphi(x_1), \varphi(x_2)$ are right independent in $(Y, \star)$, then x_1, x_2 are right independent in $(X, *)$.

Proof. Assume that x_1, x_2 are not right independent in $(X, *)$. Then there exists $u \in X$ such that $x_1 * u = x_2 * u$. It follows that

$$\varphi(x_1) \star \varphi(u) = \varphi(x_1 * u) = \varphi(x_2 * u) = \varphi(x_2) \star \varphi(u),$$

which shows that $\varphi(x_1), \varphi(x_2)$ are not right independent in $(Y, \star)$, a contradiction.

Let $X := \{a, b\}$ with the following tables:

$*_1$	a	b
a	a	b
b	b	a

$*_2$	a	b
a	b	a
b	a	b

Then $(X, *_i)$ for $i \in \{1, 2\}$ are groupoids. $E_1 = \{a\}$ and $E_2 = \{b\}$ are right independent subsets of order 1 and X is only right independent subset of order 2.

Problem 1
Suppose that $(X, *)$ is a groupoid of order n (i.e. $|X| = n \geq 3$). How many right (left) independent subsets E of order k for $1 \leq k \leq n$, are there? or we can construct on X?

Let $X := \{a, b, c\}$ with the following tables:

$*_1$	a	b	c
a	a	c	b
b	c	b	a
c	b	a	c

$*_2$	a	b	c
a	a	b	c
b	b	c	a
c	c	a	b

$*_3$	a	b	c
a	b	a	c
b	a	c	b
c	c	b	a

$*_4$	a	b	c
a	b	c	a
b	c	a	b
c	a	b	c

$*_5$	a	b	c
a	c	a	b
b	a	b	c
c	b	c	a

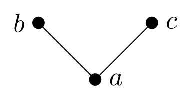
$*_6$	a	b	c
a	c	b	a
b	b	a	c
c	a	c	b

Then $(X, *_i)$ for $i \in \{1, 2, \ldots, 6\}$ are 6 commutative groupoids of order 3. $E_1 = \{a\}$, $E_2 = \{b\}$ and $E_3 = \{c\}$ are right independent subsets of order 1 and $E = X$ is right independent subset of order 3 in this case. There are more than 6 groupoids of order 3 with a right independent subset of order 3. Right independence of X in $(X;*)$ means that the right translations of $(X;*)$ are injective; if X is finite, this means that the columns of the operation table of $*$ are permutations of X. If $|X| = n$, then there are $(n!)^n$ ways of building such an operation table. For $n = 3$, we get $(3!)^3 = 216$.

4. Strongly right (left) independent

Let $(X, *)$ be a groupoid. Two elements $x, y \in X$ are said to be *strongly right independent* if $x * u \neq y * v$ for all $u, v \in X$. Similarly, x, y are said to be *strongly left independent* if $u*x \neq v*y$ for all $u, v \in X$. E is said to be *strongly independent* if it both strongly right and strongly left independent.

Example 3
Let $X := \{a, b, c\}$ be a poset with the following Hasse diagram:



Define a binary operation "$*$" on X by

$$x * y := \begin{cases} y, & \text{if } x \leq y, \\ x, & \text{otherwise.} \end{cases}$$

Then we obtain the following table:

*	a	b	c
a	a	b	c
b	b	b	b
c	c	c	c

Then it is easy to see that $\{b, c\}$ is strongly right independent, but $\{a, b\}$ (or $\{a, c\}$) is not strongly right independent, since $a * b = b = b * b$ (resp. $a * c = c = c * c$).

Proposition 6
Let $(X, *)$ be a leftoid for φ, i.e. $x * y := \varphi(x)$ for all $x, y \in X$. If $E \subseteq X$ such that φ is one-to-one on E, then E is strongly right independent.

Proof. Let $x \neq y$ in E. Since φ is one-to-one on E, we have $x * u = \varphi(x) \neq \varphi(y) = y * v$ for all $u, v \in X$. This shows that E is strongly right independent.

Proposition 7
Let $X := \mathbb{R}$ be the set of all real numbers and let $x*y := \max\{x, y\}$ for all $x, y \in X$. Then the only (strongly) right independent sets are singletons.

