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Amel Heris, Abdelkrim Salim and Mouffak Benchohra*

Some new existence results for fractional partial random nonlocal differential equations with delay

Abstract. The present paper deals with some existence results for the Darboux problem of partial fractional random differential equations with finite delay. The arguments are based on a random fixed point theorem with stochastic domain combined with the measure of noncompactness. An illustration is given to show the applicability of our results.

1. Introduction

The fractional calculus is concerned with noninteger order extensions of derivatives and integrals. Differential and integral equations of fractional order have a wide range of applications, see [1, 2, 29, 33] for more information. In recent years, there has been substantial progress in ordinary and partial fractional differential and integral equations; see the monographs of Abbas *et al.* [3, 4, 5], Ahmad *et al.* [12], Kilbas *et al.* [24], Lakshmikantham *et al.*, the papers of Abbas *et al.* [6], Ahmad and Nieto [13], Karapinar *et al.* [7, 9, 8, 22, 10], Salim *et al.* [31, 32], Vityuk and Golushkov [36], and the references therein.

The essence of a dynamic system in natural sciences or engineering is determined by the precision of the knowledge we have about the system's characteristics. A deterministic dynamical system emerges when information about a dynamic system is exact. However, most of the data obtainable for the modelling and assessment of dynamic system characteristics is incorrect, imprecise, or unclear. In other

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* Corresponding author: salim.abdelkrim@yahoo.com.

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terms, determining the parameters of a dynamical system is fraught with uncertainty. When we have statistical understanding about the parameters of a dynamic system, that is, when the knowledge is probabilistic, the most popular strategy in mathematical modelling of such systems is to employ random differential equations or stochastic differential equations. As natural extensions of deterministic differential equations, random differential equations appear in numerous applications and have been studied by several mathematicians; see the monographs [17, 25, 35] and the papers of Baleanu *et al.* [16], Boumaaza *et al.* [11, 34], Harikrishnan *et al.* [20], Karapinar *et al.* [23, 30], Liu *et al.* [26], and references therein.

Prompted by the aforementioned papers, in this paper, we consider the following problem:

$$({}^{c}D_0^{\zeta}y)(t,\eta,\delta) = \psi(t,\eta,y_{(t,\eta)},\delta), \qquad \text{if } (t,\eta)\in J := [0,\theta_1]\times[0,\theta_2],\ \delta\in\Psi, \tag{1}$$

$$y(t,\eta,\delta) = \varpi(t,\eta,\delta), \qquad \text{if } (t,\eta)\in \tilde{J} := [-\kappa_1,\theta_1]\times[-\kappa_2,\theta_2]\backslash(0,\theta_1]\times(0,\theta_2],\ \delta\in\Psi, \tag{2}$$

$$\begin{cases} y(t,0,\delta)+\varphi_1(y) = \varpi_1(t,\delta), & t\in[0,\theta_1],\\ y(0,\eta,\delta)+\varphi_2(y) = \varpi_2(\eta,\delta), & \eta\in[0,\theta_2], \end{cases} \qquad \delta\in\Psi, \tag{3}$$

where $\kappa_1,\kappa_2,\theta_1,\theta_2>0$, ${}^{c}D_0^{\zeta}$ is the standard Caputo's fractional derivative of order $\zeta=(\zeta_1,\zeta_2)\in(0,1]\times(0,1]$, $(\Psi,\mathcal{A},\nu)$ is a measurable space, $\psi\colon J\times C([-\kappa_1,0]\times[-\kappa_2,0],E)\times\Psi\to E$ is a given function, $\varpi\colon\tilde{J}\times\Psi\to E$, $\varphi_1,\varphi_2\colon C(J,E)\to E$ are given continuous functions, $\varpi_1\colon[0,\theta_1]\times\Psi\to E$, $\varpi_2\colon[0,\theta_2]\times\Psi\to E$ are given absolutely continuous functions with $\varpi_1(t,\delta)=\varpi(t,0,\delta)$, $\varpi_2(\eta,\delta)=\varpi(0,\eta,\delta)$ for each $t\in[0,\theta_1]$, $\eta\in[0,\theta_2]$, $\delta\in\Psi$, $\varphi_2(y(t,0,\delta))=\varphi_1(y(0,\eta,\delta))=0$ for each $t\in[0,\theta_1]$, $\eta\in[0,\theta_2]$, $\delta\in\Psi$. If $y\in\mathcal{S}_{(\theta_1,\theta_2)}=C([-\kappa_1,\theta_1]\times[-\kappa_2,\theta_2],E)$, then for any $(t,\eta)\in J$ define $y_{(t,\eta)}$ by

$$y_{(t,\eta)}(\tau,\varrho,\delta) = y(t+\tau,\eta+\varrho,\delta) \qquad \text{for } (\tau,\varrho)\in[-\kappa_1,0]\times[-\kappa_2,0].$$

Our research extends the existing studies on fractional problems with Caputo fractional derivatives, taking into account various conditions imposed on our problem within the abstract Banach space. It expands the scope of investigation to include problems involving delay and random variables, which introduces the need for additional requirements and different tools, such as random fixed point theorems with stochastic domains and the measure of noncompactness concept. Due to the scarcity of publications on Caputo partial fractional differential equations, our results within this specific framework are original and offer a substantial contribution to the current literature in this field of study.

