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Report of Meeting

20th International Conference on Functional Equations and Inequalities, Będlewo, Poland, September 17–23, 2023

The **20th International Conference on Functional Equations and Inequalities** (20th ICFEI) took place in the *Mathematical Research and Conference Center* in Będlewo (Poland) on September 17–23, 2023. It was organized by the *Department of Mathematics of the Pedagogical University of Krakow*.

The Scientific Committee of the 20th ICFEI consisted of Professors: Nicole Brillouët-Belluot (France), Janusz Brzdęk (Poland) – chairman, Jacek Chmieliński (Poland), Roman Ger (Poland), Zsolt Páles (Hungary), Dorian Popa (Romania), Ekaterina Shulman (Poland/Russia), Henrik Stetkær (Denmark), László Székelyhidi (Hungary) and Marek Cezary Zdun (Poland). The Organizing Committee consisted of Jacek Chmieliński (chairman), Beata Deręgowska, Zbigniew Leśniak, Paweł Pasteczka, Paweł Solarz and Paweł Wójcik.

There were 47 participants present in Będlewo. All together 13 countries were represented: Poland (24 participants), Hungary (4), Germany (3), India (3), Austria (2), Iran (2), Romania (2), Thailand (2), and – with one participant – Croatia, Italy, Japan, Namibia, United Kingdom.

The conference was officially opened on Monday, September 17, by Professor Jacek Chmieliński – chairman of the organizing committee. Then, Professor Zbigniew Leśniak presented some historical remarks, related to the jubilee of the twentieth conference of the series.

During 19 scientific sessions, 42 talks were presented. Five of them were longer invited lectures and were delivered by Professors: Jacek Chudziak, Tomasz Kochanek, Kallol Paul, Jens Schwaiger and Wutiphol Sintunavarat. The talks were devoted mainly to functional equations and inequalities, their stability, convexity, means, as well as to related topics in real analysis, functional analysis, applications of mathematics and others. Additionally, apart from regular talks, spontaneous contributions in *Problems and Remarks* sessions were delivered.

Some social events accompanied the conference. A picnic with a bonfire was organized on Tuesday evening and a banquet on Thursday. On Wednesday afternoon there was organized an excursion to Rogalin, with a visit in the 18th-century palace of the Raczyński family and in the Raczyński Art Gallery with an exhibition of Polish and foreign paintings. The trip ended with a walk through the surrounding park with 800-year-old oak trees.

The conference was closed on Saturday, September 23, by Professor Jacek Chmieliński on behalf of the Organizing and Scientific Committee. It was announced that the subsequent 21th ICFEI is planned to be organized on September 14–20, 2025, again in Będlewo.

1. Abstracts of Talks

Dennis Epebinu Abayomi *Inequalities for fractional integrals with use of stochastic ordering*

In this talk, we show the connections of inequalities involving fractional integrals with Ohlin lemma and Levin-Stechkin theorem. First, we give a new proof of the fractional version of the inequality of Hermite Hadamard type and then we extend it in two directions. Thus we compare the fractional expression occurring in this inequality with the usual integral and we obtain stronger inequalities (one of them is related to Bullen inequality).

Ulrich Abel *Characterizing convexity of functions by inequalities of Bernstein polynomials* (joint work with Dany Leviatan and Ioan Raşa)

About 30 years ago Ioan Raşa stated the following inequality involving Bernstein basis polynomials $p_{n,\nu}(x) = \binom{n}{\nu} x^\nu (1-x)^{n-\nu}$ and convex functions as an open problem:

PROBLEM

Prove or disprove:

Let $n \in \mathbb{N}$. If $f \in C[0, 1]$ is a convex function, then, for all $x, y \in [0, 1]$,

$$\sum_{i=0}^n \sum_{j=0}^n [p_{n,i}(x)p_{n,j}(x) + p_{n,i}(y)p_{n,j}(y) - 2p_{n,i}(x)p_{n,j}(y)] f\left(\frac{i+j}{2n}\right) \geq 0.$$

In 2017, Mrowiec, Rajba and Wąsowicz [5] affirmed the conjecture in positive. Their proof makes heavy use of probability theory. A short elementary proof of the above inequality and the corresponding results for Mirakyan–Favard–Szász operators and Baskakov operators can be found in [1].

In 2020, Abel and Leviatan [2] found an inequality generalizing the above one to q -monotone functions on $[0, 1]$, for any positive integer q . In 2022, they and Raşa [3] proved a converse for functions in $C[0, 1]$. The combination of both results provides a characterization of q -monotone functions in $C[0, 1]$.

Finally, we study the preservation of q -monotonicity of various Durrmeyer-type operators in $[0, 1]$ or $[0, \infty)$ as the case may be. These results yield further characterizations of continuous q -monotone functions.

References

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Ana-Maria Acu *Inequalities for Aldaz-Kounchev-Render operators*

The Aldaz-Kounchev-Render operators are positive linear operators on $C[0, 1]$ which preserve simultaneously the functions $\mathbf{1}$ and x^j for a given integer $j \geq 2$. Recently, this type of operators have been considered on finite-dimensional hypercubes and simplices; see [1], [2], [3], [4]. Besides the approximation properties we investigate their behavior with respect to the convex functions.

References

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Laddawan Aiemsomboon *Fixed Point Approach to Stability of an Additive-Quadratic Functional Equation on Orthogonality in the sense of Rätz* (joint work with Wutiphol Sintunavarat and Anurak Thanyacharoen)

Let (X, \perp) be an orthogonality Banach space in the sense of Rätz and Y be a Banach space. In this talk, we show applications of the alternative fixed point theorem for proving the Hyers-Ulam stability of the generalized orthogonally additive-quadratic functional equation of the form

$$f(ax + by) + f(ax - by) + 2b^2 f(y) = (a^2 + a)f(x) + (a^2 - a)f(-x) + b^2 f(2y)$$

for all $x, y \in X$ with $x \perp y$, where a, b are non-zero rational numbers and $f: X \rightarrow Y$ is an unknown function.