Proof. Let $\{x, y\}$ be a (strongly) right independent subsets of X and let $x \neq y$. Then $x * u \neq y * v$ for all $u, v \in X$. It follows that $\max\{x, u\} \neq \max\{y, v\}$ for all $u, v \in X$. If we let $u := v$ satisfying $x \leq u$ and $y \leq u$, then $\max\{x, u\} = u = v = \max\{y, v\}$, which leads a contradiction.

Proposition 8
Let $(X, *)$ be a commutative groupoid. Then the only strongly right independent subset of X are singletons.

Proof. Let $(X, *)$ be a commutative groupoid. Then $x * y = y * x$ for all $x, y \in X$. If $x \neq y$, then $\{x, y\}$ is not strongly right independent. Hence the only strongly right independent subsets of X are singletons.

Let $(X, *)$ be a groupoid, and let $\emptyset \neq A \subseteq X$ and $x \in X$. Define $x * A = \{x * a : a \in A\}$ (resp. $A * x = \{a * x : a \in A\}$.

Now we construct a groupoid $(X, *)$ whose 2-set to be a strongly right independent set as follows.

Theorem 3
Let X be a set with $|X| \geq 4$. Given $x \neq y \in X$, there exists a groupoid $(X, *)$ such that $\{x, y\}$ is a strongly right independent subset of X.

Proof. Let X be a set with $|X| \geq 4$. Assume $a, b, c, d \in X$ (all are distinct). Let $A \neq \emptyset \neq B$, $A \cup B = X$ and $A \cap B = \emptyset$. Define a binary operation "$*$" on X satisfying the following: Given $x \neq y \in X$, we let $x * A := \{a\}$, $x * B := \{b\}$ and $y * A := \{c\}$, $y * B := \{d\}$ and other entries are arbitrary elements. It follows that $x * u \in \{a, b\}$, $y * v \in \{c, d\}$ for all $u, v \in X$. This means that $x * u \neq y * v$ for all $u, v \in X$, which shows that $\{x, y\}$ is a strongly right independent subset of X.

Theorem 4
Let $f, g \colon X \to X$ be injective functions with $\mathrm{Im}(f) \cap \mathrm{Im}(g) = \emptyset$. Let $A \neq \emptyset \neq B$, $A \cup B = X$ and $A \cap B = \emptyset$. If $(X, *)$ is a groupoid such that $x * A = \{f(x)\}$ and $x * B = \{g(x)\}$, then $\{x, y\}$ is a strongly right independent subset of X.

Proof. If $x \neq y$, since f, g are one-to-one, we obtain

$$f(x) \neq f(y), \qquad g(x) \neq g(y)$$

Given $u \in X$, we have two cases. If $u \in A$, then $x * u = f(x)$ and

$$y * v = \begin{cases} f(y), & \text{if } v \in A, \\ g(y), & \text{if } v \in B. \end{cases}$$

Hence $y*v \in \{g(y), f(y)\}$ and $x*u = f(x) \notin \{f(y), g(y)\}$. Since $\mathrm{Im}(f) \cap \mathrm{Im}(g) = \emptyset$, we obtain $x * u \neq y * v$. If $u \in B$, then $x * u = g(x)$ and

$$y * v = \begin{cases} f(y), & \text{if } v \in A, \\ g(y), & \text{if } v \in B. \end{cases}$$

Hence $y*v \in \{g(y), f(y)\}$ and $x*u = g(x) \notin \{f(y), g(y)\}$. Since $\mathrm{Im}(f) \cap \mathrm{Im}(g) = \emptyset$, we obtain $x * u \neq y * v$. This shows that $\{x, y\}$ is a strongly right independent subset of X.

Note that, in Theorem 4.6, if $f, g\colon X \to X$ are bijective functions, then $\mathrm{Im}(f) = \mathrm{Im}(g) = X$. If we furthermore assume that $\mathrm{Im}(f) \cap \mathrm{Im}(g) = \emptyset$, then we have $X = X \cap X = \emptyset$. So the theorem only deal with the empty function $\emptyset\colon \emptyset \to \emptyset$.

Example 4
Let $X := \mathbb{Z}$ be the set of all integers. Define a binary operation "$*$" on X by

$$x * y := \begin{cases} 3x, & \text{if } y \in 2\mathbb{Z}, \\ 3x + 1, & \text{if } y \in 2\mathbb{Z} + 1. \end{cases}$$

Then every distinct two element set $\{x, y\}$ forms a strongly right independent subset of X. In fact, if we define maps $f(x) := 3x$, $g(x) := 3x + 1$ for all $x \in X$, then f, g are one-one and $\mathrm{Im}(f) \cap \mathrm{Im}(g) = \emptyset$. Define $A := 2\mathbb{Z}$, $B := 2\mathbb{Z} + 1$. Then $X = A \cup B$, $A \cap B = \emptyset$. By Theorem 4, we prove that x, y are strongly right independent.