The following is how this paper is organized. Section 2 contains definitions and lemmas that will be utilized throughout the work. Section 3 provides the existence results for the problem (1)–(3). In the final part, we present an example to demonstrate our main results.

2. Preliminaries

First, we introduce and explain the notations and concepts used in this study.

Denote by $L^1(J)$ the space of Bochner-integrable functions $y\colon J \to E$ with the norm

$$\|y\|_1 = \int_0^{\theta_1}\int_0^{\theta_2} \|y(t,\eta)\|_E d\eta dt.$$

Let $C := C(J,E)$ be the Banach space of continuous functions $y\colon J \to E$ with the norm

$$\|y\|_C = \sup_{(t,\eta)\in J} \|y(t,\eta)\|_E.$$

Denote by $L^\infty(\Psi,\nu)$ the Banach space of measurable functions $y\colon \Psi \to C$ which are essentially bounded equipped with the norm

$$\|y\|_\infty := \operatorname*{ess\,sup}_{\delta\in\Psi} \|y(\delta)\|_C = \inf\{c>0:\ \|y(\delta)\|_C \le c \ \ \nu - a.e.\ \Psi\}.$$

Consider the space $AC(J)$ of absolutely continuous functions from J into E.

Consider the σ-algebra $\mathfrak{D}_E$ of Borel subsets of E. The map $\bar{y}\colon \Psi \to E$ is measurable if for any $\Omega \in \mathfrak{D}_E$, we have

$$\bar{y}^{-1}(\Omega) = \{\delta \in \Psi:\ \bar{y}(\delta) \in \Omega\} \subset \mathcal{A}.$$

Definition 2.1
A mapping $\mathfrak{S}\colon \Psi \times E \to E$ is jointly measurable if for any $\Omega \in \mathfrak{D}_E$, we have

$$\mathfrak{S}^{-1}(\Omega) = \{(\delta,\bar{y}) \in \Psi \times E:\ \mathfrak{S}(\delta,\bar{y}) \in \Omega\} \subset \mathcal{A} \times \mathfrak{D}_E,$$

where $\mathcal{A} \times \mathfrak{D}_E$ is the direct product of the σ-algebras $\mathcal{A}$ and $\mathfrak{D}_E$ those defined in Ψ and E, respectively.

Lemma 2.2
Let $\mathfrak{S}\colon \Psi \times E \to E$ be a mapping such that $\mathfrak{S}(\cdot,\bar{y})$ is measurable for all $\bar{y} \in E$, and let $\mathfrak{S}(\delta,.)$ be continuous for all $\delta \in \Psi$. Then the map $(\delta,\bar{y}) \mapsto \mathfrak{S}(\delta,\bar{y})$ is jointly measurable.

Definition 2.3
A function $\psi\colon J \times E \times \Psi \to E$ is called random Carathéodory if the assumptions that follow are verified:

(i) the map $(t,\eta,\delta) \to \psi(t,\eta,y,\delta)$ is jointly measurable for all $y \in E$, and

(ii) $y \to \psi(t,\eta,y,\delta)$ is continuous for almost all $(t,\eta) \in J$ and $\delta \in \Psi$.

The map $\mathfrak{S}\colon \Psi \times E \to E$ is a random operator if $\mathfrak{S}(\delta,y)$ is measurable in δ for all $y \in E$ and it is given as $\mathfrak{S}(\delta)y = \mathfrak{S}(\delta,y)$. $\mathfrak{S}(\delta)$ is a random operator on E. A random operator $\mathfrak{S}(\delta)$ on E is continuous if $\mathfrak{S}(\delta,y)$ is continuous in y for all $\delta \in \Psi$. (See [21] for more details).

Definition 2.4 ([19])
Let $\mathcal{P}(\mathfrak{W})$ be the family of all nonempty subsets of $\mathfrak{W}$ and let $\mathfrak{F}$ be a mapping from Ψ into $\mathcal{P}(\mathfrak{W})$. $\mathfrak{S}\colon \{(\delta,\eta):\ \delta\in\Psi,\ \eta\in\mathfrak{F}(\delta)\}\to\mathfrak{W}$ is a random operator with stochastic domain $\mathfrak{F}$ if $\mathfrak{F}$ is measurable (i.e. for all closed $\Omega\subset\mathfrak{W}$, $\{\delta\in\Psi,\ \mathfrak{F}(\delta)\cap\Omega\neq\emptyset\}$ is measurable) and for all open $\tilde{\Omega}\subset\mathfrak{W}$ and all $\eta\in\mathfrak{W}$, $\{\delta\in\Psi:\ \eta\in\mathfrak{F}(\delta),\ \mathfrak{S}(\delta,\eta)\in\tilde{\Omega}\}$ is measurable. $\mathfrak{S}$ is continuous if every $\mathfrak{S}(\delta)$ is continuous. A mapping $\eta\colon\Psi\to\mathfrak{W}$ is a random fixed point of $\mathfrak{S}$ if for P-almost all $\delta\in\Psi$, $\eta(\delta)\in\mathfrak{F}(\delta)$ and $\mathfrak{S}(\delta)\eta(\delta)=\eta(\delta)$ and for all open $\tilde{\Omega}\subset\mathfrak{W}$, $\{\delta\in\Psi:\ \eta(\delta)\in\tilde{\Omega}\}$ is measurable.

