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q-Cesàro double sequence space $\tilde{\mathcal{L}}_s^q$ derived by q-analog

Abstract. This study includes the new Banach space $\tilde{\mathcal{L}}_s^q$ designed as the domain in $\mathcal{L}_s$ space of the 4d (4-dimensional) q-Cesàro matrix obtained as the q-analog of the well-known 4d Cesàro matrix. After showing the completeness of the aforementioned space, giving some inclusion relations, determining the fundamental set of this space and calculating the duals, finally, some matrix transformations related to the new space were characterized.

1. Introduction

Although the first q-analog studies date back to the 19th century, some mathematicians have been intensely interested in the q-analogs of the some of the known results. As in many other subjects, q-analogs have a widespread application especially in combinatorics, special functions and recently in sequence spaces. The q-analog of any number or expression is more general then these that includes a parameter q and the original number or expression is obtained when the limit is taken for $q = 1$.

All positive real numbers' set is demonsrated with $\mathbb{R}^+$. The q-analog of a $z \in \mathbb{R}^+ \cup \{0\}$ is described as

$$[z]_q = \begin{cases} \dfrac{1-q^z}{1-q}, & q \in \mathbb{R}^+ - \{1\}, \\ z, & q = 1. \end{cases}$$

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A double sequence is a function described as $F\colon \mathbb{N}\times\mathbb{N}\to\zeta$, $(t,k)\mapsto F(t,k)=u_{tk}$, where ζ represents any non-empty set and $\mathbb{N}$ is the natural numbers' set containing zero. The notion of convergence of double sequences of real numbers was presented by Pringsheim [31] at the beginning of the 20th century. A few years later, the concept of regular convergence, which requires convergence for each index separately in addition to convergence in the Pringsheim's sense, was also developed by Hardy [23]. Later, Zeltser [40] has fundamentally presented results regarding the topological structure of double sequences. All convergent in the Pringsheim's sense ($\mathcal{P}$-convergent), regular convergent, s-absolutely summable and bounded double sequences' spaces are represented by $\mathcal{C}_{\mathcal{P}}$, $\mathcal{C}_r$, $\mathcal{L}_s$ and $\mathcal{M}_u$, respectively. It is worth mentioning that, $\mathcal{P}$-convergent any double sequence can be bounded or unbounded. All double sequences' space, both $\mathcal{P}$-convergent and bounded, is represented by $\mathcal{C}_{b\mathcal{P}}$. When $s=1$ is selected, the $\mathcal{L}_u$ space introduced by Zeltser [41] is obtained from the $\mathcal{L}_s$ space described by Başar and Sever [4]. The vector space of all real valued double sequences represented by Ω.

In the rest of the study, it will be preferred to write the sum $\sum_{t,k}$ in place of $\sum_{t=0}^{\infty}\sum_{k=0}^{\infty}$. For $\vartheta\in\{\mathcal{P},b\mathcal{P},r\}$, if any $u=(u_{tk})\in\Omega$ is ϑ-convergent to a limit point L, then it is written as $\vartheta-\lim_{t,k\to\infty}u_{tk}=L$. All null double sequences' space is represented by $\mathcal{C}_{\vartheta 0}$. Zeltser [41] described the double sequences $e^{zn}=(e^{zn}_{tk})$ by

$$e^{zn}_{tk} := \begin{cases} 1, & \text{if } (z,n)=(t,k),\\ 0, & \text{elsewhere}\end{cases}$$

and e by $e=\sum_{z,n}e^{zn}$. For any 4d matrix $B=(b_{zntk})$, if $b_{zntk}=0$ for $t>z$ or $k>n$ or both for all $z,n,t,k\in\mathbb{N}$, it is said that $B=(b_{zntk})$ is a *triangular matrix* and also if $b_{znzn}\neq 0$ for all $z,n\in\mathbb{N}$, in that case the 4d triangular matrix B is entitled as *triangle*.

Let us envision that double sequence spaces Ψ and Λ, a sequence $u=(u_{tk})\in\Psi$ and the 4d matrix $B=(b_{zntk})$. For all $u=(u_{tk})$, if $(Bu)_{zn}=\vartheta-\sum_{t,k}b_{zntk}u_{tk}$ (the B-transform of u) is in Λ, then B is a matrix mapping from Ψ into Λ and we denote this situation by $B\colon\Psi\to\Lambda$. Also, $B\in(\Psi:\Lambda)$ if and only if $B_{zn}\in\Psi^{\beta(\vartheta)}$ and $Bu\in\Lambda$, where $B_{zn}=(b_{zntk})_{t,k\in\mathbb{N}}$ and $(\Psi:\Lambda)=\{B=(b_{zntk}):\ B\colon\Psi\to\Lambda\}$ for every $z,n\in\mathbb{N}$.

The set $\Psi_B^{(\vartheta)}$ described as

$$\Psi_B^{(\vartheta)} := \left\{u=(u_{tk})\in\Omega : Bu := \left(\vartheta{-}\sum_{t,k}b_{zntk}u_{tk}\right)_{z,n\in\mathbb{N}} \text{exists, } Bu\in\Psi\subset\Omega\right\} \quad (1)$$

represents the ϑ-domain of the 4d matrix B.

The 4d matrix that transforms every double sequence both bounded and $\mathcal{P}$-convergent into a $\mathcal{P}$-convergent double sequence without changing the limit is named as the RH-regular matrix [22, 32].

Both single and double sequences, their spaces and matrix domains have been seen as interesting topics in mathematics by the authors, and in recent years, many studies have been done in this area. Altay and Başar [3] have been described and studies some double series spaces, including $\mathcal{BS}$ and $\mathcal{CS}_{\vartheta}$ spaces, whose sequences of partial sums are in the spaces $\mathcal{M}_u$ and $\mathcal{C}_\vartheta$, respectively. After that, some authors such as Çapan and Başar [12, 13], Demiriz and Duyar [15, 16], Demiriz and Erdem [18, 19], Erdem and Demiriz [20, 21], Mursaleen and Başar [29], Tuğ [33, 34, 35] and Yeşilkayagil and Başar [38, 39] described the spaces obtained as the domains of some special 4d matrices and elaborated on some inclusion relations, duals and matrix transformations related these spaces. On the other hand, Aktuğlu and Bekar [2] and Bekar [6] introduced a q-analog of the well-known 2d Cesàro matrix and they examined some properties of this matrix. Furthermore, Demiriz and Şahin [17] and Yaying et al. [36] obtained new spaces as the domains of q-analog of the 2d Cesàro matrix and Yaying et al. [37] obtained new spaces as the domains of (p, q)-analog of the 2d Binomial matrix and scrutinized the aforementioned spaces. Researchers who want to get more detailed information about summability theory, 2d and 4d matrices, single and double sequence spaces, matrix domains and other related subjects can benefit from the studies [1, 5, 8, 9, 10, 11, 24, 25, 26, 27, 28, 30, 42].