5. Dynamic elements and independence

Let X be a set and let $\varphi\colon X \to X$ be a map. An element $x \in X$ is said to be *dynamic* for φ if $\varphi(x) \neq x$.

Given a groupoid $(X, *)$ and a mapping $\varphi\colon X \to X$, we define:

$$\triangle_R(\varphi) := \{x \in X :\ \varphi(x) * v \neq x * u \text{ for all } u, v \in X\}$$
$$\triangle_L(\varphi) := \{x \in X :\ v * \varphi(x) \neq u * x \text{ for all } u, v \in X\}.$$

If $(X, *)$ is a commutative groupoid, then $\triangle_R(\varphi) = \triangle_L(\varphi) = \emptyset$.

Proposition 9
If $x \in \triangle_R(\varphi)$ (resp. $x \in \triangle_L(\varphi)$), then x is dynamic.

Proof. Let $x \in \triangle_R(\varphi)$ and $\varphi(x) = x$. Then $x * u \neq \varphi(x) * v$ for all $u, v \in X$. Put $u := v$, we get $x * u = x * v = \varphi(x) * v$, a contraction.

Proposition 10
Let $(X, *)$ be a group and let $\varphi\colon X \to X$ be a map. Then $\triangle_R(\varphi) = \triangle_L(\varphi) = \emptyset$.

Proof. Assume $\triangle_R(\varphi) \neq \emptyset$. Then there exists $x \in X$ such that $x * u \neq \varphi(x) * v$ for all $u, v \in X$. It follows that $x = x * e \neq \varphi(x) * v$ for all $v \in X$, i.e. $v \neq [\varphi(x)]^{-1} * x$ for all $v \in X$, which is a contraction. Similarly, we obtain $\triangle_L(\varphi) = \emptyset$.

Proposition 11
Let $(X, *)$ be a right (left) zero semigroup and $\varphi\colon X \to X$ be a map, and let x be dynamic for φ. Then $x \in \triangle_L(\varphi)$ (resp. $x \in \triangle_R(\varphi)$).

Proof. Assume $(X, *)$ is a right zero semigroup and $\varphi\colon X \to X$ is a map, and x is a dynamic element. Let $u, v \in X$. Then

$$v * \varphi(x) = \varphi(x) \neq x = u * \varphi(x).$$

Thus, $x \in \triangle_L(\varphi)$.

Similarly, if $(X, *)$ is a left zero semigroup, then $x \in \triangle_R(\varphi)$.

Proposition 12
Let $\mathbb{R}$ be the set of all real numbers and let $x * y := \max\{x, y\}$ for all $x, y \in \mathbb{R}$. Then $\triangle_R(\varphi) = \triangle_L(\varphi) = \emptyset$.

Proof. Assume $\triangle_R(\varphi) \neq \emptyset$. Then there exists $x \in X$ such that $x * u \neq \varphi(x) * v$ for all $u, v \in X$. Take $\xi \in \mathbb{R}$ such that $\max\{x, \varphi(x)\} \leq \xi$. It follows that $\xi = \max\{x, \xi\} \neq \max\{\xi, \varphi(x)\} = \xi$, a contradiction. Similarly, $\triangle_L(\varphi) = \emptyset$.

Proposition 13
Let $X := \mathbb{R}$ be the set of all real numbers and let $a, b, c \in X$ with $bc \neq 0$. Define a map $\varphi\colon X \to X$ by $\varphi(x) := x + 1$ for all $x \in X$. Define a binary operation "$*$" by $x * y := a + bx + cy$ for all $x, y \in X$. Then $\triangle_R(\varphi) = \triangle_L(\varphi) = \emptyset$.

Proof. Assume $\triangle_R(\varphi) \neq \emptyset$. Then there exists $x \in X$ such that $x * u \neq \varphi(x) * v$ for all $u, v \in X$. It follows that $a + bx + cu \neq a + b\varphi(v) + cv$ for all $u, v \in X$. Hence $bx + cu \neq b(x + 1) + cv = bx + b + cv$, and so $cu \neq b + cv$ for all $u, v \in X$, which implies that $v \neq \frac{cu-b}{c}$ for all $v \in X$, a contradiction. Hence $\triangle_R(\varphi) = \emptyset$.