Let $\mathcal{M}_{\bar{E}}$ denote the class of all bounded subsets of a metric space $\bar{E}$.

Definition 2.5 ([14])
Let $\bar{E}$ be a complete metric space. A map $\mu\colon\mathcal{M}_{\bar{E}}\to[0,\infty)$ is a measure of noncompactness on $\bar{E}$ if for all $\Omega,\Omega_1,\Omega_2\in\mathcal{M}_{\bar{E}}$, it verifies:

(MNC.1) $\mu(\Omega)=0$ if and only if Ω is precompact (regularity),

(MNC.2) $\mu(\Omega)=\mu(\overline{\Omega})$ (invariance under closure),

(MNC.3) $\mu(\Omega_1\cup\Omega_2)=\max\{\mu(\Omega_1),\mu(\Omega_2)\}$ (semi-additivity).

Example 2.6
In every metric space $\bar{E}$, the map $\varpi\colon\mathcal{M}_{\bar{E}}\to[0,\infty)$ with $\varpi(\Omega)=0$ if Ω is relatively compact and $\varpi(\Omega)=1$ otherwise is a measure of noncompactness ([15], Example1, p. 19).

Let $\varepsilon=(0,0)$, $\zeta_1,\,\zeta_2>0$ and $\zeta=(\zeta_1,\zeta_2)$. For $\psi\in L^1(J)$, the left-sided mixed Riemann-Liouville integral of order ζ is given by:

$$(I_\varepsilon^\zeta\psi)(t,\eta)=\frac{1}{\Gamma(\zeta_1)\Gamma(\zeta_2)}\int_0^t\int_0^\eta(t-\varrho)^{\zeta_1-1}(\eta-\tau)^{\zeta_2-1}\psi(\varrho,\tau)d\tau d\varrho.$$

In particular,

$$(I_\varepsilon^0 y)(t,\eta)=y(t,\eta),\quad (I_\varepsilon^\omega y)(t,\eta)=\int_0^t\int_0^\eta y(\varrho,\tau)d\tau d\varrho;\qquad \text{for a.a. } (t,\eta)\in J,$$

where $\omega=(1,1)$ and $1-\zeta$ means $(1-\zeta_1,1-\zeta_2)\in[0,1)\times[0,1)$. Denote by $D^2_{t\eta}:=\frac{\partial^2}{\partial t\partial\eta}$, the mixed second order partial derivative.

Definition 2.7 ([36])
Let $\zeta\in(0,1]\times(0,1]$ and $y\in AC(J)$. The Caputo fractional-order derivative of order ζ of y is given by:

$${}^cD_\varepsilon^\zeta y(t,\eta)=(I_\varepsilon^{1-\zeta}D^2_{t\eta}y)(t,\eta).$$

The case $\omega=(1,1)$ is included and we have

$$({}^cD_\varepsilon^\omega y)(t,\eta)=(D^2_{t\eta}y)(t,\eta)\qquad \text{for a.a. } (t,\eta)\in J.$$

Lemma 2.8 ([18])
If $\mathfrak{W}$ is a bounded subset of Banach space $\bar{E}$, then for each $\alpha > 0$, there is a sequence $\{\eta_\beta\}_{\beta=1}^{\infty} \subset \mathfrak{W}$ such that

$$\mu(\mathfrak{W}) \leq 2\mu(\{\eta_\beta\}_{\beta=1}^{\infty}) + \alpha.$$

Lemma 2.9 ([28])
If $\{y_\beta\}_{\beta=1}^{\infty} \subset L^1(J)$, then $\mu(\{y_\beta\}_{\beta=1}^{\infty})$ is measurable and for each $(t,\eta) \in J$,

$$\mu\left(\left\{\int_0^t \int_0^\eta y_\beta(\varrho,\tau)d\tau d\varrho\right\}_{\beta=1}^{\infty}\right) \leq 2\int_0^t \int_0^\eta \mu(\{y_\beta(\varrho,\tau)\}_{\beta=1}^{\infty})d\tau d\varrho.$$

Lemma 2.10 ([27])
Consider the continuous operator $\mathfrak{S}\colon \Lambda \to \Lambda$, where $\mathfrak{S}(\Lambda)$ is bounded and Λ is a convex and closed subset of a real Banach space. If there exists a constant $\beta \in [0,1)$ such that for each bounded subset $\Omega \subset \Lambda$,

$$\mu(\mathfrak{S}(\Omega)) \leq \beta\mu(\Omega),$$

then $\mathfrak{S}$ has a fixed point in Λ.

