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Ann. Univ. Paedagog. Crac. Stud. Math. 22 (2023), 127-134
DOI: 10.2478/aupcsm-2023-0010

FOLIA 385

Annales Universitatis Paedagogicae Cracoviensis
Studia Mathematica 22 (2023)

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On absolute summability of infinite series and Fourier series

Abstract. In the present paper, a theorem on $\theta-|T;\delta|_k$ summability method of an infinite series is proved, and also by using this method, a result on summability of a trigonometric Fourier series is obtained.

1. Introduction

Let $\sum a_n$ be a given infinite series with partial sums (s_n). Let $T=(t_{nv})$ be a normal matrix, i.e. a lower triangular matrix of nonzero diagonal entries. Then T defines the sequence-to-sequence transformation, mapping the sequence $s=(s_n)$ to $Ts=(T_n(s))$, where

$$T_n(s)=\sum_{v=0}^{n} t_{nv}s_v, \qquad n=0,1,\ldots.$$

Let (θ_n) be any sequence of positive real numbers. The series $\sum a_n$ is said to be summable $\theta-|T;\delta|_k$, $k\geq 1$ and $\delta\geq 0$, if (see [11]),

$$\sum_{n=1}^{\infty}\theta_n^{\delta k+k-1}|T_n(s)-T_{n-1}(s)|^k<\infty. \tag{1}$$

Further, two lower semimatrices $\bar{T}=(\bar{t}_{nv})$ and $\hat{T}=(\hat{t}_{nv})$ are defined as follows:

$$\bar{t}_{nv}=\sum_{i=v}^{n} t_{ni}, \qquad n,v=0,1,\ldots. \tag{2}$$

AMS (2020) Subject Classification: 26D15, 40D15, 40F05, 40G99, 42A24.

Keywords and phrases: absolute matrix summability, Fourier series, summability factors, infinite series.

ISSN: 2081-545X, e-ISSN: 2300-133X.

$$\hat{t}_{00} = \bar{t}_{00} = t_{00}, \quad \hat{t}_{nv} = \bar{t}_{nv} - \bar{t}_{n-1,v}, \qquad n = 1, 2, \dots . \tag{3}$$

and

$$\bar{\Delta} T_n(s) = T_n(s) - T_{n-1}(s) = \sum_{v=0}^{n} \hat{t}_{nv} a_v. \tag{4}$$

Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad \text{as} \quad n \to \infty, \qquad (P_{-k} = p_{-k} = 0,\ k \geq 1).$$

By taking $\theta_n = \frac{P_n}{p_n}$, $\delta = 0$ and $t_{nv} = \frac{p_v}{P_n}$ in (1), we get $|\bar{N}, p_n|_k$ summability method (see [1]). For any sequence (λ_n), it should be noted that $\Delta\lambda_n = \lambda_n - \lambda_{n+1}$, $\Delta^0\lambda_n = \lambda_n$ and $\Delta^k\lambda_n = \Delta\Delta^{k-1}\lambda_n$ for $k = 1, 2, \dots$ (see [7]).

It should be noted that (y_n) is the n-th $(C, 1)$ mean of the sequence (na_n), i.e. $y_n = \frac{1}{n+1}\sum_{v=1}^{n} va_v$. Also, if we write $X_n = \sum_{v=0}^{n} \frac{p_v}{P_v}$, then (X_n) is a positive increasing sequence tending to infinity as $n \to \infty$.

2. Known Result

The following theorem has been proved in [3].

Theorem 1

Let (p_n) be a sequence of positive numbers such that

$$P_n = O(np_n) \quad as \quad n \to \infty.$$

If the conditions

$$\lambda_m = o(1) \quad as \quad m \to \infty, \tag{5}$$

$$\sum_{n=1}^{m} nX_n \left|\Delta^2\lambda_n\right| = O(1) \quad as \quad m \to \infty, \tag{6}$$

$$\sum_{n=1}^{m} \frac{p_n}{P_n} \frac{|y_n|^k}{X_n^{k-1}} = O(X_m) \quad as \quad m \to \infty$$

hold, then the series $\sum a_n\lambda_n$ is summable $|\bar{N}, p_n|_k$, $k \geq 1$.

3. Main Result

There are many papers on the field of summability, some of them are [4, 5, 8, 9, 10, 12, 13, 14, 15, 16]. In this paper, Theorem 1 is generalized to the $\theta - |T;\delta|_k$ summability method as in the following.