In parallel with the studies mentioned above, in the current study, it is obtained the new double sequence space as the domain of the 4d q-Cesàro matrix described by Çinar and Et [14] in the space $\mathcal{L}_s$, which is the q-analog of the ordinary 4d Cesàro matrix for $0 < s < \infty$. After giving some results about the newly defined space, we expressed the some duals of the space and lastly completed the article with some matrix transformations and corollaries.

2. q-Cesàro double sequence space $\tilde{\mathcal{L}}_s^q$

In the current part, it is introduced the space $\tilde{\mathcal{L}}_s^q \in \Omega$ for $0 < s < \infty$ and is shown that this space is complete and linearly isomorphic to $\mathcal{L}_s$. Then, to determine the location of the newly defined space among the other spaces, it is given inclusion relations and finally, is calculated the fundamental set of $\tilde{\mathcal{L}}_s^q$.

The ordinary first order 4d Cesàro matrix $C = (c_{zntk})$ presented by

$$c_{zntk} := \begin{cases} \dfrac{1}{(z+1)(n+1)}, & 0 \le t \le z,\ 0 \le k \le n, \\ 0, & \text{otherwise} \end{cases}$$

for all $z, n, t, k \in \mathbb{N}$.

Recently, the 4d q-Cesàro matrix $C_{(1,1)}(q) = (c_{zntk}(q))$ which is the q-analog of the first order ordinary 4d Cesàro matrix was given by Çinar and Et [14] as

$$c_{zntk}(q) := \begin{cases} \dfrac{q^{t+k}}{[z+1]_q [n+1]_q}, & 0 \le t \le z,\ 0 \le k \le n, \\ 0, & \text{otherwise} \end{cases}$$

for all $z, n, t, k \in \mathbb{N}$ and we also know from the aforementioned study that the 4d q-Cesàro matrix is RH-regular for $q \ge 1$. The inverse of the 4d q-Cesàro matrix is given by

$$c^{-1}_{zntk}(q) := \begin{cases} (-1)^{z+n-(t+k)} \dfrac{[t+1]_q[k+1]_q}{q^{z+n}}, & z-1 \leq t \leq z,\ n-1 \leq k \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

From its definition, we can understand that $C_{(1,1)}(q)$ is a triangle. Furthermore, $C_{(1,1)}(q)$-transform of a double sequence $u = (u_{tk})$ is stated as

$$\nu_{zn} := (C_{(1,1)}(q)u)_{zn} = \frac{1}{[z+1]_q[n+1]_q} \sum_{t,k=0}^{z,n} q^{t+k} u_{tk}, \qquad z, n \in \mathbb{N}. \tag{2}$$

To draw attention to an important point, for $q = 1$, $C_{(1,1)}(q)$ is turned into ordinary 4d Cesàro matrix C. So, the matrix $C_{(1,1)}(q)$ generalizes the 4d matrix C.

In the current paper, we introduce the set $\tilde{\mathcal{L}}^q_s$ of all q-Cesàro absolutely s-summable double sequences by

$$\tilde{\mathcal{L}}^q_s = \left\{ u = (u_{tk}) \in \Omega : \sum_{z,n} \left| \frac{1}{[z+1]_q[n+1]_q} \sum_{t,k=0}^{z,n} q^{t+k} u_{tk} \right|^s < \infty \right\}, \quad 0 < s < \infty.$$

In that case, the set $\tilde{\mathcal{L}}^q_s$ can be rewritten as $\tilde{\mathcal{L}}^q_s = (\mathcal{L}_s)_{C_{(1,1)}(q)}$ with the notation (1). Let Ψ is a normed double sequence space. In that case, the matrix domain $\Psi_{C_{(1,1)}(q)}$ is called as the q-Cesàro double sequence space.

It should be noted that for the case $q = 1$, $\tilde{\mathcal{L}}^q_s$ turns into the Cesàro double sequence space $\tilde{\mathcal{L}}_s$ which is described and examined by Mursaleen and Başar [29].

In the rest of the study, all terms with a negative index will be considered zero and it will be assumed that $q > 1$. Now, we are ready to give the theorem about the completeness of the set we just defined.

Theorem 1
The following statements hold:

(i) *For the case $0 < s < 1$, the set $\tilde{\mathcal{L}}^q_s$ is a complete s-normed space with*

$$\|u\|^{\wr}_{\tilde{\mathcal{L}}^q_s} = \|C_{(1,1)}(q)u\|^{\wr}_{\mathcal{L}_s} = \sum_{z,n} \left| \frac{1}{[z+1]_q[n+1]_q} \sum_{t,k=0}^{z,n} q^{t+k} u_{tk} \right|^s.$$

(ii) *For the case $1 \leq s < \infty$, the set $\tilde{\mathcal{L}}^q_s$ is a Banach space with*

$$\|u\|_{\tilde{\mathcal{L}}^q_s} = \|C_{(1,1)}(q)u\|_{\mathcal{L}_s} = \left(\sum_{z,n} \left| \frac{1}{[z+1]_q[n+1]_q} \sum_{t,k=0}^{z,n} q^{t+k} u_{tk} \right|^s \right)^{\frac{1}{s}}.$$

Proof. Since the proofs of both parts are similar, the theorem will only be proved for the second part.

We can immediately see that $\tilde{\mathcal{L}}^q_s$ is a vector space and the function $\|.\|_{\tilde{\mathcal{L}}^q_s}$ is a norm on the space $\tilde{\mathcal{L}}^q_s$ for $1 \leq s < \infty$, so we omit these.

For the aim of $\tilde{\mathcal{L}}_s^q$ is a Banach space, let us take a Cauchy sequence $u^{(m)} = \big(u_{tk}^{(m)}\big) \in \tilde{\mathcal{L}}_s^q$ for a fixed $m \in \mathbb{N}$. Then, for all $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that

$$\begin{aligned} \|u^{(m)} - u^{(l)}\|_{\tilde{\mathcal{L}}_s^q} &= \left(\sum_{z,n} \left| \frac{1}{[z+1]_q[n+1]_q} \sum_{t,k=0}^{z,n} q^{t+k}\big(u_{tk}^{(m)} - u_{tk}^{(l)}\big) \right|^s \right)^{\frac{1}{s}} \\ &= \left(\sum_{z,n} \left| \big(C_{(1,1)}(q)u^{(m)}\big)_{zn} - \big(C_{(1,1)}(q)u^{(l)}\big)_{zn} \right|^s \right)^{\frac{1}{s}} < \varepsilon \end{aligned} \tag{3}$$

for all $m, l > N$. This says us that $\big\{\big(C_{(1,1)}(q)u^{(m)}\big)_{zn}\big\}_{m\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{L}_s$. Since, the space $\mathcal{L}_s$ is a Banach space for $1 \le s < \infty$, the sequence $\big\{\big(C_{(1,1)}(q)u^{(m)}\big)_{zn}\big\}_{m\in\mathbb{N}}$ converges, that is

$$\big\{\big(C_{(1,1)}(q)u^{(m)}\big)_{zn}\big\}_{m\in\mathbb{N}} \to (C_{(1,1)}(q)u)_{zn}, \qquad m \to \infty.$$

Then, by using these infinitely limit points, it can be described the sequence $(C_{(1,1)}(q)u)_{zn}$.