3. Existence Results

Let us start by giving the following result.

Lemma 3.1 ([2, 4])
Let $E \in L^1(J)$. The linear problem:

$$\begin{cases} {}^cD_\varepsilon^\zeta y(t,\eta) = E(t,\eta) & \text{for a.a. } (t,\eta) \in J := [0,\theta_1]\times[0,\theta_2], \\ y(t,0) = \varpi_1(t), & t \in [0,\theta_1], \\ y(0,\eta) = \varpi_2(\eta), & \eta \in [0,\theta_2], \\ \varpi_1(0) = \varpi_2(0), & \end{cases}$$

has the following unique solution:

$$y(t,\eta) = \varkappa(t,\eta) + I_\varepsilon^\zeta E(t,\eta) \qquad \text{for a.a. } (t,\eta) \in J,$$

where

$$\varkappa(t,\eta) = \varpi_1(t) + \varpi_2(\eta) - \varpi_1(0).$$

Suppose that ψ is random Carathéodory on $J \times C \times \Psi$.

Lemma 3.2
Let $0 < \zeta_1, \zeta_2 \leq 1$, $\varkappa(t,\eta,\delta) = \varpi_1(t,\delta) + \varpi_2(\eta,\delta) - \varpi_1(0,\delta)$. A function $y \in \Psi \times \mathcal{S}_{(\theta_1,\theta_2)}$ is a solution of (1)-(3) if and only if y verifies (2) for $(t,\eta) \in \tilde{J}$, $\delta \in \Psi$, and the fractional integral equation:

$$y(t,\eta,\delta) \\ = \varkappa(t,\eta,\delta) - \varphi_1(y) - \varphi_2(y) + \int_0^t \int_0^\eta \frac{(t-\tau)^{\zeta_1-1}(\eta-\varrho)^{\zeta_2-1}}{\Gamma(\zeta_1)\Gamma(\zeta_2)} \psi(t,\eta,y_{(\tau,\varrho)},\delta)d\varrho d\tau$$

for $(t,\eta) \in J,\ \delta \in \Psi$.

Proof. By following the same steps employed in the proof of Lemma 3.1 and accounting for the random variable and nonlocal conditions, which do not impact the proof's progression, we can infer the outcome.

The hypotheses:

(Ax_1) The functions $\delta \mapsto \varpi_1(t,\delta)$ and $\delta \mapsto \varpi_2(\eta,\delta)$ are measurable and essentially bounded for each $t \in [0,\theta_1]$ and $\eta \in [0,\theta_2]$, respectively.

(Ax_2) The function ϖ is measurable for $(t,\eta) \in \tilde{J} \times \Psi$.

(Ax_3) The function ψ is random Carathéodory on $J \times C \times \Psi$.

(Ax_4) There exist constants $\sigma^*, \tilde{\sigma} > 0$ such that

$$\|\varphi_1(y)\|_E \leq \sigma^*(1 + \|y\|_C)$$

and

$$\|\varphi_2(y)\|_E \leq \tilde{\sigma}(1 + \|y\|_C)$$

for $y \in C(J,E)$.

(Ax_5) There exist functions $\varsigma_1, \varsigma_2 \in L^\infty(\Psi, C(J,[0,\infty)))$ such that for each $(t,\eta) \in J$,

$$\|\psi(t,\eta,y,\delta)\|_E \leq \varsigma_1(t,\eta,\delta) + \varsigma_2(t,\eta,\delta)\|y\|_C,$$

for all $y \in C$ and a.e. $\delta \in \Psi$.

(Ax_6) For each $(t,\eta) \in J$, and any bounded $B \subset C$,

$$\mu(\psi(t,\eta,B,\delta)) \leq \varsigma_2(t,\eta,\delta)\mu_c(B) \qquad \text{for a.e. } \delta \in \Psi,$$

$$\mu(\varphi_1(B)) \leq \varsigma_3(t,\eta,\delta)\mu_c(B) \qquad \text{for a.e. } \delta \in \Psi$$

and

$$\mu(\varphi_2(B)) \leq \varsigma_4(t,\eta,\delta)\mu_c(B) \qquad \text{for a.e. } \delta \in \Psi,$$

where $\varsigma_3, \varsigma_4 \in L^\infty(\Psi, C(J,[0,\infty)))$. Here μ, μ_c are respectively, the measures of noncompactness on E and C.