Roman Badora *On a separation of two functionals by an additive map*

In the talk we will deal with the problem of separating two functionals defined on a semigroup by an additive map. We examine whether the fulfillment of a given system of inequalities by two functionals guarantees the existence of an additive map separating these functionals. The classical separation theorem for subadditive and superadditive functionals provides an example of a system of inequalities guaranteeing separation by an additive map. In the talk we answer the question of how it is for other systems of inequalities.

Karol Baron *Around Hölder continuous solutions of an iterative equation*

Assume that (X, ρ) is a complete and separable metric space, (Ω, \mathcal{A}, P) is a probability space, $f: X \times \Omega \rightarrow X$ is measurable for $\mathcal{B} \otimes \mathcal{A}$, where \mathcal{B} denotes the σ -algebra of all Borel subsets of X ,

$$\int_{\Omega} \rho(f(x, \omega), f(z, \omega)) P(d\omega) \leq \lambda \rho(x, z) \quad \text{for } x, z \in X$$

with a $\lambda \in (0, 1)$, and

$$\int_{\Omega} \rho(f(x, \omega), x) P(d\omega) < \infty \quad \text{for } x \in X.$$

Given $F: X \rightarrow \mathbb{R}$ we consider the problem of the existence and uniqueness of solutions $\varphi: X \rightarrow \mathbb{R}$ of the equation

$$\varphi(x) = \int_{\Omega} \varphi(f(x, \omega)) P(d\omega) + F(x)$$

such that $\varphi - F$ is Hölder continuous.

Pál Burai *Limit theorems for Bajraktarević and Cauchy quotient means of independent identically distributed random variables* (joint work with Mátyás Barczy)

We derive strong law of large numbers and central limit theorems for Bajraktarević, Gini and exponential- (also called Beta-type) and logarithmic Cauchy quotient means of independent identically distributed (i.i.d.) random variables. The exponential- and logarithmic Cauchy quotient means of a sequence of i.i.d. random variables behave asymptotically normal with the usual square root scaling just like the geometric means of the given random variables. Somewhat surprisingly, the multiplicative Cauchy quotient means of i.i.d. random variables behave asymptotically in a rather different way: in order to get a non-trivial normal limit distribution a time dependent centering is needed.

Jacek Chmieliński *On continuity of norms*

In my talk given at the 19th ICFEI the results published in [1] were presented. Continuing this topic, we will present some new relationships between two norms

on an infinite-dimensional vector space X . We consider, in particular, the unilateral continuity of one norm with respect to the other, answering the two questions:

1. Does any norm N in X admit another norm N' in X such that N is continuous with respect to N' but N' is discontinuous with respect to N ?
2. Does any norm N in X admit a norm N' in X such that N' is continuous with respect to N but N is discontinuous with respect to N' ?

References

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Jacek Chudziak LECTURE: *On representations of the certainty equivalents for binary lotteries* (joint work with Michał Lewandowski)

Let $(x, y; p)$ denote a binary monetary lottery that pays x with probability p , and y with probability $1 - p$. Assume that $F(x, y; p)$ is the certainty equivalent of $(x, y; p)$, i.e. a certain amount, the receipt of which is as good for the decision maker as playing the lottery. We are interested in individual preferences that lead to the following representation of the certainty equivalent

$$F(x, y; p) = u^{-1}\left(w(p)u(x) + (1 - w(p))u(y)\right), \quad (1)$$

where $u: \mathbb{R} \rightarrow \mathbb{R}$ is a continuous strictly increasing utility function and the function $w: [0, 1] \rightarrow [0, 1]$ is a probability weighting function. We consider several axioms leading to (1). In particular, we characterize (1) in the domain of all binary lotteries (arbitrary payouts and probabilities) as well as in the subset of simple lotteries, where one of the two payouts of a lottery is fixed. For both domains we deal with the case of the general w and the case where w is the identity function. The latter case corresponds to the certainty equivalent of a binary lottery under the expected utility theory. In addition, for the general w , we separately characterize the case where $F(x, y; p)$ depends on the rank of the payouts (rank dependency) and the case where it does not.

Farzad Dadipour *From the triangle inequality to characterization of p -Banach spaces* (joint work with Farzane Sadeghi Boome)

In this talk, we deal with the generalized triangle inequality of the second type and its reverse in the framework of quasi Banach spaces. More precisely, by using the well-known Aoki-Rolewicz theorem and the concept of equivalent p -norms, we provide some necessary and sufficient conditions for n -tuples to satisfy the mentioned inequalities.

As applications, we improve some already known results and present some characterizations of p -Banach spaces among quasi Banach spaces. In particular,

we show that a quasi Banach space such as X is a p -Banach space if and only if for all $(\mu_1, \dots, \mu_n) \in \mathbb{R}^n$ satisfying $\mu_j > 0$ for some j and $\mu_i < 0$ for all $i \neq j$, the reverse of the generalized triangle inequality of the second type

$$\sum_{i=1}^n \frac{\|x_i\|^p}{\mu_i} \leq \left\| \sum_{i=1}^n x_i \right\|^p \quad (\text{where } x_i \in X)$$

holds only with the assumption $\mu_j \geq \max_{i \in \{1, \dots, n\} - \{j\}} \{1, |\mu_i|\}$.