Theorem 2
Let $T = (t_{nv})$ be a positive normal matrix such that

$$\bar{t}_{n0} = 1, \qquad n = 0, 1, \ldots, \tag{7}$$

$$t_{n-1,v} \geq t_{nv} \qquad for\ n \geq v + 1, \tag{8}$$

$$t_{nn} = O\Big(\frac{p_n}{P_n}\Big), \tag{9}$$

$$|\hat{t}_{n,v+1}| = O(v|\Delta_v(\hat{t}_{nv})|). \tag{10}$$

Let $\theta_n p_n = O(P_n)$. If the conditions (5), (6) and

$$\sum_{n=1}^{m} \theta_n^{\delta k-1} \frac{|y_n|^k}{X_n^{k-1}} = O(X_m) \quad as \quad m \to \infty, \tag{11}$$

$$\sum_{n=v+1}^{m+1} \theta_n^{\delta k} |\Delta_v(\hat{t}_{nv})| = O\left(\theta_v^{\delta k-1}\right) \quad as \quad m \to \infty \tag{12}$$

are satisfied, then the series $\sum a_n \lambda_n$ is summable $\theta - |T; \delta|_k$, $k \geq 1$ and $0 \leq \delta < 1/k$.

We need the following lemma to prove Theorem 2.

Lemma 1 ([2])
Under the conditions of Theorem 2, we have

$$nX_n|\Delta\lambda_n| = O(1) \quad as \quad n \to \infty, \tag{13}$$

$$\sum_{n=1}^{\infty} X_n|\Delta\lambda_n| < \infty, \tag{14}$$

$$X_n|\lambda_n| = O(1) \quad as \quad n \to \infty. \tag{15}$$

4. Proof of Theorem 2

Let (W_n) denote T-transform of the series $\sum a_n \lambda_n$. Then, by (4), we have

$$\bar{\Delta} W_n = \sum_{v=0}^{n} \hat{t}_{nv} a_v \lambda_v = \sum_{v=1}^{n} \frac{\hat{t}_{nv} \lambda_v}{v} v a_v. \tag{16}$$

Applying Abel's transformation in (16), we obtain

$$\begin{aligned}
\bar{\Delta} W_n &= \sum_{v=1}^{n-1} \Delta_v \left(\frac{\hat{t}_{nv} \lambda_v}{v} \right) \sum_{r=1}^{v} r a_r + \frac{\hat{t}_{nn} \lambda_n}{n} \sum_{r=1}^{n} r a_r \\
&= \sum_{v=1}^{n-1} \frac{v+1}{v} \Delta_v \left(\hat{t}_{nv} \right) \lambda_v y_v + \sum_{v=1}^{n-1} \frac{v+1}{v} \hat{t}_{n,v+1} \Delta \lambda_v y_v
\end{aligned}$$

$$+\sum_{v=1}^{n-1}\hat{t}_{n,v+1}\lambda_{v+1}\frac{y_v}{v}+\frac{n+1}{n}t_{nn}\lambda_n y_n$$
$$=W_{n,1}+W_{n,2}+W_{n,3}+W_{n,4}.$$

To complete the proof of Theorem 2, we will prove

$$\sum_{n=1}^{\infty}\theta_n^{\delta k+k-1}|W_{n,r}|^k<\infty \qquad \text{for } r=1,2,3,4.$$

First, applying Hölder's inequality with indices k and k', where $k>1$ and $\frac{1}{k}+\frac{1}{k'}=1$, we have

$$\sum_{n=2}^{m+1}\theta_n^{\delta k+k-1}|W_{n,1}|^k=O(1)\sum_{n=2}^{m+1}\theta_n^{\delta k+k-1}\sum_{v=1}^{n-1}|\Delta_v(\hat{t}_{nv})||\lambda_v|^k|y_v|^k \times\left(\sum_{v=1}^{n-1}|\Delta_v(\hat{t}_{nv})|\right)^{k-1}.$$

By using (2) and (3), we get $\Delta_v(\hat{t}_{nv})=t_{nv}-t_{n-1,v}$. Also, using (2), (7) and (8), we get

$$\sum_{v=1}^{n-1}|\Delta_v(\hat{t}_{nv})|=\sum_{v=1}^{n-1}(t_{n-1,v}-t_{nv})\leq t_{nn}.$$