Now, assume $\triangle_L(\varphi) \neq \emptyset$. Then there exists $x \in X$ such that $v * \varphi(x) \neq u * x$ for all $u, v \in X$. It follows that $a + bv + c\varphi(x) \neq a + bu + cx$ for all $u, v \in X$. Hence $bv + c(x + 1) = bv + cx + c \neq bu + cx$, and so $bv + c \neq bu$ for all $u, v \in X$, which implies that $v \neq \frac{bu-c}{b}$ for all $v \in X$, a contradiction. Hence $\triangle_L(\varphi) = \emptyset$. Thus, $\triangle_R(\varphi) = \triangle_L(\varphi) = \emptyset$.

Proposition 14
Let $(X, *)$ be a groupoid, $\varphi\colon X \to X$ be a map, and let $x \in \triangle_R(\varphi)$ (resp. $x \in \triangle_L(\varphi)$). Then $\{x, \varphi(x)\}$ is a strongly right (left) independent.

Proof. Assume $(X, *)$ is a groupoid, $\varphi\colon X \to X$ is a map, and let $x \in \triangle_R(\varphi)$ (resp. $x \in \triangle_L(\varphi)$). Then $x * u \neq \varphi(x) * v$ $(u * x \neq v * \varphi(x))$ for all $u, v \in X$, and so $\{x, \varphi(x)\}$ is a strongly right (left) independent.

Proposition 15
Let $(X, *)$ be a groupoid, $\varphi\colon X \to X$ be a map, and let $x \in \triangle_R(\varphi) \cap \triangle_L(\varphi)$, then $\{x, \varphi(x)\}$ is a strongly independent.

Proof. Assume $(X, *)$ is a groupoid, $\varphi\colon X \to X$ is a map, and let $x \in \triangle_R(\varphi) \cap \triangle_L(\varphi)$). Using Proposition 5.6, we get $\{x, \varphi(x)\}$ is a strongly right and left independent, and so a strongly independent.

Proposition 16
If $(X, *)$ is a groupoid and $x \in X$ is a dynamic for idempotent map φ, then $\varphi^2 \neq I$.

Proof. Assume $(X, *)$ is a groupoid and $x \in X$ is a dynamic for idempotent map φ. Then $\varphi^2(x) = \varphi(\varphi(x)) = \varphi(x) \neq x$, and so $\varphi^2 \neq I$.

Proposition 17
If $(X, *)$ is a groupoid and $x \in X$ is a dynamic for idempotent map φ, then x is a dynamic for φ^2.

Proof. Assume $(X, *)$ is a groupoid and $x \in X$ is a dynamic for idempotent map φ. Then $\varphi(x) \neq x$. Since φ is an idempotent map, we get $\varphi^2(x) = \varphi(x) \neq x$, and so x is a dynamic for φ^2.

Proposition 18
Let $(X, *)$ be an idempotent rightoid (resp. leftoid) for φ. Then $\triangle_L(\varphi) = \emptyset$ (resp. $\triangle_R(\varphi) = \emptyset$).

Proof. Assume $(X, *)$ is an idempotent rightoid for φ and $x \in X$. Using an idempotent operation, we have $u * x = \varphi(x) = \varphi(\varphi(x)) = v * \varphi(x)$ for all $u, v \in X$, and so $x \notin \triangle_L(\varphi)$.

Also, assume $(X, *)$ is an idempotent leftoid for φ and $x \in X$. Using an idempotent operation, we have $x * u = \varphi(x) = \varphi(\varphi(x)) = \varphi(x) * v$ for all $u, v \in X$, and so $x \notin \triangle_R(\varphi)$.

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Akbar Rezaei
Department of Mathematics
Payame Noor University
P. O. Box 19395-3697 Tehran
Iran
E-mail: rezaei@pnu.ac.ir

Hee Sik Kim
Research Institute for Natural Sci.
Department of Mathematics
Hanyang University
Seoul, 04763
Korea
E-mail: heekim@hanyang.ac.kr

Arsham Borumand Saeid
Department of Pure Mathematics
Faculty of Mathematics and Computer
Shahid Bahonar University of Kerman
Kerman
Iran
E-mail: arsham@uk.ac.ir

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