Set

$$\varkappa^*(\delta) = \sup_{(t,\eta)\in J} \|\varkappa(t,\eta,\delta)\|_E.$$

Theorem 3.3
Suppose that (Ax_1)–(Ax_6) *are met. If*

$$\gamma := 2(\|\varsigma_3\|_\infty + \|\varsigma_4\|_\infty) + \frac{4\|\varsigma_2\|_\infty \theta_1^{\zeta_1}\theta_2^{\zeta_2}}{\Gamma(1+\zeta_1)\Gamma(1+\zeta_2)} < 1,$$

then (1)–(3) *admit a random solution on* $[-\kappa_1, \theta_1] \times [-\kappa_2, \theta_2]$.

Proof. Define the operator $\mathfrak{T}\colon \Psi \times \mathcal{S}_{(\theta_1,\theta_2)} \to \mathcal{S}_{(\theta_1,\theta_2)}$ by

$$(\mathfrak{T}(\delta)y)(t,\eta) = \begin{cases} \varpi(t,\eta,\delta), & (t,\eta) \in \tilde{J}, \\ \varkappa(t,\eta,\delta) - \varphi_1(y) - \varphi_2(y) \\ \quad + \dfrac{1}{\Gamma(\zeta_1)\Gamma(\zeta_2)} \displaystyle\int_0^t \int_0^\eta (t-\tau)^{\zeta_1-1}(\eta-\varrho)^{\zeta_2-1} \\ \quad \times \psi(\tau,\varrho,y_{(\tau,\varrho)},\delta)\,d\varrho\, d\tau, & (t,\eta) \in J, \end{cases} \qquad \delta \in \Psi.$$

Since the functions ϖ_1, ϖ_2 and φ_1, H and ψ are absolutely continuous, the function $\varkappa$ and the indefinite integral are absolutely continuous for all $\delta \in \Psi$ and almost all $(t,\eta) \in J$. And, since $\varkappa$ is continuous for all $\delta \in \Psi$, then $\mathfrak{T}(\delta)$ defines a mapping $\mathfrak{T}\colon \Psi \times \mathcal{S}_{(\theta_1,\theta_2)} \to \mathcal{S}_{(\theta_1,\theta_2)}$. Hence y is a solution for (1)–(3) if and only if $y = (\mathfrak{T}(\delta))y$. We will demonstrate in three steps that $\mathfrak{T}$ verifies all the requirements of Lemma 2.10.

Claim 1: $\mathfrak{T}(\delta)$ is a random operator with stochastic domain on $\mathcal{S}_{(\theta_1,\theta_2)}$. Since $\psi(t,\eta,y,\delta)$ is random Carathéodory, the map $\delta \to \psi(t,\eta,y,\delta)$ is measurable in view of Definition 2.1. Also, the product $(t-\tau)^{\zeta_1-1}(\eta-\varrho)^{\zeta_2-1}\psi(\tau,\varrho,y_{(\tau,\varrho)},\delta)$ is measurable. Then

$$\begin{aligned} \delta \mapsto{} & \varkappa(t,\eta,\delta) - \varphi_1(y) - \varphi_2(y) \\ & + \frac{1}{\Gamma(\zeta_1)\Gamma(\zeta_2)} \int_0^t \int_0^\eta (t-\tau)^{\zeta_1-1}(\eta-\varrho)^{\zeta_2-1}\psi(\tau,\varrho,y_{(\tau,\varrho)},\delta)\,d\varrho\, d\tau, \end{aligned}$$

is measurable. Consequently, $\mathfrak{T}$ is a random operator on $\Psi \times \mathcal{S}_{(\theta_1,\theta_2)}$ into $\mathcal{S}_{(\theta_1,\theta_2)}$.

Consider the operator $\mathfrak{X}\colon \Psi \to \mathcal{P}(\mathcal{S}_{(\theta_1,\theta_2)})$ given by

$$\mathfrak{X}(\delta) = \{y \in \mathcal{S}_{(\theta_1,\theta_2)} : \ \|y\|_\infty \le R(\delta)\},$$

where $R(\cdot)$ is a measurable function such that

$$R(\delta) \ge \frac{\rho_1}{\rho_2},$$

where

$$\rho_1 = \varkappa^*(\delta) + \tilde{\sigma} + \sigma^* + \frac{\|\varsigma_1\|_\infty \theta_1^{\zeta_1}\theta_2^{\zeta_2}}{\Gamma(1+\zeta_1)\Gamma(1+\zeta_2)},$$

$$\rho_2 = 1 - \frac{\|\varsigma_2\|_\infty \theta_1^{\zeta_1}\theta_2^{\zeta_2}}{\Gamma(1+\zeta_1)\Gamma(1+\zeta_2)} - \tilde{\sigma} - \sigma^*,$$

and

$$\frac{\|\varsigma_2\|_\infty \theta_1^{\zeta_1}\theta_2^{\zeta_2}}{\Gamma(1+\zeta_1)\Gamma(1+\zeta_2)} + \tilde{\sigma} + \sigma^* < 1,$$

and by (Ax_1), (Ax_2) and (Ax_3) we have that $\mathfrak{X}(\delta)$ is closed, convex, bounded and solid for all $\delta \in \Psi$. Then $\mathfrak{X}$ is measurable (see [19]). Let $\delta \in \Psi$ be fixed, then from (Ax_4) and (Ax_5), for any $y \in \mathfrak{X}(\delta)$, we obtain