Now, we must show the relation $(C_{(1,1)}(q)u)_{zn} \in \mathcal{L}_s$ in the rest of the proof. Since $\big\{\big(C_{(1,1)}(q)u^{(m)}\big)_{zn}\big\}_{m\in\mathbb{N}} \in \mathcal{L}_s$, then we can write the inequality

$$\big(\sum_{z,n} \big|\big(C_{(1,1)}(q)u^{(m)}\big)_{zn}\big|^s\big)^{\frac{1}{s}} < \infty.$$

Thus, it is easy to reach the fact $(C_{(1,1)}(q)u)_{zn} \in \mathcal{L}_s$ from the following inequality by applying limit on (3) for $l \to \infty$,

$$\begin{aligned} \|(C_{(1,1)}(q)u)_{zn}\|_{\mathcal{L}_s} &= \left(\sum_{z,n} |(C_{(1,1)}(q)u)_{zn}|^s \right)^{\frac{1}{s}} \\ &\le \left(\sum_{z,n} \big|(C_{(1,1)}(q)u^{(m)})_{zn} - (C_{(1,1)}(q)u)_{zn}\big|^s \right)^{\frac{1}{s}} \\ &\quad + \left(\sum_{z,n} \big|(C_{(1,1)}(q)u^{(m)})_{zn}\big|^s \right)^{\frac{1}{s}} < \infty. \end{aligned}$$

Consequently, $u \in \tilde{\mathcal{L}}_s^q$ and $\tilde{\mathcal{L}}_s^q$ is complete with $\|.\|_{\tilde{\mathcal{L}}_s^q}$ for $1 \le s < \infty$.

Theorem 2
The spaces $\tilde{\mathcal{L}}_s^q$ and $\mathcal{L}_s$ are linearly norm isomorphic for $1 \le s < \infty$.

Proof. For the proof, it must be shown that there is a norm-preserving bijection between the aforementioned spaces. The linearity of the function described for this purpose as $L\colon \tilde{\mathcal{L}}_s^q \to \mathcal{L}_s$, $L(u) = C_{(1,1)}(q)u$ can be seen immediately. Besides this, from the proposition $L(u) = 0 \Rightarrow u = 0$, L is decided to be an injection.

By taking into account the sequences $\nu = (\nu_{tk}) \in \mathcal{L}_s$ and $u = (u_{tk})$ whose terms are

$$u_{zn} = \frac{1}{q^{z+n}} \sum_{t=z-1}^{z} \sum_{k=n-1}^{n} (-1)^{z+n-(t+k)} [t+1]_q [k+1]_q \nu_{tk}, \qquad z, n \in \mathbb{N},$$

we reach the surjectivity of L from the following expression

$$\begin{aligned} \|u\|_{\tilde{\mathcal{L}}_s^q} &= \left(\sum_{z,n} \left| \frac{1}{[z+1]_q [n+1]_q} \sum_{t,k=0}^{z,n} q^{t+k} u_{tk} \right|^s \right)^{\frac{1}{s}} \\ &= \left(\sum_{z,n} \left| \frac{1}{[z+1]_q [n+1]_q} \sum_{t,k=0}^{z,n} q^{t+k} \right.\right. \\ &\quad \left.\left. \times \sum_{t=z-1}^{z} \sum_{k=n-1}^{n} \frac{1}{q^{t+k}} (-1)^{t+k-(i+j)} [i+1]_q [j+1]_q \nu_{ij} \right|^s \right)^{\frac{1}{s}} \\ &= \left(\sum_{z,n} |\nu_{zn}|^s \right)^{\frac{1}{s}} = \|\nu\|_{\mathcal{L}_s} < \infty. \end{aligned}$$

Since the relation $\|u\|_{\tilde{\mathcal{L}}_s^q} = \|\nu\|_{\mathcal{L}_s}$ holds for $1 \leq s < \infty$, then L keeps the norm.

Now, we may present following findings regarding inclusion relations.

Theorem 3
The inclusion $\mathcal{L}_s \subset \tilde{\mathcal{L}}_s^q$ is valid for $1 \leq s < \infty$.

Proof. Let us take a sequence $u = (u_{tk}) \in \mathcal{L}_s$ for $1 < s < \infty$. Then, from the Holder's inequality and the relation (2), it is achieved that

$$\begin{aligned} |\nu_{zn}|^s &= \left| \frac{1}{[z+1]_q [n+1]_q} \sum_{t,k=0}^{z,n} q^{t+k} u_{tk} \right|^s \\ &\leq \left(\frac{1}{[z+1]_q [n+1]_q} \sum_{t,k=0}^{z,n} q^{t+k} |u_{tk}|^s \right) \\ &\quad \times \left(\frac{1}{[z+1]_q [n+1]_q} \sum_{t,k=0}^{z,n} q^{t+k} \right)^{s-1} \\ &= \frac{1}{[z+1]_q [n+1]_q} \sum_{t,k=0}^{z,n} q^{t+k} |u_{tk}|^s. \end{aligned} \tag{4}$$

By taking sum over the $z, n \in \mathbb{N}$ on the inequality (4), it is seen that

$$\sum_{z,n} |\nu_{zn}|^s \leq \sum_{z,n} \left(\frac{1}{[z+1]_q [n+1]_q} \sum_{t,k=0}^{z,n} q^{t+k} |u_{tk}|^s \right)$$

$$= \sum_{t,k=0}^{\infty,\infty} |u_{tk}|^s \left(\sum_{z=t}^{\infty} \sum_{n=k}^{\infty} \frac{q^{t+k}}{[z+1]_q [n+1]_q} \right).$$

Thus, it is seen that $\|u\|^s_{\tilde{\mathcal{L}}_s^q} \leq M \|u\|^s_{\mathcal{L}_s} < \infty$, where

$$M = \sup_{t,k\in\mathbb{N}} \left(\sum_{z=t}^{\infty} \sum_{n=k}^{\infty} \frac{q^{t+k}}{[z+1]_q [n+1]_q} \right).$$

This implies that $u \in \tilde{\mathcal{L}}_s^q$, that is $\mathcal{L}_s \subset \tilde{\mathcal{L}}_s^q$ which is desired result. The case $s = 1$ can also be represented in the same way.

Theorem 4
The inclusion $\tilde{\mathcal{L}}_s \subset \tilde{\mathcal{L}}_s^q$ holds. Moreover, if $s = 1$, then the inclusion is strict.