References

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Beata Deręgowska *On maximal projection constants*

Let X be a Banach space over \mathbb{K} , where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Let $Y \subseteq X$ be a subspace. By $\mathcal{P}(X, Y)$ we denote the set of all linear and continuous projections from X onto Y , recalling that an operator $P: X \rightarrow Y$ is called a *projection* onto Y if $P|_Y = \text{Id}_Y$. We define the *relative projection constant of subspace Y* of space X by

$$\lambda(Y, X) := \inf \{ \|P\| : P \in \mathcal{P}(X, Y) \}.$$

Now we can define the *absolute projection constant of Y* by

$$\lambda(Y) := \sup \{ \lambda(Y, X) : Y \subseteq X \}.$$

The ultimate goal of researchers in this area is to determine the exact value of *maximal absolute projection constant*, which is defined by

$$\lambda_{\mathbb{K}}(m) := \sup \{ \lambda(Y) : \dim(Y) = m \}.$$

In 1994, H. König and N. Tomczak-Jaegermann stated the following estimation.

THEOREM 1 (stated in [3]; proved in [1])

Let $m > 1$ then

1. $\lambda_{\mathbb{R}}(m) \leq \frac{2}{m+1} \left(1 + \frac{m-1}{2} \sqrt{m+2} \right),$
2. $\lambda_{\mathbb{C}}(m) \leq \frac{1}{m} \left(1 + (m-1) \sqrt{m+1} \right).$

Unfortunately, their proof is based on an erroneous lemma, as was pointed out in [2]. In this talk, we will present the correct proof of the latter. Moreover relying on this result we provide the exact values of $\lambda_{\mathbb{K}}(m)$ in the cases where the maximal equiangular tight frame exists in \mathbb{K}^m .

References

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Włodzimierz Fechner *Some remarks on the Casino Inequality of Dubins and Savage*

L.E. Dubins and L.J. Savage in their seminal book *How to gamble if you must. Inequalities for stochastic processes* [1] provided formal definitions of many fundamental notions of gambling, like gambles and lotteries, gambling house (as a family of gambles available for the gambler at a given moment) and a casino (a special type of a gambling house). They introduced the *casino inequality*:

$$U(xy + (1-x)z) \geq U(x)U(y) + (1-U(x))U(z), \quad (1)$$

with $x \in [0, 1]$, $0 \leq z \leq y \leq 1$, where $U: [0, \infty) \rightarrow [0, 1]$ satisfies $U(0) = 0$ and $U(x) = 1$ for $x \geq 1$. The function U represents the optimal probability of victory in a casino of a fixed goal of 1 (i.e. of reaching 1 starting with a given fortune) and x, y, z are possible fortunes. The fundamental theorem [1, Theorem 4.2.1, p. 64] establishes an equivalence between solutions of (1) and optimal probabilities of attaining a fixed goal in a casino.

During the talk we collect fundamental properties of the inequality (1). Further, we exploit its connections with other functional inequalities, in particular with the Hosszú inequality.

References

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Gian Luigi Forti *Some classes of C^1 solutions of an alternative Cauchy functional equation in three unknown functions*

During ICFEI 19 it has been proved that if the functions f, g, h are real analytic solutions of the alternative Cauchy equation

$$[f(x+y) - f(x) - f(y)][g(x+y) - g(x) - g(y)][h(x+y) - h(x) - h(y)] = 0,$$

then at least one of them must be linear (see [1]). The natural open question is to find a way to construct less regular solutions of the previous equation. By using the integral representation of the Cauchy difference, some classes of C^1 solutions of the alternative Cauchy functional equation are presented. Then, some open problems are presented.

References

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Dorota Głazowska *Subcommutativity of integrals similar to integral quasi-arithmetic means* (joint work with Paolo Leonetti, Janusz Matkowski and Salvatore Tringali)

Let $(X, \mathcal{L}, \lambda)$ and (Y, \mathcal{M}, μ) be finite non-degenerate measure spaces, $I \subseteq \mathbb{R}$ be a non-empty interval, and $f, g: I \rightarrow \mathbb{R}$ be continuous injections. We look for conditions on f and g under which the inequality

$$f^{-1}\left(\int_X f\left(g^{-1}\left(\int_Y g \circ h \, d\mu\right)\right) d\lambda\right) \leq g^{-1}\left(\int_Y g\left(f^{-1}\left(\int_X f \circ h \, d\lambda\right)\right) d\mu\right) \quad (1)$$

is satisfied for every $h: X \times Y \rightarrow I$ in a suitable class of $\mathcal{L} \otimes \mathcal{M}$ -measurable simple functions, taking for granted that each side of the above inequality is well defined. Notice that setting

$$\mathcal{M}_f(h) = f^{-1}\left(\int_X f \circ h \, d\lambda\right), \quad \mathcal{M}_g(h) = g^{-1}\left(\int_Y g \circ h \, d\mu\right)$$

for the $\mathcal{L} \otimes \mathcal{M}$ -measurable simple functions $h: X \times Y \rightarrow I$, the inequality (1) holds if

$$\mathcal{M}_f \circ \mathcal{M}_g \leq \mathcal{M}_g \circ \mathcal{M}_f,$$

and it can be interpreted as the "subcommutativity of the pair $(\mathcal{M}_f, \mathcal{M}_g)$ " or "supercommutativity of the pair $(\mathcal{M}_g, \mathcal{M}_f)$ ".

Moreover, if $(X, \mathcal{L}, \lambda)$ and (Y, \mathcal{M}, μ) are probability spaces and the image $h(X \times Y)$ is contained in a compact subset of I for every test function h , which is especially the case when $h: X \times Y \rightarrow I$ is an $\mathcal{L} \otimes \mathcal{M}$ -measurable simple function, then each side of the inequality (1) is well posed and can be interpreted as "partially mixed integral quasi-arithmetic means".