$$\begin{aligned}
&\|(\mathfrak{T}(\delta)y)(t,\eta)\|_E \\
&\quad \leq \|\varkappa(t,\eta,\delta)\|_E + \|\varphi_1(y)\|_E + \|\varphi_2(y)\|_E \\
&\qquad + \int_0^t\int_0^\eta \frac{(t-\tau)^{\zeta_1-1}(\eta-\varrho)^{\zeta_2-1}}{\Gamma(\zeta_1)\Gamma(\zeta_2)} \|\psi(\tau,\varrho,y_{(\tau,\varrho)},\delta)\|_E d\varrho d\tau \\
&\quad \leq \|\varkappa(t,\eta,\delta)\|_E + \tilde{\sigma}(1+\|y\|_C) + \sigma^*(1+\|y\|_C) \\
&\qquad + \frac{1}{\Gamma(\zeta_1)\Gamma(\zeta_2)} \int_0^t\int_0^\eta (t-\tau)^{\zeta_1-1}(\eta-\varrho)^{\zeta_2-1}\varsigma_1(\tau,\varrho,\delta) d\varrho d\tau \\
&\qquad + \frac{1}{\Gamma(\zeta_1)\Gamma(\zeta_2)} \int_0^t\int_0^\eta (t-\tau)^{\zeta_1-1}(\eta-\varrho)^{\zeta_2-1}\varsigma_2(\tau,\varrho,\delta)\|y_{(\tau,\varrho)}\|_\infty d\varrho d\tau \\
&\quad \leq \varkappa^*(\delta) + (\tilde{\sigma}+\sigma^*)(1+R(\delta)) \\
&\qquad + \frac{\|\varsigma_1\|_\infty}{\Gamma(\zeta_1)\Gamma(\zeta_2)} \int_0^t\int_0^\eta (t-\tau)^{\zeta_1-1}(\eta-\varrho)^{\zeta_2-1} d\varrho d\tau \\
&\qquad + \frac{\|\varsigma_2\|_\infty R(\delta)}{\Gamma(\zeta_1)\Gamma(\zeta_2)} \int_0^t\int_0^\eta (t-\tau)^{\zeta_1-1}(\eta-\varrho)^{\zeta_2-1} d\varrho d\tau \\
&\quad \leq \varkappa^*(\delta) + (\tilde{\sigma}+\sigma^*)(1+R(\delta)) + \frac{(\|\varsigma_1\|_\infty + \|\varsigma_2\|_\infty R(\delta))\theta_1^{\zeta_1}\theta_2^{\zeta_2}}{\Gamma(1+\zeta_1)\Gamma(1+\zeta_2)} \\
&\quad \leq R(\delta).
\end{aligned}$$

Thus, $\mathfrak{T}$ is a random operator with stochastic domain $\mathfrak{X}$ and $\mathfrak{T}(\delta)\colon \mathfrak{X}(\delta) \to \mathfrak{T}(\delta)$. Moreover, $\mathfrak{T}(\delta)$ maps bounded sets into bounded sets in $\mathcal{S}_{(\theta_1,\theta_2)}$.

Claim 2: $\mathfrak{T}(\delta)$ is continuous.

Consider the sequence $\{y_n\}$, where $y_n \to y$ in $\mathcal{S}_{(\theta_1,\theta_2)}$. Then, for each $(t,\eta) \in J$ and $\delta \in \Psi$, we have

$$\begin{aligned}
\|(\mathfrak{T}(\delta)y_n)(t,\eta) - &(\mathfrak{T}(\delta)y)(t,\eta)\|_E \\
&\leq \|\varphi_1(y_n) - \varphi_1(y)\|_E + \|\varphi_2(y_n) - \varphi_2(y)\|_E \\
&\quad + \frac{1}{\Gamma(\zeta_1)\Gamma(\zeta_2)} \int_0^t\int_0^\eta (t-\tau)^{\zeta_1-1}(\eta-\varrho)^{\zeta_2-1} \\
&\quad \times \|\psi(\tau,\varrho,y_{n(\tau,\varrho)},\delta) - \psi(\tau,\varrho,y_{(\tau,\varrho)},\delta)\|_E d\varrho d\tau.
\end{aligned}$$

Thus

$$\|\mathfrak{T}(\delta)y_n - \mathfrak{T}(\delta)y\|_C \quad \to \quad 0 \text{as } n \to \infty.$$

As a result, we can deduce that $\mathfrak{T}(\delta)\colon \mathfrak{X}(\delta) \to \mathfrak{T}(\delta)$ is a continuous random operator with stochastic domain $\mathfrak{X}$, and $\mathfrak{T}(\delta)(\mathfrak{X}(\delta))$ is bounded.

