sciendo

Ann. Univ. Paedagog. Crac. Stud. Math. 22 (2023), 99-109
DOI: 10.2478/aupcsm-2023-0008

FOLIA 385

Annales Universitatis Paedagogicae Cracoviensis
Studia Mathematica 22 (2023)

*Noureddine Azzouz, Bouharket Benaissa and Abdelkader Senouci*

# Some Hardy type integral inequalities with two parameters of summation

**Abstract.** In the present work, some Hardy-type integral inequalities were proved for two parameters of summation $q \leq p < 0$ and $p < 0$, $q > 0$. In addition, some two-sided estimates are obtained.

## 1. Introduction

Hardy type inequalities were studied by a large number of authors during the 20th century and have inspired some important research, which is now at work. Over the two past decades, a large number of papers have been published with various generalizations and applications.

The following statements were established in [5, Th.330, Th.347]. If $p > 0$, $r \neq 1$, $f(t) \geq 0$ and $0 < \int_0^\infty t^{-r}(tf(t))^p dt < \infty$,

$$F(x) = \int_0^x f(t)dt \quad \text{for } r > 1, \qquad F(x) = \int_x^\infty f(t)dt \quad \text{for } r < 1,$$

then for $p > 1$, one has

$$\int_0^\infty x^{-r} F^p(x)dx < \Big(\frac{p}{|r-1|}\Big)^p \int_0^\infty x^{-r}(xf(x))^p dx,$$

for $0 < p < 1$, one has

$$\int_0^\infty x^{-r} F^p(x)dx > \Big(\frac{p}{|r-1|}\Big)^p \int_0^\infty x^{-r}(xf(x))^p dx,$$

AMS (2010) Subject Classification: 26D10, 26D15.
Keywords and phrases: Hölder type inequalities, Hardy type inequalities, parameters of summation.
ISSN: 2081-545X, e-ISSN: 2300-133X.

where the constant $\left(\frac{p}{|r-1|}\right)^p$ is the best possible.

Bicheng Yang [4] proved the similar Hardy type integral inequalities for one negative parameter of summation $p < 0$ in the following Theorem.

Theorem 1 (see [4])
*If* $p < 0$, $r \neq 1$, $f(t) \geq 0$ *and* $0 < \int_0^\infty t^{-r}(tf(t))^p dt < \infty$,

$$F(x) = \int_0^x f(t)dt \quad \text{for } r < 1, \qquad F(x) = \int_x^\infty f(t)dt \quad \text{for } r > 1,$$

*then one has*

$$\int_0^\infty x^{-r} F^p(x)dx \leq \Big(\frac{-p}{|r-1|}\Big)^p \int_0^\infty x^{-r}(xf(x))^p dx, \tag{1.1}$$

*where the constant* $\left(\frac{-p}{|r-1|}\right)^p$ *is the best possible.*

In several cases the integral Hardy inequalities or Hardy type inequalities are applied to functions or weighted functions satisfying additional conditions. In [3, Theorem 1-3], the authors gave a generalization of inequality (1.1) by introducing a monotone weighted function $w$.

Theorem 2 (see [3])
*Let* $p < 0$, $f, w > 0$ *and* $\int_0^\infty w(x)^{-r}(xf(x))^p dx < \infty$, *then*

(a) *If* $\frac{x}{w(x)}$ *is non-decreasing and* $r > 1$,

$$\int_0^\infty F_1^p(x)w^{-r}(x)dx \leq \Big(\frac{p}{1-r}\Big)^p \int_0^\infty (xf(x))^p w^{-r}(x)dx, \tag{1.2}$$

*where* $F_1(t) = \int_x^\infty f(t)dt$.

(b) *If* $\frac{x}{w(x)}$ *is non-increasing and* $0 \leq r < 1$,

$$\int_0^\infty F_2^p(x)w^{-r}(x)dx \leq \Big(\frac{p}{r-1}\Big)^p \int_0^\infty (xf(x))^p w^{-r}(x)dx, \tag{1.3}$$

*where* $F_2(t) = \int_0^x f(t)dt$.