Proof. Since, the space $\tilde{\mathcal{L}}_s^q$ reduced to the space $\tilde{\mathcal{L}}_s$ whenever q approaches 1, the inclusion part is clear. For the case $s = 1$, by the aid of the sequence e^{00}, we obtain $(C_{(1,1)}(q)e^{00})_{zn} = \left(\frac{1}{[z+1]_q[n+1]_q}\right)_{zn} \in \mathcal{L}_u$, that is $e^{00} \in \tilde{\mathcal{L}_u}^q$ but $(Ce^{00})_{zn} = \left(\frac{1}{(z+1)(n+1)}\right)_{zn} \notin \mathcal{L}_u$, that is $e^{00} \notin \tilde{\mathcal{L}}_u$. These say us that the inclusion $\tilde{\mathcal{L}}_s \subset \tilde{\mathcal{L}}_s^q$ strict for $s = 1$, as claimed.

Theorem 5
The inclusion $\tilde{\mathcal{L}}_s^q \subset \tilde{\mathcal{L}}_{s_1}^q$ is strict for $1 \leq s < s_1 < \infty$.

Proof. Consider the sequence $u = (u_{tk}) \in \tilde{\mathcal{L}}_s^q$ such that $C_{(1,1)}(q)u \in \mathcal{L}_s$. We know from the Başar and Sever [4] that if $1 \leq s < s_1$, then $\mathcal{L}_s \subset \mathcal{L}_{s_1}$. Therefore, $C_{(1,1)}(q)u \in \mathcal{L}_{s_1}$, that is $u = (u_{tk}) \in \tilde{\mathcal{L}}_{s_1}^q$. Thus, inclusion part holds.

For the strictness part, let us take a sequence $\tilde{\nu} \in \mathcal{L}_{s_1} \setminus \mathcal{L}_s$ and consider the sequence $\tilde{u} = (\tilde{u}_{zn})$ described as

$$\begin{aligned} \tilde{u}_{zn} = & \frac{[z+1]_q[n+1]_q\tilde{\nu}_{z,n} - [z]_q[n+1]_q\tilde{\nu}_{z-1,n}}{q^{z+n}} \\ & + \frac{-[z+1]_q[n]_q\tilde{\nu}_{z,z-1} + [z]_q[n]_q\tilde{\nu}_{z-1,n-1}}{q^{z+n}}. \end{aligned}$$

Then, it is concluded that $C_{(1,1)}(q)\tilde{u} = \tilde{\nu}$ and thus $\tilde{u} \in \tilde{\mathcal{L}}_{s_1}^q \setminus \tilde{\mathcal{L}}_s^q$. Consequently, the inclusion is strict.

"A non-empty subset X of a locally convex space Ψ is called fundamental set if the closure of the linear span of X equals Ψ [7, Boss]." In [38], Yeşilkayagil and Başar have been proved that the set $S = \{e^{tk} : \ t,k \in \mathbb{N}\}$ is the fundamental set of $\mathcal{L}_s$ for $0 < s < \infty$.

Using this fact, we describe the double sequence $d^{tk} = (d^{tk}_{zn})$ by the following way:

$$d^{tk}_{zn} := \begin{cases} \dfrac{[t+1]_q[k+1]_q}{q^{t+k}}, & z=t,\ n=k, \\ -\dfrac{[t+1]_q[k+1]_q}{q^{t+k+1}}, & z=t,\ n=k+1, \\ -\dfrac{[t+1]_q[k+1]_q}{q^{t+k+1}}, & z=t+1,\ n=k, \\ \dfrac{[t+1]_q[k+1]_q}{q^{t+k+2}}, & z=t+1,\ n=k+1, \\ 0, & \text{otherwise} \end{cases}$$

for all $z, n, t, k \in \mathbb{N}$. In that case, $\{d^{tk}:\ t, k \in \mathbb{N}\}$ is the fundamental set of $\tilde{\mathcal{L}}^q_s$ for $0 < s < \infty$ because $C_{(1,1)}(q)d^{tk} = e^{tk}$.

3. Dual Spaces

Now, it is calculated the α-, $\beta(\vartheta)$- and γ-duals of the space $\tilde{\mathcal{L}}^q_s$. If Ψ and Λ are two double sequence spaces, then the set $D(\Psi : \Lambda)$ is described as

$$D(\Psi : \Lambda) = \left\{x = (x_{zn}) \in \Omega :\ xu = (x_{zn}u_{zn}) \in \Lambda \text{ for all } (u_{zn}) \in \Psi\right\}.$$

In that case, α-, $\beta(\vartheta)$- and γ-duals of the space Ψ are described as

$$\Psi^{\alpha} = D(\Psi : \mathcal{L}_u), \quad \Psi^{\beta(\vartheta)} = D(\Psi : \mathcal{CS}_{\vartheta}) \quad \text{and} \quad \Psi^{\gamma} = D(\Psi : \mathcal{BS}).$$

Now, we may give the following conditions and a table collected from the studies [12, 13, 38] to characterize some 4d matrix classes:

$$\sup_{z,n,t,k\in\mathbb{N}} |b_{zntk}| < \infty, \tag{5}$$

$$\sup_{z,n\in\mathbb{N}} \sum_{t,k} |b_{zntk}|^{s'} < \infty, \qquad \frac{1}{s} + \frac{1}{s'} = 1, \tag{6}$$

$$\sup_{t,k\in\mathbb{N}} \sum_{z,n} |b_{zntk}|^{s_1} < \infty, \tag{7}$$

$$\exists\ (b_{tk}) \in \Omega \ni \vartheta - \lim_{z,n\to\infty} b_{zntk} = b_{tk}, \tag{8}$$

$$B^{tk} = (b_{zntk})_{z,n\in\mathbb{N}} \in \mathcal{C}_{\vartheta 0}. \tag{9}$$

The symbol "$\bullet$" represents unknown conditions for $(\Psi : \Lambda)$.

Table 1: Characterizations of $(\mathcal{L}_s : \Lambda)$, where $\Lambda \in \{\mathcal{M}_u, \mathcal{C}_\vartheta, \mathcal{C}_{\vartheta 0}, \mathcal{L}_{s_1}\}$.

$(\Psi \downarrow : \Lambda \to)$	$\mathcal{M}_u$	$\mathcal{C}_\vartheta$	$\mathcal{C}_{\vartheta 0}$	$\mathcal{L}_{s_1}(0 < s_1 < \infty)$
$\mathcal{L}_s(0 < s \leq 1)$	(5)	(5),(8)	(5),(9)	(7)
$\mathcal{L}_s(1 < s < \infty)$	(6)	(6),(8)	(6),(9)	•

Theorem 6
Consider the set $\varpi_1(q)$ described as

$$\varpi_1(q) = \left\{ x = (x_{zn}) \in \Omega : \sup_{t,k \in \mathbb{N}} \sum_{z,n=0}^{\infty} g_{zntk} \right\}$$

where the 4d matrix $G = (g_{zntk})$ described by

$$g_{zntk} := \begin{cases} \dfrac{(-1)^{z+n-(t+k)}}{q^{z+n}} [t+1]_q [k+1]_q x_{zn}, & z-1 \leq t \leq z,\ n-1 \leq k \leq n, \\ 0, & \textit{otherwise}. \end{cases}$$

In that case, $\{\tilde{\mathcal{L}}_s^q\}^\alpha = \varpi_1(q)$ for $0 < s \leq 1$.