Chaitanya Gopalakrishna *On the nonexistence of iterative roots of functions* (joint work with B. V. Rajarama Bhat)

An *iterative root of order* $n \geq 2$ of a self-map f on a nonempty set X is a self-map g on X such that $g^n = f$. We discuss a new result on the nonexistence of iterative roots of self-maps on arbitrary sets and use it to prove that every nonempty open set of the space $\mathcal{C}([0, 1]^m)$ of all continuous self-maps on the unit cube $[0, 1]^m$ in \mathbb{R}^m contains a map that has not even discontinuous iterative roots of order $n \geq 2$. This, in particular, proves that continuous self-maps on $[0, 1]^m$ with no continuous iterative roots are dense in $\mathcal{C}([0, 1]^m)$. The talk is based on our recent works [1] and [2].

References

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Szymon Ignaciuk *Relationship between some angular inequalities and Kaplan classes for complex polynomials* (joint work with Maciej Parol)

The presented results concern a certain class of finite products of the form

$$\mathbb{D} \ni z \mapsto F_n(z; T_n; P_n) := \prod_{k=1}^n (1 - ze^{-it_k})^{p_k} \quad (1)$$

where $\mathbb{N} \ni n \mapsto T_n := (t_1, t_2, \dots, t_n)$ is an increasing sequence of values from $[0; 2\pi)$ such that $t_1 := 0$ and $\mathbb{N} \ni n \mapsto P_n := (p_1, p_2, \dots, p_n)$ is a sequence of real numbers of the same sign. The first results for polynomials with all zeros on unit circle \mathbb{T} (when P_n is a sequence of natural numbers) were given by Jahangiri [4] in terms of a gap condition. Complete membership to Kaplan classes for the polynomials was presented in [1]. In the paper [2] the authors carried out complete membership to Kaplan classes of finite products of the form similar to (1), but with zeros symmetrically situated in \mathbb{T} . The presented results from the paper [3] generalize ones from [1] and [2]. An interpretation of the obtained gap condition in terms of mass and density is given in the case where all zeros of the studied functions are situated in \mathbb{T} .

An open problem is a more sophisticated physical interpretation that allows for the consideration of zeros outside the unit circle. A visualization of the obtained gap condition in terms of angular inequalities between vectors in \mathbb{R}^2 is given.

References

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Justyna Jarczyk *On a linear iterative equation of finite order and its solutions with a prescribed asymptotics*

Fix a number $a \in (0, +\infty]$. We start with solutions $\varphi: (0, a) \rightarrow \mathbb{R}$ of the equation

$$\varphi(x) = \sum_{j=1}^n p_j(x) \varphi(f_j(x)), \quad (1)$$

where $p_1, \dots, p_n: (0, a) \rightarrow [0, +\infty)$ are given and $f_1, \dots, f_n: (0, a) \rightarrow (0, a)$ satisfy some natural assumptions and there exists a $\lambda \in \mathbb{N}_0$ such that

$$\sum_{j=1}^n p_j(x) \left(\frac{f_j(x)}{x} \right)^\lambda = 1$$

for all admissible x . Theorem 6.1.3 from the monograph [1] by Marek Kuczma, Bogdan Choczewski and Roman Ger shows that if φ satisfies (1) and the limit

$$\lim_{x \rightarrow 0} \frac{\varphi(x)}{x^\lambda} =: c$$

exists and is finite, then

$$\varphi(x) = cx^\lambda, \quad x \in (0, a).$$

We present a generalization of this result where it is enough to assume that λ is an arbitrary real number. It turns out that in fact something more can be proved. As consequences we obtain extensions of some results by Ernő Vincze, Marek Cezary Zdun, John A. Baker and, finally, Justyna and Witold Jarczyk.

References

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Witold Jarczyk *Iterative roots of piecewise monotonic functions extended from the characteristic interval* (joint work with Veerapazham Murugan, Murugan Suresh Kumar and Justyna Jarczyk)

We consider the problem of finding iterative roots of order less than the number of farts of continuous piecewise monotonic functions with nonmonotonicity height greater than 1. We present sufficient conditions to extend iterative roots of piecewise monotonic functions from the characteristic interval which determines the behaviour of the function under iteration.

Divya Khurana *Birkhoff-James orthogonality and its pointwise symmetry in some Banach spaces*

Let X be a real Banach space. If $x, y \in X$, then we say that x is *Birkhoff-James orthogonal to y* , written as $x \perp_B y$, if $\|x + \lambda y\| \geq \|x\|$ for all $\lambda \in \mathbb{R}$. Generally, the Birkhoff-James orthogonality need not be symmetric, that is, $x \perp_B y$ does not necessarily imply that $y \perp_B x$. An element $x \in X$ is said to be *left-symmetric* (resp. *right-symmetric*), if the following condition holds:

$$x \perp_B y \text{ (resp. } y \perp_B x) \Rightarrow y \perp_B x \text{ (resp. } x \perp_B y) \text{ for all } y \in X.$$

In this talk, we will discuss symmetric, left-symmetric, and right-symmetric points in some Banach spaces.

Tibor Kiss *Some solutions of the Járαι-Maksa-Páles functional equation*

The talk is about the solutions of the functional equation

$$F\left(\frac{x+y}{2}\right) + f_1(x) + f_2(y) = G(g_1(x) + g_2(y)), \quad x, y \in I,$$

where I stands for an open interval and all functions are assumed to be unknown and continuously differentiable. The talk covers irregular solutions.