(c) *If* $\frac{x}{w(x)}$ *is non-decreasing and* $r < 0$,

$$\int_0^\infty F_2^p(x)w^{-r}(x)dx \leq \Big(\frac{p}{r-1}\Big)^p \int_0^\infty (xf(x))^p w^{-r}(x)dx. \tag{1.4}$$

In this work, two generalizations of Theorem 2 and the Theorem 1 are established, first replacing $(0, +\infty)$ by $(a, b)$, where $0 \leq a < b \leq +\infty$, second by applying two negative parameters $p$, $q$. In addition, in the case of the cone of monotone functions, two sided integral inequalities are obtained for $p < 0$, $q > 0$.

We need the following Hölder's inequality for $p < 0$ (see [6], [1]).

Lemma 1
*If* $p < 0$, $\frac{1}{p} + \frac{1}{p'} = 1$, $f \in L_p(E)$, $g \in L_{p'}(E)$, *where* $E$ *is Lebesgue measurable set,* $f(t), g(t) \geq 0$, *then*

$$\int_E f(t)g(t)dt \geq \Big(\int_E f^p(t)dt\Big)^{\frac{1}{p}} \Big(\int_E g^{p'}(t)dt\Big)^{\frac{1}{p'}}, \tag{1.5}$$

*where the equality holds if and only if there exists constants c and d, such they are not all zero and*

$$c\, f^p(t) = d\, g^{p'}(t), \qquad a.e.\ in\ E.$$

## 2. Main results

The following Lemma is useful in the proofs of the first result.

Lemma 2
*Let* $p < 0$, $0 \leq a < b \leq +\infty$ *and* $f$ *be a positive measurable function on* $(a, b)$, *defined* $F_1(x) = \int_x^b f(t)dt$ *and* $F_2(x) = \int_a^x f(t)dt$.

(a) *If* $r > 1$, *then*

$$F_1^p(x) \leq \Big(\frac{p}{1-r}\Big)^{p-1} \big(x^{\frac{r-1}{p}} - b^{\frac{r-1}{p}}\big)^{p-1} \int_x^b t^{\frac{1+p-r}{p'}} f^p(t)dt. \tag{2.1}$$

(b) *If* $r < 1$, *then*

$$F_2^p(x) \leq \Big(\frac{p}{r-1}\Big)^{p-1} \big(x^{\frac{r-1}{p}} - a^{\frac{r-1}{p}}\big)^{p-1} \int_a^x t^{\frac{1+p-r}{p'}} f^p(t)dt. \tag{2.2}$$

*Proof.* Let $p < 0$. (a) Suppose that $r > 1$, using the Hölder inequality (1.5) for $\frac{1}{p} + \frac{1}{p'} = 1$, we get

$$\begin{aligned} F_1(x) &= \int_x^b t^{-\frac{1+p-r}{p'p}} t^{\frac{1+p-r}{p'p}} f(t)dt \\ &\geq \Big(\int_x^b t^{-\frac{1+p-r}{p}} dt\Big)^{\frac{1}{p'}} \Big(\int_x^b t^{\frac{1+p-r}{p'}} f^p(t)dt\Big)^{\frac{1}{p}} \\ &= \Big(\frac{p}{1-r}\Big)^{\frac{1}{p'}} \big(x^{\frac{r-1}{p}} - b^{\frac{r-1}{p}}\big)^{\frac{1}{p'}} \Big(\int_x^b t^{\frac{1+p-r}{p'}} f^p(t)dt\Big)^{\frac{1}{p}}, \end{aligned}$$

which yields

$$F_1^p(x) \leq \Big(\frac{p}{1-r}\Big)^{p-1} \big(x^{\frac{r-1}{p}} - b^{\frac{r-1}{p}}\big)^{p-1} \int_x^\infty t^{\frac{1+p-r}{p'}} f^p(t)dt.$$

(b) Suppose now $r < 1$, applying (1.5), we obtain

$$\begin{aligned} F_2(x) &= \int_a^x t^{-\frac{1+p-r}{p'p}} t^{\frac{1+p-r}{p'p}} f(t)dt \\ &\geq \Big(\int_a^x t^{-\frac{1+p-r}{p}} dt\Big)^{\frac{1}{p'}} \Big(\int_a^x t^{\frac{1+p-r}{p'}} f^p(t)dt\Big)^{\frac{1}{p}} \\ &= \Big(\frac{p}{r-1}\Big)^{\frac{1}{p'}} \big(x^{\frac{r-1}{p}} - a^{\frac{r-1}{p}}\big)^{\frac{1}{p'}} \Big(\int_a^x t^{\frac{1+p-r}{p'}} f^p(t)dt\Big)^{\frac{1}{p}}, \end{aligned}$$

thus

$$F_2^p(x) \leq \Big(\frac{p}{r-1}\Big)^{p-1} \big(x^{\frac{r-1}{p}} - a^{\frac{r-1}{p}}\big)^{p-1} \int_a^x t^{\frac{1+p-r}{p'}} f^p(t)dt.$$

Now, we state and prove the first generalization on $(a,b)$.