Proof. By using (2), we obtain that

$$\begin{aligned} x_{zn} u_{zn} &= x_{zn} \left(\sum_{t=z-1}^{z} \sum_{k=n-1}^{n} \frac{(-1)^{z+n-(t+k)}}{q^{z+n}} [t+1]_q [k+1]_q \nu_{tk} \right) \\ &= \sum_{t=z-1}^{z} \sum_{k=n-1}^{n} \left(\frac{(-1)^{z+n-(t+k)}}{q^{z+n}} [t+1]_q [k+1]_q x_{zn} \right) \nu_{tk} \\ &= (G\nu)_{zn} \end{aligned} \tag{10}$$

for $u \in \tilde{\mathcal{L}}_s^q$. Hence, by the aid of the equality (10) it is obtained that "$xu = (x_{zn} u_{zn}) \in \mathcal{L}_u$ when $u \in \tilde{\mathcal{L}}_s^q$ if and only if $G\nu \in \mathcal{L}_u$ when $\nu \in \mathcal{L}_s$". In that case it is reached the biconditional statement "$x \in \{\tilde{\mathcal{L}}_s^q\}^\alpha$ if and only if $G \in (\mathcal{L}_s : \mathcal{L}_u)$". By taking into consideration the condition of the class $(\mathcal{L}_s : \mathcal{L}_{s_1})$ for $0 < s \leq 1$ and $s_1 = 1$ in Table 1 together with $G = (g_{zntk})$ in place of $B = (b_{zntk})$, it is reached that $\{\tilde{\mathcal{L}}_s^q\}^\alpha = \varpi_1(q)$ for $0 < s \leq 1$.

Now, we may describe the sets $\varpi_2(q)$, $\varpi_3(q)$, $\varpi_4(q)$ and $\varpi_5(q)$ which will be utilized in Theorem 7.

$$\varpi_2(q) = \left\{ x = (x_{tk}) \in \Omega : \sum_{t,k} \left| [t+1]_q [k+1]_q \Delta_{11} \left(\frac{x_{tk}}{q^{t+k}} \right) \right|^{s'} < \infty \right\},$$

$$\varpi_3(q) = \left\{ x = (x_{tk}) \in \Omega : \sup_{n \in \mathbb{N}} \sum_{t} \left| [t+1]_q [n+1]_q \Delta_{10} \left(\frac{x_{tn}}{q^{t+n}} \right) \right|^{s'} < \infty \right\},$$

$$\varpi_4(q) = \left\{ x = (x_{tk}) \in \Omega : \sup_{r \in \mathbb{N}} \sum_{k} \left| [z+1]_q [k+1]_q \Delta_{01} \left(\frac{x_{zk}}{q^{z+k}} \right) \right|^{s'} < \infty \right\},$$

$$\varpi_5(q) = \left\{ x = (x_{tk}) \in \Omega : \sup_{z,n\in\mathbb{N}} \left| \frac{[z+1]_q[n+1]_q x_{zn}}{q^{z+n}} \right|^{s'} < \infty \right\},$$

where

$$\begin{aligned}
\Delta_{11}\Big(\frac{x_{tk}}{q^{t+k}}\Big) &= \Big(\frac{x_{tk}}{q^{t+k}} - \frac{x_{t+1,k}+x_{t,k+1}}{q^{t+k+1}} + \frac{x_{t+1,k+1}}{q^{t+k+2}}\Big),\\
\Delta_{10}\Big(\frac{x_{tn}}{q^{t+n}}\Big) &= \Big(\frac{x_{tn}}{q^{t+n}} - \frac{x_{t+1,n}}{q^{t+n+1}}\Big),\\
\Delta_{01}\Big(\frac{x_{zk}}{q^{z+k}}\Big) &= \Big(\frac{x_{zk}}{q^{z+k}} - \frac{x_{z,k+1}}{q^{z+k+1}}\Big)
\end{aligned}$$

and $\frac{1}{s} + \frac{1}{s'} = 1$.

THEOREM 7

$\left\{\tilde{\mathcal{L}}_s^q\right\}^{\beta(b\mathcal{P})} = \bigcap_{k=2}^{5} \varpi_k(q)$ *for* $1 < s < \infty$.

Proof. Let us choose two sequences $x = (x_{tk}) \in \Omega$ and $u \in \tilde{\mathcal{L}}_s^q$ such that $\nu \in \mathcal{L}_s$ with the relation (2). In that case, we reach that

$$\begin{aligned}
\sigma_{zn} &= \sum_{t,k=0}^{z,n} x_{tk}u_{tk}\\
&= \sum_{t,k=0}^{z,n} x_{tk}\Big(\frac{1}{q^{t+k}} \sum_{m=t-1}^{t}\sum_{l=k-1}^{k} (-1)^{t+k-(m+l)}[m+1]_q[l+1]_q\nu_{ml}\Big)\\
&= \sum_{t=0}^{z-1}[t+1]_q[n+1]_q\Delta_{10}\Big(\frac{x_{tn}}{q^{t+n}}\Big)\nu_{tn}\\
&\quad + \sum_{k=0}^{n-1}[z+1]_q[k+1]_q\Delta_{01}\Big(\frac{x_{zk}}{q^{z+k}}\Big)\nu_{zk}\\
&\quad + \sum_{t=0}^{z-1}\sum_{k=0}^{n-1}[t+1]_q[k+1]_q\Delta_{11}\Big(\frac{x_{tk}}{q^{t+k}}\Big)\nu_{tk}\\
&\quad + \frac{[z+1]_q[n+1]_q x_{zn}\nu_{zn}}{q^{z+n}} = (O\nu)_{zn},
\end{aligned} \tag{11}$$

where the 4d matrix $O = (o_{zntk})$ is described as

$$o_{zntk} := \begin{cases} [t+1]_q[k+1]_q\Delta_{11}\Big(\dfrac{x_{tk}}{q^{t+k}}\Big), & 0 \le t \le z-1,\ 0 \le k \le n-1, \\ [t+1]_q[n+1]_q\Delta_{10}\Big(\dfrac{x_{tn}}{q^{t+n}}\Big), & 0 \le t \le z-1,\ k = n, \\ [z+1]_q[k+1]_q\Delta_{01}\Big(\dfrac{x_{zk}}{q^{z+k}}\Big), & 0 \le k \le n-1,\ t = z, \\ \dfrac{[z+1]_q[n+1]_q x_{zn}}{q^{z+n}}, & k = n,\ t = z, \\ 0, & \text{elsewhere} \end{cases} \tag{12}$$

