Ann. Univ. Paedagog. Crac. Stud. Math. 23 (2024), 13-17
DOI: 10.2478/aupcsm-2024-0002

FOLIA 386

Annales Universitatis Paedagogicae Cracoviensis
Studia Mathematica 23 (2024)

Patrycja Liana

On maximizing line arrangements in the complex plane

Abstract. In this note we provide a complete classification of weak combinatorics of the so-called maximizing line arrangements in the complex projective plane.

1. Introduction

The main aim of this short note is to provide a complete classification of maximizing line arrangements in the complex projective plane. This notion comes from the theory of algebraic surfaces and is due to Persson [5]. Let $C : f = 0$ be a reduced plane curve in $\mathbb{P}^2_{\mathbb{C}}$ of degree $d \geq 3$ having only ADE singularities, and for the completeness of the note, let us recall the classification of ADE singularities by presenting their local normal forms:

$$
\begin{aligned}
A_k \text{ with } k \geq 1 &: x^2 + y^{k+1} = 0,\\
D_k \text{ with } k \geq 4 &: y^2 x + x^{k-1} = 0,\\
E_6 &: x^3 + y^4 = 0,\\
E_7 &: x^3 + xy^3 = 0,\\
E_8 &: x^3 + y^5 = 0.
\end{aligned}
$$

All reduced plane curves that admit only ADE singularities will be called *simply singular*. We will need the following definition.

AMS (2020) Subject Classification: 14N20, 14C20.
Keywords and phrases: line arrangements, maximizing curves.
ISSN: 2081-545X, e-ISSN: 2300-133X.

Definition 1.1
Let p be an isolated singularity of a polynomial $f \in \mathbb{C}[x,y]$. Since we can change the local coordinates, we may assume that $p = (0,0)$.

The number

$$\tau_p = \dim_{\mathbb{C}} \left(\mathbb{C}[x,y] / \left\langle f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \right)$$

is called the *Tjurina number* of f at p.

For a projective situation, with a point $p \in \mathbb{P}^2_{\mathbb{C}}$ and a homogeneous polynomial $f \in \mathbb{C}[x,y,z]$, we take local affine coordinates such that $p = (0,0,1)$ and then the dehomogenization of f.

In the projective setting, the total Tjurina number of a given reduced curve $C \subset \mathbb{P}^2_{\mathbb{C}}$ is defined as

$$\tau(C) = \sum_{p \in \mathrm{Sing}(C)} \tau_p.$$

Now we are ready to formulate the mains objects of our studies.

Definition 1.2 ([5, Definition 1.6.])
We say that a reduced simply singular curve $C : f = 0$ in $\mathbb{P}^2_{\mathbb{C}}$ of even degree $n = 2m \geq 4$ is *maximizing*, if

$$\tau(C) = 3m(m-1) + 1.$$

Very recently, Dimca and Pokora proved in [2] a theorem saying that maximizing curves are exactly free curves with some additional property regarding the so-called exponents of a given free curve. Moreover, they introduced the notion of maximizing curves in the odd-degree case.

Definition 1.3 ([2, Definition 5.2])
Let $C : f = 0$ be a reduced simply singular curve in $\mathbb{P}^2_{\mathbb{C}}$ of odd degree $n = 2m+1 \geq 3$. We say that C is a *maximizing* curve if

$$\tau(C) = 3m^2 + 1.$$

The main aim of this note is to provide a classification of maximizing line arrangements, both in odd and even degrees. Let us explain that the condition that a line arrangement in the complex plane has ADE singularities means that it has only double and triple intersections as singularities.

To formulate our main result, we need the following technical definition.

Definition 1.4
Let $\mathcal{L} \subset \mathbb{P}^2_{\mathbb{C}}$ be an arrangement of $n \geq 2$ lines. Denote by t_i the number of i-fold intersection points of lines, i.e. points in the plane where exactly i lines from the arrangement meet. The weak combinatorics of $\mathcal{L}$ is defined to be the vector of the form $(n; t_2, \ldots, t_d)$.

Here is our main result of this note.

Main Theorem
Let $\mathcal{L} \subset \mathbb{P}^2_{\mathbb{C}}$ be a maximizing line arrangement of $n \geq 3$ lines. Then $\mathcal{L}$ is one of the following weak combinatorial types:

$$(n; t_2, t_3) \in \{(3; 0, 1), (4; 3, 1), (6; 3, 4)\}.$$

Let us point out here that our proof is purely combinatorial and we do not use any advanced theory of plane curves.

2. Proof of Main Theorem

We will use the notation and basic definitions covered in [1]. We start with a small preparatory result.