Claim 3: For each bounded subset B of $\mathfrak{X}(\delta)$ we have $\mu_c(\mathfrak{T}(\delta)B) \le \gamma\mu_c(B)$. Let $\delta \in \Psi$. For any $B \subset \mathfrak{X}$ and any $\alpha > 0$, and by Lemmas 2.8 and 2.9, there exists a sequence $\{y_n\}_{n=0}^{\infty} \subset B$, where for all $(t,\eta) \in J$, and by (Ax_6), we get

$$
\begin{aligned}
&\mu_c((\mathfrak{T}(\delta)B)(t,\eta))\\
&\quad= \mu\bigg(\bigg\{\varkappa(t,\eta) - \varphi_1(y) - \varphi_2(y)\\
&\qquad + \int_0^t\int_0^\eta \frac{(t-\tau)^{\zeta_1-1}(\eta-\varrho)^{\zeta_2-1}}{\Gamma(\zeta_1)\Gamma(\zeta_2)}\psi(\tau,\varrho,y_{(\tau,\varrho)},\delta)d\varrho d\tau:\ y\in B\bigg\}\bigg)\\
&\quad\le 2\mu\bigg\{-\varphi_1(y_n) - \varphi_2(y_n)\\
&\qquad + \int_0^t\int_0^\eta \frac{(t-\tau)^{\zeta_1-1}(\eta-\varrho)^{\zeta_2-1}}{\Gamma(\zeta_1)\Gamma(\zeta_2)}\psi(\tau,\varrho,y_{n(\tau,\varrho)},\delta)d\varrho d\tau\bigg\}_{n=1}^{\infty} + \alpha\\
&\quad\le 2\mu\{\varphi_1(y_n)\} + 2\mu\{\varphi_2(y_n)\}\\
&\qquad + 4\int_0^t\int_0^\eta \mu\bigg(\bigg\{\frac{(t-\tau)^{\zeta_1-1}(\eta-\varrho)^{\zeta_2-1}}{\Gamma(\zeta_1)\Gamma(\zeta_2)}\psi(\tau,\varrho,y_{n(\tau,\varrho)},\delta)\bigg\}_{n=1}^{\infty}\bigg)d\varrho d\tau + \alpha\\
&\quad\le 2\varsigma_3(\tau,\varrho,\delta)\mu(\{y_n\}_{n=1}^{\infty}) + 2\varsigma_4(\tau,\varrho,\delta)\mu(\{y_n\}_{n=1}^{\infty})\\
&\qquad + 4\int_0^t\int_0^\eta \frac{(t-\tau)^{\zeta_1-1}(\eta-\varrho)^{\zeta_2-1}}{\Gamma(\zeta_1)\Gamma(\zeta_2)}\mu(\{\psi(\tau,\varrho,y_{n(\tau,\varrho)},\delta)\}_{n=1}^{\infty})d\varrho d\tau + \alpha\\
&\quad\le 2\varsigma_3(\tau,\varrho,\delta)\mu_c(B) + 2\varsigma_4(\tau,\varrho,\delta)\mu_c(B)\\
&\qquad + 4\int_0^t\int_0^\eta \frac{(t-\tau)^{\zeta_1-1}(\eta-\varrho)^{\zeta_2-1}}{\Gamma(\zeta_1)\Gamma(\zeta_2)}\varsigma_2(\tau,\varrho,\delta)\mu(\{y_{n(\tau,\varrho)}\}_{n=1}^{\infty})d\varrho d\tau + \alpha\\
&\quad\le 2(\|\varsigma_3\|_\infty + \|\varsigma_4\|_\infty)\mu_c(B)\\
&\qquad + \bigg(4\int_0^t\int_0^\eta \frac{(t-\tau)^{\zeta_1-1}(\eta-\varrho)^{\zeta_2-1}}{\Gamma(\zeta_1)\Gamma(\zeta_2)}\varsigma_2(\tau,\varrho,\delta)d\tau d\varrho\bigg)\mu(\{y_n\}_{n=1}^{\infty}) + \alpha\\
&\quad\le 2(\|\varsigma_3\|_\infty + \|\varsigma_4\|_\infty)\mu_c(B)\\
&\qquad + \bigg(4\int_0^t\int_0^\eta \frac{(t-\tau)^{\zeta_1-1}(\eta-\varrho)^{\zeta_2-1}}{\Gamma(\zeta_1)\Gamma(\zeta_2)}\varsigma_2(\tau,\varrho,\delta)d\varrho d\tau\bigg)\mu_c(B) + \alpha\\
&\quad\le \bigg(2(\|\varsigma_3\|_\infty + \|\varsigma_4\|_\infty) + \frac{4\|\varsigma_2\|_\infty{\theta_1}^{\zeta_1}{\theta_2}^{\zeta_2}}{\Gamma(1+\zeta_1)\Gamma(1+\zeta_2)}\bigg)\mu_c(B) + \alpha\\
&\quad= \gamma\mu_c(B) + \alpha.
\end{aligned}
$$

Since $\alpha > 0$ is arbitrary, then

$$\mu_c(\mathfrak{T}(B)) \le \gamma\mu_c(B).$$

Lemma 2.10 implies that for each $\delta \in \Psi$, $\mathfrak{T}$ has at least one fixed point in $\mathfrak{X}$. Since $\bigcap_{\delta\in\Psi} \mathrm{int}\mathfrak{X}(\delta) \neq \emptyset$, and we have the existence of a measurable selector of $int\mathfrak{X}$, Lemma 2.10 implies that $\mathfrak{T}$ has a stochastic fixed point, i.e. the problem (1)–(3) has at least one random solution on $\mathcal{S}_{(\theta_1,\theta_2)}$.