for $z, n, t, k \in \mathbb{N}$. In that case, from the relation (11), it is infered that $xu = (x_{tk}u_{tk}) \in \mathcal{CS}_{b\mathcal{P}}$ whenever $u = (u_{tk}) \in \tilde{\mathcal{L}}_s^q$ if and only if $\sigma = (\sigma_{zn}) \in \mathcal{C}_{b\mathcal{P}}$ whenever $\nu \in \mathcal{L}_s$. This implies that $x \in \{\tilde{\mathcal{L}}_s^q\}^{\beta(b\mathcal{P})}$ if and only if $O \in (\mathcal{L}_s : \mathcal{C}_{b\mathcal{P}})$. Hence, in view of Table 1, the following statement

$$\begin{aligned} \sup_{z,n\in\mathbb{N}} \sum_{t,k=0} |o_{zntk}|^{s'} = \sup_{z,n\in\mathbb{N}} \Bigg\{ & \sum_{t=0}^{z-1}\sum_{k=0}^{n-1} \Big|[t+1]_q[k+1]_q\Delta_{11}\Big(\frac{x_{tk}}{q^{t+k}}\Big)\Big|^{s'} \\ & + \sum_{t=0}^{z-1} \Big|[t+1]_q[n+1]_q\Delta_{10}\Big(\frac{x_{tn}}{q^{t+n}}\Big)\Big|^{s'} \\ & + \sum_{k=0}^{n-1} \Big|[z+1]_q[k+1]_q\Delta_{01}\Big(\frac{x_{zk}}{q^{z+k}}\Big)\Big|^{s'} \\ & + \Big|\frac{[z+1]_q[n+1]_q x_{zn}}{q^{z+n}}\Big|^{s'} \Bigg\} < \infty \end{aligned}$$

holds and we reach that

$$\begin{aligned} \sum_{t,k} \Big|[t+1]_q[k+1]_q\Delta_{11}\Big(\frac{x_{tk}}{q^{t+k}}\Big)\Big|^{s'} &< \infty \\ \sup_{n\in\mathbb{N}} \sum_{t} \Big|[t+1]_q[n+1]_q\Delta_{10}\Big(\frac{x_{tn}}{q^{t+n}}\Big)\Big|^{s'} &< \infty \\ \sup_{z\in\mathbb{N}} \sum_{k} \Big|[z+1]_q[k+1]_q\Delta_{01}\Big(\frac{x_{zk}}{q^{z+k}}\Big)\Big|^{s'} &< \infty \\ \Big|\frac{[z+1]_q[n+1]_q x_{zn}}{q^{z+n}}\Big|^{s'} &\in \mathcal{M}_u. \end{aligned}$$

Thus, $\{\tilde{\mathcal{L}}_s^q\}^{\beta(b\mathcal{P})} = \bigcap_{k=2}^{5} \varpi_5(q)$ for $1 < s < \infty$.

Since its proof can be done similarly to Theorem 7, the following theorem will be given without proof.

Theorem 8
The $\beta(\vartheta)$-duals of $\tilde{\mathcal{L}}_s^q$ is the set $\{x = (x_{zn}) \in \Omega : \ O = (o_{zntk}) \in (\mathcal{L}_s : \mathcal{C}_\vartheta)\}$, where $O = (o_{zntk})$ is described by (12) and $\vartheta \in \{\mathcal{P}, r\}$.

4. Matrix Transformations

Current part aims to present the classes $(\tilde{\mathcal{L}}_s^q : \Lambda)$, where $\Lambda \in \{\mathcal{M}_u, \mathcal{C}_\vartheta, \mathcal{C}_{\vartheta 0}, \mathcal{L}_{s_1}\}$ for $0 < s, s_1 < \infty$.

Theorem 9
Let $B = (b_{zntk})$ and $H = (h_{zntk})$ 4d matrices be given by the relation

$$h_{zntk} = [t+1]_q[k+1]_q \Delta_{11}^{tk}\Big(\frac{b_{zntk}}{q^{z+n}}\Big). \tag{13}$$

$B \in (\tilde{\mathcal{L}}_s^q : \Lambda)$ if and only if $H \in (\mathcal{L}_s : \Lambda)$ and

$$B_{zn} \in (\tilde{\mathcal{L}}_s^q)^{\beta(\vartheta)} \tag{14}$$

for $0 < s, s_1 < \infty$, where $\Lambda \in \{\mathcal{M}_u, \mathcal{C}_\vartheta, \mathcal{C}_{\vartheta 0}, \mathcal{L}_{s_1}\}$.

Proof. If $B \in (\tilde{\mathcal{L}}_s^q : \Lambda)$, $Bu \in \Lambda$ for all $u \in \tilde{\mathcal{L}}_s^q$ such that $\nu = C_{(1,1)}(q)u \in \mathcal{L}_s$ for $0 < s, s_1 < \infty$, where $\Lambda \in \{\mathcal{M}_u, \mathcal{C}_\vartheta, \mathcal{C}_{\vartheta 0}, \mathcal{L}_{s_1}\}$. This says us that $B_{zn} \in (\tilde{\mathcal{L}}_s^q)^{\beta(\vartheta)}$. As the (i,j)-th partial sums of the series $\sum_{t,k} b_{zntk}u_{tk}$ we get

$$\begin{aligned}(Bu)_{zn}^{[i,j]} &= \sum_{t,k=0}^{i,j} b_{zntk}u_{tk}\\ &= \sum_{t=0}^{i-1}\sum_{k=0}^{j-1}[t+1]_q[k+1]_q\Delta_{11}^{tk}\Big(\frac{b_{zntk}}{q^{t+k}}\Big)\nu_{tk}\\ &\quad + \sum_{t=0}^{i-1}[t+1]_q[j+1]_q\Delta_{10}^{tj}\Big(\frac{b_{zntj}}{q^{t+j}}\Big)\\ &\quad + \sum_{k=0}^{j-1}[i+1]_q[k+1]_q\Delta_{01}^{ik}\Big(\frac{b_{znik}}{q^{i+k}}\Big) + \frac{[i+1]_q[j+1]_q}{q^{i+j}}b_{znij}\end{aligned} \tag{15}$$

for all $z, n, i, j \in \mathbb{N}$. If we describe the 4d infinite matrix $H_{zn} = (h_{ijtk}^{[z,n]})$ as

$$h_{ijtk}^{[z,n]} := \begin{cases}[t+1]_q[k+1]_q\Delta_{11}^{tk}\Big(\frac{b_{zntk}}{q^{t+k}}\Big), & 0 \le t \le i-1,\ 0 \le k \le j-1,\\ [t+1]_q[j+1]_q\Delta_{10}^{tj}\Big(\frac{b_{zntj}}{q^{t+j}}\Big), & 0 \le t \le i-1,\ k = j,\\ [i+1]_q[k+1]_q\Delta_{01}^{ik}\Big(\frac{b_{znik}}{q^{i+k}}\Big), & 0 \le k \le j-1,\ t = i,\\ \frac{[i+1]_q[j+1]_q}{q^{i+j}}b_{znij}, & k = j,\ t = i,\\ 0, & \text{otherwise}\end{cases}$$

the relation (15) can be restated as

$$(Bu)_{zn}^{[i,j]} = (H_{zn}\nu)_{[i,j]}. \tag{16}$$

Furthermore, when it is taken ϑ-limit on $H_{zn} = (h_{ijtk}^{[z,n]})$ as $i, j \to \infty$, it is seen that

$$\vartheta - \lim_{i,j\to\infty} h_{ijtk}^{[z,n]} = [t+1]_q[k+1]_q\Delta_{11}^{tk}\Big(\frac{b_{zntk}}{q^{t+k}}\Big). \tag{17}$$

By the aid of (17), we can describe the 4d matrix $H = (h_{zntk})$ as

$$h_{zntk} = [t+1]_q[k+1]_q\Delta_{11}^{tk}\Big(\frac{b_{zntk}}{q^{t+k}}\Big)$$

and also by taking ϑ-limit on (16) for $i, j \to \infty$, it is reached that $Bu = H\nu$. So, $H\nu \in \Lambda$ when $\nu \in \mathcal{L}_s$ and $H \in (\mathcal{L}_s : \Lambda)$.