Proposition 2.1
Let $\mathcal{L} \subset \mathbb{P}^2_{\mathbb{C}}$ be an arrangement of lines with only t_2 double and t_3 triple intersections. Then the total Tjurina number of $\mathcal{L}$ has the following form

$$\tau(\mathcal{L}) = t_2 + 4t_3.$$

Proof. It follows from the fact that for line arrangements in the plane one has

$$\tau(\mathcal{L}) = \sum_{p \in \mathrm{Sing}(\mathcal{L})} (\mathrm{mult}_p - 1)^2 = \sum_{p \in \mathrm{Sing}(\mathcal{L})} (r-1)^2 t_r = t_2 + 4t_3,$$

where mult_p denotes the multiplicity of a singular point $p \in \mathrm{Sing}(\mathcal{L})$.

We will need also the following version of Hirzebruch's inequality [4].

Theorem 2.2 (Hirzebruch)
Let $\mathcal{L} \subset \mathbb{P}^2_{\mathbb{C}}$ be an arrangement of $n \geq 4$ lines such that $t_n = t_{n-1} = 0$. Then one has

$$t_2 + \frac{3}{4}t_3 \geq n + \sum_{r \geq 5} (r-4)t_r.$$

Now we are ready to present our proof of Main Theorem.

Proof. We will divide our discussion into two parts

(even case): If $n = 2m$ with $m \geq 2$, then we have the following constraints:

$$(\star): \quad \binom{2m}{2} = t_2 + 3t_3 \quad \text{and} \quad t_2 + 4t_3 = 3m(m-1) + 1.$$

Observe that these constraints give us that

$$t_3 = 3m^2 - 3m + 1 - 2m^2 + m = (m-1)^2,$$
$$t_2 = 2m^2 - m - 3(m-1)^2 = -m^2 + 5m - 3.$$

Since $t_2 \geq 0$, it implies that $m \in \{2, 3, 4\}$. The collected data gives us the following cases.

1. For $m = 2$ we have $t_2 = 3$ and $t_3 = 1$. This weak combinatorics corresponds to a Hirzebruch's quasi-pencil of 4 lines.
2. For $m = 3$ we have $t_2 = 3$ and $t_3 = 4$. This weak combinatorics can be realized as a simplicial $\mathcal{A}_1(6)$ arrangement, cf. this notation with Grünbaum catalogue [3].
3. For $m = 4$ we have $t_2 = 1$ and $t_3 = 9$. Observe that

$$t_2 + \frac{3}{4}t_3 = 1 + \frac{3}{4} \cdot 9 = \frac{31}{4} < n = 8,$$

so we have a violation of the Hirzebruch's inequality and a contradiction.

(odd case): If $n = 2m + 1$ with $m \geq 1$, then we have the following constraints:

$$(\triangle): \quad \binom{2m+1}{2} = t_2 + 3t_3 \quad \text{and} \quad t_2 + 4t_3 = 3m^2 + 1.$$

Based on that combinatorial identities, we arrive at

$$t_3 = m^2 - m + 1,$$
$$t_2 = -m^2 + 4m - 3.$$

Since $t_2 \geq 0$, we have that $m \in \{1, 2, 3\}$. The collected data gives us the following cases.

1. For $m = 1$ we have $t_3 = 1$ and $t_2 = 0$, so this is a pencil of 3 lines.
2. For $m = 2$ we have that $t_3 = 3$ and $t_2 = 1$. Observe that

$$t_2 + \frac{3}{4}t_3 = 1 + 3 \cdot \frac{3}{4} = \frac{13}{4} < 5,$$

so we have a contradiction with respect to Hirzebruch's inequality.
3. For $m = 3$ we have $t_3 = 7$ and $t_2 = 0$, so this is the weak combinatorics of the Fano plane that cannot be constructed over the complex numbers. More concretely, observe that

$$t_2 + \frac{3}{4}t_3 = 0 + 7 \cdot \frac{3}{4} = \frac{21}{4} < 7,$$

and again we have a contradiction with respect to Hirzebruch's inequality.

References

[1] Dimca, Alexandru. *Hyperplane arrangements. An introduction.* Cham: Springer, 2017. Cited on 15.

[2] Dimca, Alexandru and Piotr Pokora. "Maximizing curves viewed as free curves." *Int. Math. Res. Not. IMRN* 2023, no. 22 (2023): 19156-19183. Cited on 14.

[3] Grünbaum, Branko. "A catalogue of simplicial arrangements in the real projective plane." *Ars Math. Contemp.* 2, no. 1 (2009): 1-25. Cited on 16.

[4] Hirzebruch, Friedrich. "Arrangements of lines and algebraic surfaces." *Arithmetic and geometry, Vol. II*, 113-140. Vol. 36 of *Progress in Mathematics*. Boston: Birkhäuser, 1983. Cited on 15.

[5] Persson, Ulf. "Horikawa surfaces with maximal Picard numbers." *Math. Ann.* 259, no. 3 (1982): 287-312. Cited on 13 and 14.

Patrycja Liana
Department of Mathematics
University of the National Education Commission Krakow
Podchorążych 2
PL-30-084 Kraków
Poland
E-mail: patrycja.liana@student.up.krakow.pl

Received: October 19, 2023; final version: November 30, 2023;
available online: January 25, 2024.