Theorem 3

*Let $p < 0$, $r > 1$, $0 \leq a < b \leq \infty$ and $f$, $w$ be positive measurable functions on $(a,b)$, $F_1(x) = \int_x^b f(t)dt$. If $\frac{x}{w(x)}$ is non-decreasing, then*

$$\int_a^b w^{-r}(x) F_1^p(x)dx \leq \Big(\frac{p}{1-r}\Big)^p \int_a^b w^{-r}(x)\Big(1 - \Big(\frac{x}{b}\Big)^{\frac{1-r}{p}}\Big)^{p-1} (xf(x))^p dx. \quad (2.3)$$

*Proof.* Applying inequality (2.1) and the Fubini's theorem, we get

$$\begin{aligned} &\int_a^b w^{-r}(x) F_1^p(x)dx \\ &\leq \Big(\frac{p}{1-r}\Big)^{p-1} \int_a^b \int_x^b w^{-r}(x)\big(x^{\frac{r-1}{p}} - b^{\frac{r-1}{p}}\big)^{p-1} t^{\frac{1+p-r}{p'}} f^p(t)dtdx \\ &= \Big(\frac{p}{1-r}\Big)^{p-1} \int_a^b t^{\frac{1+p-r}{p'}} f^p(t)\Big(\int_a^t \frac{1}{w^r(x)} x^{\frac{r-1}{p}(p-1)}\Big(1 - \Big(\frac{b}{x}\Big)^{\frac{r-1}{p}}\Big)^{p-1} dx\Big)dt \\ &= \Big(\frac{p}{1-r}\Big)^{p-1} \int_a^b t^{\frac{1+p-r}{p'}} f^p(t)\Big(\int_a^t \Big(\frac{x}{w(x)}\Big)^r x^{\frac{1-r}{p}-1}\Big(1 - \Big(\frac{x}{b}\Big)^{\frac{1-r}{p}}\Big)^{p-1} dx\Big)dt. \end{aligned}$$

Since $\big(\frac{x}{w(x)}\big)^r$ and $\big(1-\big(\frac{x}{b}\big)^{\frac{1-r}{p}}\big)^{p-1}$ are non-decreasing functions on $(a,t)$, therefore

$$\begin{aligned} &\int_a^b w^{-r}(x) F_1^p(x)dx \\ &\leq \Big(\frac{p}{1-r}\Big)^{p-1} \int_a^b t^{\frac{1+p-r}{p'}} f^p(t)\Big(\frac{t}{w(t)}\Big)^r \Big(1 - \Big(\frac{t}{b}\Big)^{\frac{1-r}{p}}\Big)^{p-1} \Big(\int_a^t x^{\frac{1-r}{p}-1} dx\Big)dt \\ &= \Big(\frac{p}{1-r}\Big)^{p} \int_a^b t^{\frac{1+p-r}{p'}} f^p(t)\Big(\frac{t}{w(t)}\Big)^r \Big(1 - \Big(\frac{t}{b}\Big)^{\frac{1-r}{p}}\Big)^{p-1} \big(t^{\frac{1-r}{p}} - a^{\frac{1-r}{p}}\big)dt \end{aligned}$$

$$\leq \left(\frac{p}{1-r}\right)^p \int_a^b t^{\frac{1+p-r}{p'}} f^p(t) \left(\frac{t}{w(t)}\right)^r \left(1-\left(\frac{t}{b}\right)^{\frac{1-r}{p}}\right)^{p-1} t^{\frac{1-r}{p}} dt$$
$$= \left(\frac{p}{1-r}\right)^p \int_a^b w^{-r}(t)\Big(1-\left(\frac{t}{b}\right)^{\frac{1-r}{p}}\Big)^{p-1} (tf(t))^p dt.$$

Remark 1

Taking $b=+\infty$ in Theorem 3, we get

$$\int_a^{+\infty} w^{-r}(x)F_1^p(x)dx \leq \left(\frac{p}{1-r}\right)^p \int_a^{+\infty} w^{-r}(x)(xf(x))^p dx, \tag{2.4}$$

where $F_1(x)=\int_x^{+\infty} f(t)dt$.
The inequality (2.4) which is a particular case of (2.3) is an extension of inequality (1.2) and they coincide when $a=0$.