On the other hand, consider that $B_{zn} \in (\tilde{\mathcal{L}}_s^q)^{\beta(\vartheta)}$ and $H \in (\mathcal{L}_s : \Lambda)$ for $0 < s, s_1 < \infty$, where $\Lambda \in \{\mathcal{M}_u, \mathcal{C}_\vartheta, \mathcal{C}_{\vartheta 0}, \mathcal{L}_{s_1}\}$. Let us take the sequence $u \in \tilde{\mathcal{L}}_s^q$ such that $\nu = C_{(1,1)}(q)u \in \mathcal{L}_s$. In that case, Bu exists. Also, from the (i, j)-th partial sums of $\sum_{t,k} b_{zntk}u_{tk}$ we reach that

$$\sum_{t,k=0}^{i,j} b_{zntk}u_{tk} = \sum_{t,k=0}^{i,j} h_{ijtk}^{[z,n]}\nu_{tk}$$

for all $z, n, t, k \in \mathbb{N}$. By letting ϑ-limit as $i, j \to \infty$ on the equality above, it is concluded that $Bu = H\nu$. Thus, $B \in (\tilde{\mathcal{L}}_s^q : \Lambda)$.

Corollary 1

Let $B = (b_{zntk})$ and $H = (h_{zntk})$ 4d matrices be given by the relation (13). *In that case, in addition to providing the condition* (14) *for each classes, the necessary and sufficient conditions for the classes $(\tilde{\mathcal{L}}_s^q : \Lambda)$ can be seen from the Table* 2.

Table 2: Characterizations of $(\tilde{\mathcal{L}}_s^q : \Lambda)$, where $\Lambda \in \{\mathcal{M}_u, \mathcal{C}_\vartheta, \mathcal{C}_{\vartheta 0}, \mathcal{L}_{s_1}\}$.

$(\Psi \downarrow: \Lambda \to)$	$\mathcal{M}_u$	$\mathcal{C}_\vartheta$	$\mathcal{C}_{\vartheta 0}$	$\mathcal{L}_{s_1}(0 < s_1 < \infty)$
$\tilde{\mathcal{L}}_s^q(0 < s \le 1)$	(5)	(5),(8)	(5),(9)	(7)
$\tilde{\mathcal{L}}_s^q(1 < s < \infty)$	(6)	(6),(8)	(6),(9)	$\bullet$

holds with h_{zntk} instead of b_{zntk}.

Lemma 1 ([38])

Consider that $\Psi,\Lambda \subset \Omega$, a 4d matrix $B = (b_{zntk})$ and 4d triangle $F = (f_{zntk})$. Then, $B \in (\Psi : \Lambda_F)$ if and only if $FB \in (\Psi : \Lambda)$.

Now, by keeping in mind the Lemma just mentioned above, we can give the final result of our study.

Corollary 2

Let $B = (b_{zntk})$ and $W = (w_{zntk})$ 4d matrices be given by the relation

$$w_{zntk} = \sum_{t,k=0}^{z,n} c_{znij}(q)b_{ijtk}.$$

Then, the necessary and sufficient conditions for the classes $(\mathcal{L}_s : \Lambda_{C_{(1,1)}(q)})$ can be read from the Table 3.

Table 3: Characterizations of $(\mathcal{L}_s : \Lambda_{C_{(1,1)}(q)})$, where $\Lambda \in \{\mathcal{M}_u, \mathcal{C}_\vartheta, \mathcal{C}_{\vartheta 0}, \mathcal{L}_{s_1}\}$.

$(\Psi \downarrow : \Lambda \rightarrow)$	$(\mathcal{M}_u)_{C_{(1,1)}(q)}$	$(\mathcal{C}_\vartheta)_{C_{(1,1)}(q)}$	$(\mathcal{C}_{\vartheta 0})_{C_{(1,1)}(q)}$	$(\mathcal{L}_{s_1})_{C_{(1,1)}(q)}$
$\mathcal{L}_s(0 < s \leq 1)$	(5)	(5),(8)	(5),(9)	(7)
$\mathcal{L}_s(1 < s < \infty)$	(6)	(6),(8)	(6),(9)	$\bullet$

where $0 < s_1 < \infty$ *holds with* w_{zntk} *instead of* b_{zntk}.

References

[1] Adams C. Raymond. "On non-factorable transformations of double sequences." *Proc. Natl. Acad. Sci. USA* 19, no. 5 (1933): 564-567. Cited on 113.

[2] Aktuğlu, Hüseyin, and Bekar Şerife. "q-Cesáro matrix and q-statistical convergence." *J. Comput. Appl. Math.* 235, no. 16 (2011): 4717-4723. Cited on 113.

[3] Altay, Bilâl, and Başar Feyzi. "Some new spaces of double sequences." *J. Math. Anal. Appl.* 309, no. 1 (2005): 70-90. Cited on 112.

[4] Başar, Feyzi. "The space $\mathcal{L}_q$ of double sequences." *Math. J. Okayama Univ.* 51 (2009): 149-157. Cited on 112 and 117.

[5] Başarir, Metin. "On the strong almost convergence of double sequences." *Period. Math. Hungar.* 30, no. 3 (1995): 177-181. Cited on 113.

[6] Bekar, Şerife. *q-matrix summability methods.* Ph.D. Dissertation. Eastern Mediterranean University, 2010. Cited on 113.

[7] Boos, Johann. *Classical and Modern Methods in Summability. Oxford Mathematical Monographs.* New York: Oxford University Press Inc., 2000. Cited on 117.

[8] Candan, Murat. "Domain of the double sequential band matrix in the spaces of convergent and null sequences." *Adv. Difference Equ.* (2014): Art. no. 163. Cited on 113.

[9] Candan, Murat. "A new sequence space isomorphic to the $\ell(p)$ and compact operators." *J. Math. Comput. Sci.* 4, no. 2 (2014): 306-334. Cited on 113.