References

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Milica Klaričić Bakula *Jensen-Steffensen Inequality: Accentuate the Negative*

Let $f: I \rightarrow \mathbb{R}$, where I is an interval in \mathbb{R} , be a convex function and let $\mathbf{x} = (x_1, \dots, x_n) \in I^n$. If $\mathbf{p} = (p_1, \dots, p_n)$ is a nonnegative real n -tuple such that $P_n = \sum_{i=1}^n p_i > 0$, then the well-known Jensen inequality

$$f\left(\frac{1}{P_n} \sum_{i=1}^n p_i x_i\right) \leq \frac{1}{P_n} \sum_{i=1}^n p_i f(x_i) \quad (1)$$

holds. To Steffensen's credit, it is known that the assumption " \mathbf{p} is a nonnegative real n -tuple" can be relaxed at the expense of further restrictions on the n -tuple \mathbf{x} . Namely, if $\mathbf{x} \in I^n$ is a monotonic (increasing or decreasing) n -tuple, then for any real n -tuple \mathbf{p} such that

$$0 \leq P_j = \sum_{i=1}^j p_i \leq P_n, \quad j = 1, \dots, n, \quad P_n > 0, \quad (2)$$

we get the condition

$$\bar{x} = \frac{1}{P_n} \sum_{i=1}^n p_i x_i \in I,$$

and (1) still holds. The inequality (1) under the conditions (2) is known as the Jensen-Steffensen inequality for convex functions.

One can say that the Jensen-Steffensen inequality is "the ugly sister" of the Jensen inequality: not much admired and usually "not invited to the party". Our goal here is to show that "she" has many hidden beauties and that "she" can proudly walk hand in hand with her well-known sister.

Paweł A. Kluza *Inequalities for Jensen–Sharma–Mittal and Jeffreys–Sharma–Mittal type f -divergences*

In this talk, we present new divergences called Jensen–Sharma–Mittal and Jeffreys–Sharma–Mittal in relation to convex functions. Some theorems, which give the lower and upper bounds for two new introduced divergences are provided. The obtained results imply some new inequalities corresponding to known divergences. Some examples, which show that it is generalization of Rényi, Tsallis and Kullback–Leibler types of divergences are provided in order to show a few applications of new divergences.

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Tomasz Kochanek LECTURE: *Approximate multiplicativity and zero-product preserving in Banach algebras*

Through last several decades, various aspects of Ulam stability have become fruitful areas of research in Banach algebra theory. Among them, there are three general subjects which appeared to be of special importance, and which are all underpinned by the two main themes mentioned in the title:

- approximately multiplicative maps and the theory of AMNM pairs,
- approximate representations and $*$ -homomorphisms,
- almost zero-product-preserving maps.

The first subject was initiated by B.E. Johnson in *Memoirs AMS* (1972) by introducing cohomology theory of Banach algebras and notably revealing its connections with the notion of amenability, generalized from group theory to the Banach algebras setting. These new tools were then widely developed in his two seminal papers published in *J. London Math. Soc.* (1986–88), where he defined AMNM pairs as those pairs of Banach algebras for which an appropriate Ulam's stability effect holds true for approximate multiplicativity. The second subject has its roots in D. Kazhdan's stability theorem on approximate unitary representations (*Israel J. Math.* 1982), most noticeably used by I. Farah in his celebrated paper on automorphisms of the Calkin algebra published in *Ann. Math.* (2011). The last mentioned subject is a wide area of study concerning zero-product structures of Banach algebras, disjointness preserving operators and order zero maps, with some remarkable results obtained in commutative case by J. Araujo and J.J. Font (*J. London Math. Soc.* 2009–10), and for Fourier algebras, group algebras and C^* -algebras by J. Alaminos, J. Extremera and A.R. Villena (*Israel J. Math.* 2010).

During the talk, we shall present an overview of the most important classical results in these topics, as well as a discussion of the newest results published in various papers during recent years by J. Alaminos, Y. Choi, J. Extremera, M. Ghandehari, M.L.C. Godoy, B. Horváth, N.J. Laustsen, H.L. Pham, A.R. Villena, and the author.

Dawid Komorek *Continuous dependence of the weak limit of iterates of some random-valued vector functions*

Given a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, a complete separable Banach space X with the σ -algebra $\mathcal{B}(X)$ of all its Borel subsets, an operator $\Lambda: \Omega \rightarrow L(X, X)$ and $\xi: \Omega \rightarrow X$ we consider the $\mathcal{B}(X) \otimes \mathcal{A}$ -measurable function $f: X \times \Omega \rightarrow X$ given by $f(x, \omega) = \Lambda(\omega)x + \xi(\omega)$ and investigate the continuous dependence of the weak limit π^f of the sequence of iterates $(f^n(x, \cdot))_{n \in \mathbb{N}}$ of f , defined by $f^0(x, \omega) = x$, $f^{n+1}(x, \omega) = f(f^n(x, \omega), \omega_{n+1})$ for $x \in X$ and $\omega = (\omega_1, \omega_2, \dots)$. Moreover for X taken as a Hilbert space we characterize π^f via the functional equation

$$\varphi^f(u) = \int_{\Omega} \varphi^f(\Lambda(\omega)u) \varphi^\xi(u) \mathbb{P}(d\omega)$$

with the aid of its characteristic function φ^f . We also indicate the continuous dependence of the solution of that equation.

References

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Zbigniew Leśniak *On foliations consisting of invariant lines of a Brouwer homeomorphism*

We study properties of a Brouwer homeomorphism f for which there exists a foliation of the plane with leaves being invariant lines of f . For such a homeomorphism, we find the form of limit sets of strongly irregular points. Assuming the matching property for the leaves of the foliation contained in the set of strongly irregular points, we can construct continuous iterative roots of the Brouwer homeomorphism f .