Theorem 4

*Let $p<0$, $r<1$, $0\leq a<b\leq\infty$ and $f$, $w$ be positive measurable functions on $(a,b)$, $F_2(x)=\int_a^x f(t)dt$.*

(a) *If $0\leq r<1$ and $\frac{x}{w(x)}$ is non-increasing, then*

$$\int_a^b w^{-r}(x)F_2^p(x)dx \leq \left(\frac{p}{r-1}\right)^p \int_a^b w^{-r}(x)\Big(1-\left(\frac{a}{x}\right)^{\frac{r-1}{p}}\Big)^{p-1}(xf(x))^p dx. \tag{2.5}$$

(b) *If $r<0$ and $\frac{x}{w(x)}$ is a non-decreasing function, the inequality (2.5) holds.*

*Proof.* (a). Suppose that $0\leq r<1$ and $\frac{x}{w(x)}$ is a non-increasing function. Using the inequality (2.2) of Lemma 2 and the Fubini's theorem, we obtain

$$\int_a^b w^{-r}(x)F_2^p(x)dx$$
$$\leq \left(\frac{p}{r-1}\right)^{p-1} \int_a^b \int_a^x w^{-r}(x)\big(x^{\frac{r-1}{p}}-a^{\frac{r-1}{p}}\big)^{p-1} t^{\frac{1+p-r}{p'}} f^p(t)dtdx$$
$$= \left(\frac{p}{r-1}\right)^{p-1} \int_a^b t^{\frac{1+p-r}{p'}} f^p(t)\left(\int_t^b \left(\frac{x}{w(x)}\right)^r x^{\frac{1-r}{p}-1}\Big(1-\left(\frac{a}{x}\right)^{\frac{r-1}{p}}\Big)^{p-1} dx\right)dt.$$

Since $\left(\frac{x}{w(x)}\right)^r$ and $\big(1-\left(\frac{a}{x}\right)^{\frac{r-1}{p}}\big)^{p-1}$ are non-increasing functions on $(t,b)$, it follows that

$$\begin{aligned}
&\int_a^b w^{-r}(x)F_2^p(x)dx \\
&\quad\le \Big(\frac{p}{r-1}\Big)^{p-1}\int_a^b t^{\frac{1+p-r}{p'}} f^p(t)\Big(\frac{t}{w(t)}\Big)^r\Big(1-\Big(\frac{a}{t}\Big)^{\frac{r-1}{p}}\Big)^{p-1}\Big(\int_t^b x^{\frac{1-r}{p}-1}dx\Big)dt \\
&\quad= \Big(\frac{p}{r-1}\Big)^{p}\int_a^b t^{\frac{1+p-r}{p'}} f^p(t)\Big(\frac{t}{w(t)}\Big)^r\Big(1-\Big(\frac{a}{t}\Big)^{\frac{r-1}{p}}\Big)^{p-1}\big(t^{\frac{1-r}{p}}-b^{\frac{1-r}{p}}\big)dt \\
&\quad\le \Big(\frac{p}{r-1}\Big)^{p}\int_a^b t^{\frac{1+p-r}{p'}} f^p(t)\Big(\frac{t}{w(t)}\Big)^r\Big(1-\Big(\frac{a}{t}\Big)^{\frac{r-1}{p}}\Big)^{p-1} t^{\frac{1-r}{p}}dt \\
&\quad= \Big(\frac{p}{r-1}\Big)^{p}\int_a^b w^{-r}(t)\Big(1-\Big(\frac{a}{t}\Big)^{\frac{r-1}{p}}\Big)^{p-1}(tf(t))^p dt.
\end{aligned}$$

(b). The proof is similar if $r<0$ and $\frac{x}{w(x)}$ is a non-decreasing function.

Remark 2
Setting $a=0$ in Theorem 4, we obtain

$$\int_0^b w^{-r}(x)F_2^p(x)dx \le \Big(\frac{p}{r-1}\Big)^p\int_0^b w^{-r}(x)(xf(x))^p dx, \tag{2.6}$$

where $F_2(x)=\int_0^b f(t)dt$.
The inequality (2.6) which is a special case of (2.5) is an extension of inequalities (1.3) and (1.4), additionally they coincide when $b=+\infty$.