Then, by using (9), (12), (15), we have

$$\begin{aligned}\sum_{n=2}^{m+1}\theta_n^{\delta k+k-1}|W_{n,1}|^k&=O(1)\sum_{n=2}^{m+1}\theta_n^{\delta k+k-1}t_{nn}^{k-1}\sum_{v=1}^{n-1}|\Delta_v(\hat{t}_{nv})||\lambda_v|^k|y_v|^k\\&=O(1)\sum_{n=2}^{m+1}\theta_n^{\delta k}\sum_{v=1}^{n-1}|\Delta_v(\hat{t}_{nv})||\lambda_v|^k|y_v|^k\\&=O(1)\sum_{v=1}^{m}|\lambda_v|^k|y_v|^k\sum_{n=v+1}^{m+1}\theta_n^{\delta k}|\Delta_v(\hat{t}_{nv})|\\&=O(1)\sum_{v=1}^{m}\theta_v^{\delta k-1}|\lambda_v|^{k-1}|\lambda_v||y_v|^k\\&=O(1)\sum_{v=1}^{m}\theta_v^{\delta k-1}|\lambda_v|\frac{|y_v|^k}{X_v^{k-1}}.\end{aligned}$$

Now, by applying Abel's transformation and using the conditions (11), (14) and (15), we get

$$\begin{aligned}
&\sum_{n=2}^{m+1}\theta_n^{\delta k+k-1}|W_{n,1}|^k\\
&\qquad = O(1)\sum_{v=1}^{m-1}\Delta|\lambda_v|\sum_{r=1}^{v}\theta_r^{\delta k-1}\frac{|y_r|^k}{X_r^{k-1}}+O(1)|\lambda_m|\sum_{v=1}^{m}\theta_v^{\delta k-1}\frac{|y_v|^k}{X_v^{k-1}}\\
&\qquad = O(1)\sum_{v=1}^{m-1}|\Delta\lambda_v|X_v+O(1)|\lambda_m|X_m\\
&\qquad = O(1)\quad \text{as} \quad m\to\infty.
\end{aligned}$$

Now, using (10), (9), (12), (13) and applying Hölder's inequality, we obtain

$$\begin{aligned}
&\sum_{n=2}^{m+1}\theta_n^{\delta k+k-1}|W_{n,2}|^k\\
&\qquad = O(1)\sum_{n=2}^{m+1}\theta_n^{\delta k+k-1}\left(\sum_{v=1}^{n-1}v|\Delta_v(\hat{t}_{nv})||\Delta\lambda_v||y_v|\right)^k\\
&\qquad = O(1)\sum_{n=2}^{m+1}\theta_n^{\delta k+k-1}\sum_{v=1}^{n-1}(v|\Delta\lambda_v|)^k|\Delta_v(\hat{t}_{nv})||y_v|^k\times\left(\sum_{v=1}^{n-1}|\Delta_v(\hat{t}_{nv})|\right)^{k-1}\\
&\qquad = O(1)\sum_{n=2}^{m+1}\theta_n^{\delta k}\sum_{v=1}^{n-1}(v|\Delta\lambda_v|)^k|\Delta_v(\hat{t}_{nv})||y_v|^k\\
&\qquad = O(1)\sum_{v=1}^{m}(v|\Delta\lambda_v|)^{k-1}(v|\Delta\lambda_v|)|y_v|^k\sum_{n=v+1}^{m+1}\theta_n^{\delta k}|\Delta_v(\hat{t}_{nv})|\\
&\qquad = O(1)\sum_{v=1}^{m}\theta_v^{\delta k-1}v|\Delta\lambda_v|\frac{|y_v|^k}{X_v^{k-1}}.
\end{aligned}$$

Here, applying Abel's transformation and using the conditions (11), (6), (14) and (13), we get

$$\begin{aligned}
&\sum_{n=2}^{m+1}\theta_n^{\delta k+k-1}|W_{n,2}|^k\\
&\qquad = O(1)\sum_{v=1}^{m-1}\Delta(v|\Delta\lambda_v|)\sum_{r=1}^{v}\theta_r^{\delta k-1}\frac{|y_r|^k}{X_r^{k-1}}+O(1)m|\Delta\lambda_m|\sum_{v=1}^{m}\theta_v^{\delta k-1}\frac{|y_v|^k}{X_v^{k-1}}\\
&\qquad = O(1)\sum_{v=1}^{m-1}\Delta(v|\Delta\lambda_v|)X_v+O(1)m|\Delta\lambda_m|X_m
\end{aligned}$$

$$= O(1)\sum_{v=1}^{m-1} v|\Delta^2\lambda_v|X_v + O(1)\sum_{v=1}^{m-1}|\Delta\lambda_v|X_v + O(1)m|\Delta\lambda_m|X_m$$
$$= O(1) \quad \text{as} \quad m\to\infty.$$