References

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Joanna Markowicz *García-Falset coefficient and Opial property in direct sums of Banach spaces*

We present some results on the García-Falset coefficient and the Opial property in direct sums of Banach spaces. Constructing a general direct sum, we use a function lattice called a *substitution space* E . The *García-Falset coefficient* $R(X)$ of a Banach space X is a classical geometric parameter used in metric fixed point theory. In [1] J. García-Falset introduced it by

$$R(X) = \sup \left\{ \liminf_{n \rightarrow \infty} \|x + x_n\| : x, x_n \in B(X), x_n \xrightarrow{weak} 0 \right\},$$

where $B(X)$ denotes the unit ball of X (see [1]). It characterizes the presence of a weak fixed point property for nonexpansive mappings in Banach spaces. We present an estimate for the García-Falset coefficient in the direct sum $Y = (\sum X_i)_E$ in terms of the supremum of $R(X_i)$ and the Riesz angle $\alpha(E)$ of the lattice E . Here, the Opial property is studied in the context of direct sums. We say that a Banach space X has the *non-strict Opial property* if

$$\liminf_{n \rightarrow \infty} \|x_n\| \leq \liminf_{n \rightarrow \infty} \|x_n - x\|$$

for every weakly null sequence (x_n) in X and every $x \in X$. If

$$\liminf_{n \rightarrow \infty} \|x_n\| < \liminf_{n \rightarrow \infty} \|x_n - x\|$$

for every weakly null sequence (x_n) in X and $x \in X \setminus \{0\}$, we say that X has the *Opial property*. We show that these properties are preserved under passing to direct sums. We present a new modulus corresponding to uniform Opial property and provide an estimate for the modulus of the direct sum in terms of the moduli of the spaces X_i and the modulus of monotonicity of E . These results contribute to a deeper understanding of direct sums of Banach spaces.

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Kazuki Okamura *Properties of quasi-arithmetic means of random variables* (joint work with Yuichi Akaoka and Yoshiki Otobe)

We will deal with quasi-arithmetic means of i.i.d. random variables. Here, we allow the target space to be the complex plane \mathbb{C} . This is useful for non-integrable heavy-tailed distributions supported on \mathbb{R} such as Cauchy distributions. There are similarities and differences between the quasi-arithmetic mean of i.i.d. random variables and the arithmetic mean of i.i.d. random variables. We will focus on the limit theorem for variances and integrability. This talk is based on the papers [1] and [2].

References

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Andrzej Olbryś *On approximate convexity*

Let D be a convex subset of a real linear space X . In this talk we examine the properties of functions $f: D \rightarrow \mathbb{R}$ satisfying the inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + \phi(t(x-y)) - t\phi(x-y),$$

for all $x, y \in D$, $t \in [0, 1]$, where $\phi: X \rightarrow \mathbb{R}$ is a given function.

Adam J. Ostaszewski *Homomorphisms from Functional Equations: The Goldie Equation Revisited* (joint work with Nicholas H. Bingham)

For X, Y real topological vector spaces, we characterize the continuous solutions of functional equation of the Levi-Civita-like equation

$$K(u + h(u)v) = K(u) + g(u)K(v) \quad (u, v \in X), \quad (GGE)$$

(cf. Goldie's equation in regular variation theory) as homomorphisms,

$$K(u \circ_{\rho} v) = K(u) \circ_{\sigma} K(v),$$

between 'Popa groups' generated by continuous linear maps $\rho \in X^*$, $\sigma \in Y^*$:

$$\mathbb{G}_{\rho}(X) := \{x \in X : h(x) = 1 + \rho(x) > 0\},$$

$$\mathbb{G}_{\sigma}(Y) := \{y \in Y : g(y) = 1 + \sigma(y) > 0\}.$$

The binary operation $u \circ_{\rho} v = u + (1 + \rho(u))v$ creates group structure. The key to the characterization are two collaborative dichotomies:

- (i) homeomorphy implies one or other of the inclusions $K(\mathcal{N}(\rho)) \subseteq \mathcal{N}(\sigma)$ or $K(\mathcal{N}(\rho)) \subseteq \text{Lin}K(w)$ for some w ; and
- (ii) the intersection of two hyperplanes is of codimension 1 or 2.

Some of the results require the inner auxiliary function h to be Gateaux differentiable at the origin.

Zdzisław Otachel *Jensen-Jessen inequality for sublinear functionals*

The notion of convex function ϕ is inextricably linked with the inequality

$$\phi\left(\sum_{k=1}^n w_k x_k\right) \leq \sum_{k=1}^n w_k \phi(x_k),$$

where ϕ is defined on an interval containing real numbers x_k , while w_k , ($k = 1, \dots, n$) are positive weights with the sum equals 1, in the sense that the inequality can be treated as a definition of a convex function itself. Counterparts of the inequality for integrals also hold true. The introduced facts were established by Jensen [2].

Jessen [3] unified these results replacing sums and integrals by an abstract linear and positive functional A acting on a linear space of real functions g including constant functions, such that $A(J) = 1$, where J is the function constantly equals 1. More precisely, he showed that

$$\phi(A(g)) \leq A(\phi(g)).$$

Functionals of that type are called *linear means*.

The Jensen-Jessen inequalities remain true for the wider class of functionals than linear means, namely, sublinear isotonic functionals preserving constants. This result comes from Dragomir, Pearce and Pečarić [1].

We show that Jensen-Jessen inequalities are met with weaker assumptions about functionals than those made in [1]. This generalizes Jessen's inequality and complements the result by Dragomir, Pearce and Pečarić.