We need the following lemma from [2] to prove the second result.

Lemma 3 (see [2])
*Let $-\infty<q\le p<0$ and $\phi$, $\psi$ be measurable positive functions on $(a,b)$ and suppose that $0<\int_a^b\phi^q(x)\psi(x)dx<\infty$, then*

$$\int_a^b \phi^p(x)\psi(x)dx \le \Big(\int_a^b\psi(x)dt\Big)^{\frac{q-p}{q}}\Big(\int_a^b\phi^q(x)\psi(x)dx\Big)^{\frac{p}{q}}. \tag{2.7}$$

*Inequality (2.7) holds for $0<p\le q<\infty$ and is reversed for $0<q\le p<\infty$.*

Let $-\infty<q\le p<0$, $r\ne1$, putting $\phi(x)=xf(x)$ and $\psi(x)=w^{-r}(x)$ in (2.7), we get

$$\int_a^b (xf(x))^p w^{-r}(x)dx \le \Big(\int_a^b w^{-r}(x)dt\Big)^{\frac{q-p}{q}}\Big(\int_a^b (xf(x))^q w^{-r}(x)dx\Big)^{\frac{p}{q}}. \tag{2.8}$$

The next result includes the second generalization with two negative parameters of summation.

Corollary 1
*Let $-\infty<q\le p<0$, $r\ne1$ and $f$, $w$ be positive measurable functions on $(0,+\infty)$, $F_1(x)=\int_x^{+\infty}f(t)dt$, $F_2(x)=\int_0^x f(t)dt$.*

(a) *If* $r > 1$ *and* $\frac{x}{w(x)}$ *is non-decreasing, then*

$$\int_0^{+\infty} F_1^p(x)w^{-r}(x)dx \le \Big(\frac{p}{1-r}\Big)^p \Big(\int_0^{+\infty} w^{-r}(x)dx\Big)^{1-\frac{p}{q}} \times \Big(\int_0^{+\infty} (xf(x))^q w^{-r}(x)dx\Big)^{\frac{p}{q}}. \tag{2.9}$$

(b) *If* $0 \le r < 1$ *and* $\frac{x}{w(x)}$ *is non-increasing, then*

$$\int_0^{+\infty} F_2^p(x)w^{-r}(x)dx \le \Big(\frac{p}{r-1}\Big)^p \Big(\int_0^{+\infty} w^{-r}(x)dx\Big)^{1-\frac{p}{q}} \times \Big(\int_0^{+\infty} (xf(x))^q w^{-r}(x)dx\Big)^{\frac{p}{q}}. \tag{2.10}$$

(c) *If* $r < 0$ *and* $\frac{x}{w(x)}$ *is a non-decreasing function, inequality* (2.10) *holds.*

*Proof.* The proof of (2.9) and (2.10) follows from Theorem 3 and from Theorem 4 by setting $a = 0$, $b = +\infty$ and applying (2.8).

Remark 3
Taking $p = q$ in the Corollary 1, we obtain inequalities (1.2), (1.3) and (1.4).

### 2.1. Case of monotone functions

We present here some estimates and two-sided estimates for $\int_a^b F_2^p(x)w^{-r}(x)dx$ and $\int_a^b F_1^p(x)w^{-r}(x)dx$ in the case of monotonic functions, for this we need the next Lemma.

Lemma 4
*Let* $p < 0$, $q > 0$, $0 \le a \le b \le \infty$, $\phi$ , $\psi$ *be positive measurable functions on* $(a, b)$, *then*

$$\int_a^b \phi(x)\psi^p(x)dx \ge \Big(\int_a^b \psi^q(x)\phi(x)dx\Big)^{\frac{p}{q}} \Big(\int_a^b \phi(x)dx\Big)^{1-\frac{p}{q}}, \tag{2.11}$$

$$\int_a^b \phi(x)\psi^q(x)dx \ge \Big(\int_a^b \psi^p(x)\phi(x)dx\Big)^{\frac{q}{p}} \Big(\int_a^b \phi(x)dx\Big)^{1-\frac{q}{p}}. \tag{2.12}$$