4. An Example

Let $E = \mathbb{R}$, $\Psi = (-\infty, 0)$ be equipped with the usual σ-algebra consisting of Borel measurable subsets of $(-\infty, 0)$. Consider the following problem:

$$({}^cD_0^\zeta y)(t,\eta,\delta) = \psi(t,\eta,y_{(t,\eta)},\delta), \qquad \text{if } (t,\eta) \in J := [0,1]\times[0,1],\ \delta \in \Psi, \tag{4}$$

$$\varpi(t,\eta,\delta) = t\sin\delta + \eta^2\cos\delta \quad \text{if } (t,\eta) \in \tilde{J} := [-1,1]\times[-2,1]\backslash(0,1]\times(0,1],\ \delta\in\Psi, \tag{5}$$

$$\begin{cases} y(t,0,\delta) + \varphi_1(y) = t\sin\delta, & t\in[0,1], \\ y(0,\eta,\delta) + \varphi_2(y) = \eta^2\cos\delta, & \eta\in[0,1], \end{cases} \qquad \delta\in\Psi \tag{6}$$

for $(t,\eta) \in [0,1]\times[0,1]$, $\delta \in \Psi$, where $\zeta = (\zeta_1,\zeta_2) \in (0,1]\times(0,1]$, function $\psi\colon [0,1]\times[0,1]\times C([-1,1]\times[-2,1],E)\times\Psi \to E$ is such that

$$\psi(t,\eta,y_{(t,\eta)},\delta) = 1 + \frac{ct\eta^2|y(t-1,\eta-2,\delta)|}{(1+\delta^2)e^{t+\eta+5}},$$

with $c > 0$, and

$$\begin{cases} (\varphi_1 y)(t,\eta,\delta) = \dfrac{d_1\eta}{1+\delta^2}(1+\|y\|_C), & t\in[0,1], \\ (\varphi_2 y)(t,\eta,\delta) = \dfrac{d_2 t}{1+\delta^2}(1+\|y\|_C), & \eta\in[0,1], \end{cases}$$

with $d_1, d_2 > 0$. We can observe that the hypotheses (Ax_1), (Ax_2) and (Ax_4) are satisfied.

The function $(t,\eta,\delta) \mapsto \psi(t,\eta,y,\delta)$ is jointly continuous for all $y \in C([-1,1]\times[-2,1],E)$ and hence jointly measurable for all $y \in C([-1,1]\times[-2,1],E)$. Also the map $y \mapsto \psi(t,\eta,y,\delta)$ is continuous for all $(t,\eta)\in[0,1]\times[0,1]$ and $\delta\in\Psi$. Thus, ψ is Carathéodory and (Ax_3) is verified. For each $y\in\mathbb{R}$, $(t,\eta)\in[0,1]\times[0,1]$ and $\delta\in\Psi$ we get

$$|\psi(t,\eta,y,\delta)| \le 1 + \frac{c|y|}{e^5}.$$

Hence, (Ax_5) and (Ax_6) are verified with

$$\|\varsigma_1\|_\infty = 1, \quad \|\varsigma_2\|_\infty = ce^{-5}, \quad \varsigma_3 = \frac{d_1\eta}{1+\delta^2}, \quad \varsigma_4 = \frac{d_2 t}{1+\delta^2}.$$

A simple computations show that all conditions of Theorem 3.3 are satisfied for a good choice of the constants c, d_1 and d_2. It follows that the random problem (4)–(6) has at least one random solution on $[-1,1]\times[-2,1]$.

Declarations

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Amel Heris
Laboratory of Mathematics
Djillali Liabes University of Sidi Bel-Abbes
P.O. Box 89, Sidi Bel-Abbes 22000
Algeria
E-mail: herisamel@gmail.com

Abdelkrim Salim
Laboratory of Mathematics
Djillali Liabes University of Sidi Bel-Abbes
P.O. Box 89, Sidi Bel-Abbes 22000
Faculty of Technology
Hassiba Benbouali University of Chlef
P.O. Box 151 Chlef 02000
Algeria
E-mail: a.salim@univ-chlef.dz

Mouffak Benchohra
Laboratory of Mathematics
Djillali Liabes University of Sidi Bel-Abbes
P.O. Box 89, Sidi Bel-Abbes 22000
Algeria
E-mail: benchohra@yahoo.com

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