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Paweł Pasteczka *Invariance property for extended means*

We study the properties of the mean-type mappings $\mathbf{M}: I^p \rightarrow I^p$ of the form

$$\mathbf{M}(x_1, \dots, x_p) := (M_1(x_{\alpha_{1,1}}, \dots, x_{\alpha_{1,d_1}}), \dots, M_p(x_{\alpha_{p,1}}, \dots, x_{\alpha_{p,d_p}})),$$

where p and d_i -s are positive integers, each M_i is a d_i -variable mean on an interval $I \subseteq \mathbb{R}$, and $\alpha_{i,j}$ -s are elements from $\{1, \dots, p\}$.

We show that under some natural assumption on M_i -s, the problem of existing the unique \mathbf{M} -invariant mean can be reduced to the ergodicity of the directed graph with vertices $\{1, \dots, p\}$ and edges $\{(\alpha_{i,j}, i) : i, j \text{ admissible}\}$.

This talk is based on the recent article [1].

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Kallol Paul LECTURE: *Geometry of the space of linear operators through the lens of Birkhoff-James orthogonality*

The Birkhoff-James orthogonality generalizes the concept of usual orthogonality in an Euclidean space. In this talk we discuss the Birkhoff-James orthogonality of bounded linear operators defined between Banach spaces and investigate the role of Birkhoff-James orthogonality in the study of geometric properties of the space of bounded linear operators. We also plan to talk about some open problems in this area.

Teresa Rajba *On the Raşa, Hermite-Hadamard and Jensen inequalities for Popoviciu's box-(m, n)-convex functions* (joint work with Andrzej Komisarski)

This talk is based on the article [3]. We give the integral representation of Popoviciu's [4] box-(m, n)-convex functions. Based on this integral representation, we obtain a characterization of box-(m, n)-convex functions, which we then use in the proofs of the Raşa, Hermite-Hadamard type and Jensen inequalities for box-(m, n)-convex functions.

We extend the results obtained by S. G. Gal and C. P. Niculescu [1], and by B. Gavrea and I. Gavrea [2].

References

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Ioan Raşa *Kantorovich type operators and convexity*

We introduce a sequence of Kantorovich type operators which preserve the affine functions. They are defined as

$$\mathcal{K}_n f(x) = f(0)p_{n,0}(x) + f(1)p_{n,n}(x) + \frac{1}{2a_n} \sum_{k=1}^{n-1} p_{n,k}(x) \int_{\frac{k}{n}-a_n}^{\frac{k}{n}+a_n} f(t)dt,$$

where $f \in C[0, 1]$, $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$, $x \in [0, 1]$ and $0 < a_n \leq \frac{1}{n}$.

If n is fixed, $a_n = \frac{\theta}{n}$ and $\theta \rightarrow 0$, we recover the classical Bernstein operator.

Some approximation properties are investigated. Moreover, we present inequalities involving the new operators and convex functions.

Jens Schwaiger LECTURE: *The Sincov equation, history, applications, and new developments*

After a short excursion to the history of the Sincov equation as described by Detlef Gronau in [*A remark on Sincov's functional equation*, Notices of the South African Mathematical Society, **31**(1), (2000), 1–8], some applications, old and recent ones, are presented. Finally a result of Prasanna K. Sahoo on the Sincov like functional equation

$$f_1(x, y) + f_2(x, z) + f_3(z, x) = f(x + y + z)$$

is generalized by considering

$$\sum_{i=1}^n f_i(x_i, x_{i+1}) = f\left(\sum_{i=1}^n x_i\right) \quad \text{with } x_{n+1} = x_1,$$

where f_i, f are functions defined on an Abelian group G and taking values in an Abelian group H , and by determining its solutions.

Wutiphol Sintunavarat LECTURE: *The modification of fuzzy numbers with its impact on the stability of fuzzy number-valued functional equations*

In this talk, we will provide remarks on the definition of fuzzy numbers and discuss various stability results of fuzzy number-valued functional equations. Building upon these observations, we will introduce a slight modification of fuzzy numbers and explore its implications on numerous stability results of fuzzy number-valued functional equations.

Aleksandra Świątczak *Functional equations characterizing differential operators* (joint work with Włodzimierz Fechner and Eszter Gselmann)

If $k \geq 2$ is a positive integer and $\Omega \subseteq \mathbb{R}^N$ is a domain, then by the well-known properties of the Laplacian and the gradient, we have

$$\Delta(f \cdot g) = g\Delta f + f\Delta g + 2\langle \nabla f, \nabla g \rangle$$

for all $f, g \in C^k(\Omega, \mathbb{R})$. Due to the results of König–Milman [1], the converse is also true under some assumptions. Thus the main aim of this talk is to study the equation

$$T(f \cdot g) = fT(g) + T(f)g + 2B(A(f), A(g)) \quad (f, g \in P),$$

where Q and R are commutative rings and P is a subring of Q , further $T: P \rightarrow Q$ and $A: P \rightarrow R$ are additive mappings, while $B: R \times R \rightarrow Q$ is a symmetric and bi-additive mapping.

References

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László Székelyhidi *New Results on Spectral Synthesis*

In this talk we present some new results related to spectral synthesis on not necessarily discrete groups. We exhibit a new method of proving spectral synthesis on varieties which depends on the Fourier algebra of the underlying group. We show that derivations on the Fourier algebra play a deciding role when investigating spectral synthesis. We show how known results can be proved using this method, and also how new results can be obtained.

Patricia Szokol *G-majorization and Schur-type convexity* (joint work with Pál Burai)

In this presentation we recall the standard majorization of vectors (introduced by Hardy, Littlewood and Pólya), that is closely related to Schur-convex functions. We present G -majorization, i.e. majorization induced by a linear group operating on \mathbb{R}^n . As a particular case (when G is the group of cyclic permutations), we obtain the notion of cyclic majorization.

As a consequence, we get the definition of cyclically convex functions that can be considered as a generalization of Schur-convex functions. Finally, we present characterization theorems of such kind of functions.