*Proof.* By applying the Hölder inequality (1.5) with exponent $\frac{q}{p} < 0$ and its conjugate $(\frac{q}{p})' = \frac{q}{q-p}$, we have

$$\begin{aligned}\int_a^b \phi(x)\psi^p(x)dx &= \int_a^b \psi^p(x)(\phi(x))^{p/q}(\phi(x))^{1-p/q}dx \\ &\ge \Big(\int_a^b \psi^q(x)\phi(x)dx\Big)^{\frac{p}{q}} \Big(\int_a^b \phi(x)dx\Big)^{1-\frac{p}{q}}.\end{aligned}$$

By choosing the exponent $\frac{p}{q} < 0$ instead of $\frac{q}{p} < 0$, inequality (2.12) is proved similarly.

Now we state and prove the following result.

THEOREM 5
*Let $p < 0$, $q > 0$, $0 \leq a < b < \infty$, $r \neq 1$ and let $f$, $w$ be positive measurable functions on $(a,b)$. If moreover $F_1(x) = \int_x^b f(t)dt$ and $F_2(x) = \int_a^x f(t)dt$, then*

(a) *If $f$ is non-decreasing then*

$$\left(\int_a^b F_1^q(x)w^{-r}(x)dx\right)^{\frac{1}{q}} \geq \left(\int_a^b w^{-r}(x)dx\right)^{\frac{1}{q}-\frac{1}{p}} \left(\int_a^b ((b-x)f(x))^p w^{-r}(x)dx\right)^{\frac{1}{p}}. \tag{2.13}$$

$$\left(\int_a^b F_2^p(x)w^{-r}(x)dx\right)^{\frac{1}{p}} \leq \left(\int_a^b w^{-r}(x)dx\right)^{\frac{1}{p}-\frac{1}{q}} \left(\int_a^b ((x-a)f(x))^q w^{-r}(x)dx\right)^{\frac{1}{q}}. \tag{2.14}$$

(b) *If $f$ is non-increasing then*

$$\left(\int_a^b F_1^p(x)w^{-r}(x)dx\right)^{\frac{1}{p}} \leq \left(\int_a^b w^{-r}(x)dx\right)^{\frac{1}{p}-\frac{1}{q}} \left(\int_a^b ((b-x)f(x))^q(x) w^{-r}(x)dx\right)^{\frac{1}{q}}. \tag{2.15}$$

$$\left(\int_a^b F_2^q(x)w^{-r}(x)dx\right)^{\frac{1}{q}} \geq \left(\int_a^b w^{-r}(x)dx\right)^{\frac{1}{q}-\frac{1}{p}} \left(\int_a^b ((x-a)f(x))^p \ w^{-r}(x)dx\right)^{\frac{1}{p}}. \tag{2.16}$$

*Proof.* (a). Let $f$ be a positive non-decreasing function,

$$\int_x^b f(t)dt \geq f(x)\int_x^b dt = \big(b-x\big)f(x),$$

then

$$\int_a^b F_1^q(x)w^{-r}(x)dx \geq \int_a^b ((b-x)f(x))^q w^{-r}(x)dx.$$

Applying (2.12) on the right hand side of the above inequality, we get

$$\begin{aligned}\int_a^b & F_1^q(x)w^{-r}(x)dx \\ & \geq \left(\int_a^b ((b-x)f(x))^p(x)w^{-r}(x)dx\right)^{\frac{q}{p}}\left(\int_a^b w^{-r}(x)dx\right)^{1-\frac{q}{p}},\end{aligned} \tag{2.17}$$

then, we get (2.13). The proof of (2.16) is similar to that of (2.13).
(b). Let be $f$ a positive non-decreasing function,

$$F_2(x) = \int_a^x f(t)dt \leq (x-a)f(x),$$

since $p < 0$, we obtain

$$\int_a^b F_2^p(x)w^{-r}(x)dx \geq \int_a^b ((x-a)f(x))^p w^{-r}(x)dx.$$

Applying (2.11) on the right hand side of the above inequality, we deduce

$$\begin{aligned}\int_a^b & F_2^p(x)w^{-r}(x)dx \\ & \geq \left(\int_a^b ((x-a)\,f(x))^q\; w^{-r}(x)dx\right)^{\frac{p}{q}}\left(\int_a^b w^{-r}(x)dx\right)^{1-\frac{p}{q}},\end{aligned} \tag{2.18}$$

thus, the desired inequality (2.14) is obtained. The proof (2.15) is similar to that of (2.14).