Now, using (10), (9), (12), (15), we have

$$\sum_{n=2}^{m+1}\theta_n^{\delta k+k-1}|W_{n,3}|^k$$
$$= O(1)\sum_{n=2}^{m+1}\theta_n^{\delta k+k-1}\sum_{v=1}^{n-1}|\Delta_v(\hat{t}_{nv})||\lambda_{v+1}|^k|y_v|^k \times \left(\sum_{v=1}^{n-1}|\Delta_v(\hat{t}_{nv})|\right)^{k-1}$$
$$= O(1)\sum_{n=2}^{m+1}\theta_n^{\delta k}\sum_{v=1}^{n-1}|\Delta_v(\hat{t}_{nv})||\lambda_{v+1}|^k|y_v|^k$$
$$= O(1)\sum_{v=1}^{m}|\lambda_{v+1}|^{k-1}|\lambda_{v+1}||y_v|^k\sum_{n=v+1}^{m+1}\theta_n^{\delta k}|\Delta_v(\hat{t}_{nv})|$$
$$= O(1)\sum_{v=1}^{m}\theta_v^{\delta k-1}|\lambda_{v+1}|\frac{|y_v|^k}{X_v^{k-1}}.$$

Then, as in $W_{n,1}$, we get

$$\sum_{n=2}^{m+1}\theta_n^{\delta k+k-1}|W_{n,3}|^k = O(1) \quad \text{as} \quad m\to\infty.$$

Finally, as in $W_{n,1}$, we have

$$\sum_{n=1}^{m}\theta_n^{\delta k+k-1}|W_{n,4}|^k$$
$$= O(1)\sum_{n=1}^{m}\theta_n^{\delta k+k-1}t_{nn}^k|\lambda_n|^k|y_n|^k$$
$$= O(1)\sum_{n=1}^{m}\theta_n^{\delta k-1}|\lambda_n|^{k-1}|\lambda_n||y_n|^k$$
$$= O(1)\sum_{n=1}^{m}\theta_n^{\delta k-1}|\lambda_n|\frac{|y_n|^k}{X_n^{k-1}} = O(1) \quad as \quad m\to\infty.$$

This completes the proof of Theorem 2.

5. An application to Fourier series

Let f be a periodic function with period 2π and Lebesgue integrable over $(-\pi, \pi)$. The trigonometric Fourier series of f is defined as

$$f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) = \sum_{n=0}^{\infty} T_n(x),$$

where

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx,$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx,$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx.$$

Write

$$\phi(y) = \frac{1}{2}\{f(x+y) + f(x-y)\} \quad \text{and} \quad \phi_1(y) = \frac{1}{y}\int_0^y \phi(u)du.$$

If $\phi_1(y) \in \mathcal{BV}(0,\pi)$, then $y_n(x) = O(1)$, where $y_n(x)$ is the n-th $(C,1)$ mean of the sequence $(nT_n(x))$ (see [6]). By using this fact, the following theorem on absolute summability of the trigonometric Fourier series is obtained in [3].

Theorem 3
If $\phi_1(y) \in \mathcal{BV}(0,\pi)$, and the sequences (p_n), (λ_n) and (X_n) satisfy the conditions of Theorem 1, then the series $\sum T_n(x)\lambda_n$ is summable $|\bar{N}, p_n|_k$, $k \geq 1$.

Now, Theorem 3 is generalized to the $\theta - |T;\delta|_k$ summability method as in the following form.

Theorem 4
Let $T = (t_{nv})$ be a positive normal matrix which satisfies the conditions (7)-(10). If $\phi_1(y) \in \mathcal{BV}(0,\pi)$, and the sequences (p_n), (λ_n), (θ_n) and (X_n) satisfy the conditions of Theorem 2, then the series $\sum T_n(x)\lambda_n$ is summable $\theta - |T;\delta|_k$, $k \geq 1$ and $0 \leq \delta < 1/k$.

If we take $\theta_n = \frac{P_n}{p_n}$, $\delta = 0$ and $t_{nv} = \frac{p_v}{P_n}$ in Theorem 2 and Theorem 4, then we get Theorem 1 and Theorem 3, respectively.

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Received: February 23, 2023; final version: July 7, 2023;
available online: August 30, 2023.