Bettina Wilkens *An alternative additive-quadratic equation* (joint work with Gian-Luigi Forti)

Parts of the result have been presented at ISFE in Innsbruck last year. Let G be a group and H be an abelian group. A map $f: G \rightarrow H$ is said to satisfy the *alternative additive and quadratic equation* if

$$f(xy) = f(x) + f(y) \text{ or } f(xy) + f(xy^{-1}) = 2f(x) + 2f(y) \quad (*)$$

holds for $x, y \in G$. In the talk we present a complete description of the case where G , too, is abelian.

THEOREM

Let G and H be abelian groups and let $f: G \rightarrow H$ be a map satisfying equation (*). Assume that $H = \langle f(G) \rangle$. The map f satisfies the additive or the quadratic functional equation unless $H \cong \langle \alpha \rangle + K$ with $2K = 0$, $\langle \alpha \rangle \cong \mathbb{Z}/8\mathbb{Z}$ and there is an

element x of G such that

$$f(kx) = \begin{cases} 0 & \text{if } k \equiv 0 \pmod{4}, \\ \alpha & \text{if } k \equiv 1 \pmod{4}, \\ 4\alpha & \text{if } k \equiv 2 \pmod{4}, \\ 5\alpha & \text{if } k \equiv 3 \pmod{4}. \end{cases}$$

We will provide details of the proof as well as more detailed description of the one "truly alternative" case. Possible applications to some nonabelian cases are briefly discussed.

References

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Sebastian Wójcik *Zero utility principle under uncertainty*

A process of insurance contract pricing consists on assigning to any risk, represented by a non-negative essentially bounded random variable on a given probability space, a non-negative real number, being a premium for the risk. In a literature one can find various methods of insurance contracts pricing. In this paper we consider the method, known as the zero utility principle, introduced by H. Bühlmann [1]. It presents the problem from the point of view of an insurance company, assuming that the premium for a given risk is determined in such a way that the company is indifferent between entering into contract and rejecting it.

We study the zero utility principle in the cumulative prospect theory (cf. [2]) under uncertainty. In this setting, the risks are represented by measurable function defined on a given measurable space (S, \mathcal{F}) . The principle for a risk X is defined as a real number $H_{(u, \mu, \nu)}(X)$ satisfying the equation

$$E_{\mu\nu} [u(H_{(u, \mu, \nu)}(X) - X)] = 0,$$

where $E_{\mu\nu}$ is the Choquet integral with respect to $\mu, \nu: \mathcal{F} \rightarrow [0, 1]$.

We establish a necessary and sufficient condition for existence of the principle. Furthermore, we characterize its important properties: the comparability, equality, positive homogeneity and comonotonic additivity.

References

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Marek Cezary Zdun *On Koopmans recursion*

Let I be an interval and X be a metric space. Assume that $U: X^\infty \rightarrow I$ is a continuous surjection satisfying the *Koopmans recursion*

$$U(x_0, x_1, x_2, \dots) = f_{x_0}(U(x_1, x_2, \dots)), \quad (x_0, x_1, \dots) \in X^\infty,$$

where $f_x: I \rightarrow I$ are strictly increasing functions such that the map $(x, t) \rightarrow f_x(t)$, $x \in X$, $t \in I$ is continuous. The mapping U is said to be the *utility function*. We show that every function f_x , $x \in X$ has a unique attractive fixed point and $U(x_0, x_1, \dots) = \lim_{n \rightarrow \infty} (f_{x_0} \circ f_{x_1} \circ \dots \circ f_{x_n})(t)$ for every $t \in I$. Moreover, define on X^∞ the order relation

$$(x_0, x_1, \dots) \succeq (y_0, y_1, \dots) \Leftrightarrow U(x_0, x_1, \dots) \geq U(y_0, y_1, \dots).$$

This order is called the *preference relation represented by U* . The existence of homeomorphic solutions of the system of simultaneous functional equations

$$\varphi(f_x(t)) = \alpha_x \varphi(t) + \beta_x, \quad x \in X, t \in I \quad (1)$$

is equivalent to a special property of the preference relation \succeq represented by U . We determine the form of all f_x which ensures the existence of homeomorphic solution of (1). We show that the homeomorphic solution φ of (1) and coefficients α_x , β_x determine the form of the utility function U . The problem of the uniqueness of homeomorphic solutions of system (1) is considered. Some applications in the preference theory in economics will be presented.

2. A remark**1. Remark**

Let X be Banach space and $x, y \in S_X$. Consider $\min\{\|x + y\|, \|x - y\|\}$ and take $\sup\{\min\{\|x + y\|, \|x - y\|\} : x, y \in S_X\}$. This constant is defined as the *James constant*

$$\mathcal{J}(X) = \sup\{\min\{\|x + y\|, \|x - y\|\} : x, y \in S_X\}.$$

In general, $\sqrt{2} \leq \mathcal{J}(X) \leq 2$. Moreover:

if X is an inner product space, then $\mathcal{J}(X) = \sqrt{2}$.

The converse is not true if $\dim X = 2$.

X is *uniformly non-square* if $\mathcal{J}(X) < 2$.

Let X be a normed space such that $\mathcal{J}(X) = \min\{\|u + v\|, \|u - v\|\}$. Then $u \perp_I v$, i.e. $\|u + v\| = \|u - v\|$.

Let X be a two-dimensional polyhedral Banach space. Then there exists an extreme point z of the unit ball $B_X := \{x \in X : \|x\| \leq 1\}$ such that

$$\mathcal{J}(X) = \|z + y\|, \text{ where } z \perp_I y.$$

The study is still open for polyhedral Banach space with $\dim X \geq 3$.

This discussion is based on the article [1]

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Kallol Paul

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