Now we present some two sided integral inequalities.

Corollary 2
*Let $p < 0$, $q > 0$, $0 \leq a < b < \infty$, $r < 1$ and $f$ be a positive non-decreasing and $w$ a positive measurable functions on $(a,b)$, $F_2(x) = \int_a^x f(t)dt$.*

(a) *If $0 \leq r < 1$ and $\frac{x}{w(x)}$ is non-increasing, then*

$$\begin{aligned}\left(\int_a^b w^{-r}(x)dx\right)^{1-\frac{p}{q}} & \left(\int_a^b ((x-a)f(x))^q w^{-r}(x)dx\right)^{\frac{p}{q}} \\ & \leq \int_a^b F_2^p(x)w^{-r}(x)dx \\ \leq \left(\frac{p}{r-1}\right)^p & \int_a^b w^{-r}(x)\left(1-\left(\frac{a}{x}\right)^{\frac{r-1}{p}}\right)^{p-1}(xf^p(x))^p dx.\end{aligned} \tag{2.19}$$

(b) *If $r < 0$ and $\frac{x}{w(x)}$ is non-decreasing then* (2.19) *holds.*

*Proof.* Inequality (2.19) is a direct consequence of (2.18) and (2.5).

Combining the inequalities (2.17) and (2.3) we get the following corollary.

Corollary 3

*Let $p < 0$, $q > 0$, $0 \leq a < b < \infty$, $r > 1$, $f$ be a positive non-increasing and let $w$ be a positive measurable functions on $(a, b)$, $F_1(x) = \int_x^b f(t)dt$. If $\frac{x}{w(x)}$ is non-decreasing, then*

$$\left(\int_a^b ((b-x)f(x))^q(x)w^{-r}(x)dx\right)^{\frac{p}{q}}\left(\int_a^b w^{-r}(x)dx\right)^{1-\frac{p}{q}}$$

$$\leq \int_a^b F_1^p(x)w^{-r}(x)dx$$

$$\leq \left(\frac{p}{1-r}\right)^p \int_a^b w^{-r}(x)\Big(1-\left(\frac{x}{b}\right)^{\frac{1-r}{p}}\Big)^{p-1}(xf^p(x))^r dx.$$

## References


[1] Benaissa, Bouharket. "On the Reverse Minkowski's Integral Inequality." *Kragujevac J. Math.* 46, no. 3 (2022): 407-416. DOI: 10.46793/kgjmat2203.407b. Cited on 100.

[2] Benaissa, Bouharket. "A further generalization of the reverse Minkowski type inequality via Hölder and Jensen inequalities." *J. Siberian Fed. Univ. Math. Phys.* 15, no. 3 (2022): 319-328. DOI:10.17516/1997-1397-2022-15-3-319-328. Cited on 104.

[3] Benaissa, Bouharket, and Mehmet Zeki Sarikaya. "Generalisation of some Hardy-type integral inequality with negative parameter." *Bulletin of the Transilvania University of Brasov* 13 (2020): 69-76. DOI: 10.31926/but.mif.2020.13.62.1.6. Cited on 100.

[4] Bicheng, Yang. "On a new Hardy integral inequalities." *Int. Math. Forum* 2, no. 65-68 (2007): 3317-3322. Cited on 100.

[5] Hardy, Godfrey H., John E. Littlewood, and George Pólya. *Inequalities.* Cambridge: Cambridge University Press, 1952. Cited on 99.

[6] Kuang, Ji Chang. *Applied Inequalities.* Changsha: Human Education Press, 1993. Cited on 100.



*Noureddine Azzouz*
*Faculty of Sciences*
*University Center Nour Bachir El Bayadh*
*Laboratory of Informatics and Mathematics*
*University of Tiaret*
*Algeria*
*E-mail:* azzouz_n@live.fr

*Bouharket Benaissa*
*Faculty of Material Sciences*
*Laboratory of Informatics and Mathematics*
*University of Tiaret*
*Algeria*
*E-mail:* bouharket.benaissa@univ-tiaret.dz

*Abdelkader Senouci*
*Faculty of Mathematics and Informatics*
*Laboratory of Informatics and Mathematics*
*University of Tiaret*
*Algeria*
*E-mail:* kamer295@yahoo.fr

*Received: March 5, 2023; final version: May 16, 2023;*
*available online: July 21, 2023.*