[10] Candan, Murat. "Some Characteristics of Matrix Operators on Generalized Fibonacci Weighted Difference Sequence Space." *Symmetry* 14, no. 7 (2022): Art. no. 1283. Cited on 113.

[11] Cooke, Richard George. *Infinite Matrices and Sequence Spaces.* London: Macmillan & Co. Limited, 1950. Cited on 113.

[12] Çapan, Hüsamettin, and Başar Feyzi. "On the paranormed space $\mathcal{L}(t)$ of double sequences." *Filomat* 32, no. 3 (2018): 1043-1053. Cited on 113 and 118.

[13] Çapan, Hüsamettin, and Başar Feyzi. "On some spaces isomorphic to the space of absolutely q-summable double sequences." *Kyungpook Math. J.* 58, no. 2 (2018): 271-289. Cited on 113 and 118.

[14] Çinar, Muhammed, and Et Mikail. "q-double Cesaro matrices and q-statistical convergence of double sequences." *Nat. Acad. Sci. Lett.* 43, no. 1 (2020): 73-76. Cited on 113.

[15] Demiriz, Serkan, and Duyar Osman. "Domain of difference matrix of order one in some spaces of double sequences." *Gulf J. Math.* 3, no. 3 (2015): 85-100. Cited on 113.

[16] Demiriz, Serkan, and Duyar Osman. "Domain of the generalized double Cesàro matrix in some paranormed spaces of double sequences." *Tbilisi Math. J.* 10, no. 2 (2017): 43-56. Cited on 113.

[17] Demiriz, Serkan, and Adem Şahin. "q-Cesàro Sequence Spaces Derived by q-analogue." *Adv. Math. Sci. J.* 5, no. 2 (2016): 97-110. Cited on 113.

[18] Demiriz, Serkan, and Sezer Erdem. "On the New Double Binomial Sequence Space" *Turk. J. Math. Comput. Sci.* 12, no. 2 (2020): 101-111. Cited on 113.

[19] Demiriz, Serkan, and Erdem Sezer. "Domain of binomial matrix in some spaces of double sequences." *Punjab Univ. J. Math. (Lahore)* 52, no. 11 (Lahore): 65-79. Cited on 113.

[20] Erdem, Sezer, and Demiriz Serkan. "Almost convergence and 4-dimensional binomial matrix." *Konuralp J. Math.* 8, no. 2 (2020): 329-336. Cited on 113.

[21] Erdem, Sezer, and Demiriz Serkan. "A new RH-regular matrix derived by Jordan's function and its domains on some double sequence spaces." *J. Funct. Spaces*, to appear. Cited on 113.

[22] Hamilton, Hugh J. "Transformations of multiple sequences." *Duke Math. J.* 2, no. 1 (1936): 29-60. Cited on 112.

[23] Hardy, Godfrey Harold. "On the Convergence of Certain Multiple Series." *Proc. London Math. Soc. (2)* 1 (1904): 124-128. Cited on 112.

[24] İlkhan, Merve, Alp Pınar Zengin, and Kara Emrah Evren. "On the spaces of linear operators acting between asymmetric cone normed spaces." *Mediterr. J. Math.* 15, no. 3 (2018): Paper No. 136. Cited on 113.

[25] İlkhan, Merve, and Kara Emrah Evren. "A new Banach space defined by Euler totient matrix operator." *Oper. Matrices* 13, no. 2 (2019): 527-544. Cited on 113.

[26] Kac, Victor, and Pokman Cheung. *Quantum Calculus.* Springer: New York, 2002. Cited on 113.

[27] Móricz, Ferenc, and Rhoades Billy E. "Almost convergence of double sequences and strong regularity of summability matrices." *Math. Proc. Cambridge Philos. Soc.* 104, no. 2 (1988): 283-294. Cited on 113.

[28] Mursaleen, Mohammad. "Almost strongly regular matrices and a core theorem for double sequences." *J. Math. Anal. Appl.* 293, no. 2 (2004): 523-531. Cited on 113.

[29] Mursaleen, Mohammad, and Başar Feyzi. "Domain of Cesàro mean of order one in some spaces of double sequences." *Studia Sci. Math. Hungar.* 51, no. 3 (2014): 335-356. Cited on 113 and 114.

[30] Ng, Peng Nung. "Cesàro sequence spaces of non-absolute type." *Comment. Math. Prace Mat.* 20, no. 2 (1977/78): 429-433. Cited on 113.

[31] Pringsheim, Alfred. "Zur Theorie der zweifach unendlichen Zahlenfolgen." *Math. Ann.* 53, no. 3 (1900): 289-321. Cited on 112.

[32] Robison, George Merritt. "Divergent double sequences and series." *Trans. Amer. Math. Soc.* 28, no. 1 (1926): 50-73. Cited on 112.

[33] Tuğ, Orhan. "Four-dimensional generalized difference matrix and some double sequence spaces." *J. Inequal. Appl.* (2017): Paper No. 149. Cited on 113.

[34] Tuğ, Orhan. "On almost B-summable double sequence spaces." *J. Inequal. Appl.* (2018): Paper No. 9. Cited on 113.

[35] Tuğ, Orhan. "The spaces of $B(r, s, t, u)$ strongly almost convergent double sequences and matrix transformations." *Bull. Sci. Math.* 169 (2021): Paper No. 102989. Cited on 113.

[36] Yaying, Taja, Hazarika Bipan, and Mursaleen Mohammad. "On sequence space derived by the domain of q-Cesàro matrix in ell_p space and the associated operator ideal." *J. Math. Anal. Appl.* 493, no. 1 (2021): Paper No. 124453. Cited on 113.

[37] Yaying, Taja, Hazarika Bipan, and Mursaleen Mohammad. "On generalized (p, q)-Euler matrix and associated sequence spaces." *J. Funct. Spaces* (2021): Art. ID 8899960. Cited on 113.

[38] Yeşilkayagil, Medine, and Başar Feyzi. "On the domain of Riesz mean in the space $\mathcal{L}_s^*$." *Filomat* 31, no. 4 (2017): 925-940. Cited on 113, 117, 118 and 123.

[39] Yeşilkayagil, Medine, and Başar Feyzi. "Domain of Euler mean in the space of absolutely p-summable double sequences with $0 < p < 1$." *Anal. Theory Appl.* 34, no. 3 (2018): 241-252. Cited on 113.

[40] Zeltser, Maria. *Investigation of double sequence spaces by soft and hard analitic methods.* Vol. 25 of *Dissertationes Mathematicae Universtaties Tartuensis.* Tatru: Tartu University Press, 2001. Cited on 112.

[41] Zeltser, Maria. "On conservative matrix methods for double sequence spaces." *Acta Math. Hungar.* 95, no. 3 (2002): 225-242. Cited on 112.

[42] Zeltser, Maria, Mursaleen Mohammad, and Mohiuddine Syed Abdul. "On almost conservative matrix methods for double sequence spaces." *Publ. Math. Debrecen* 75, no. 3-4 (2009): 387-399. Cited on 113.

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