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Ann. Univ. Paedagog. Crac. Stud. Math. 23 (2024), 19-27
DOI: 10.2478/aupcsm-2024-0003

FOLIA 386

Annales Universitatis Paedagogicae Cracoviensis
Studia Mathematica 23 (2024)

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Rings with centrally-extended Jordan epimorphisms

Abstract. The aim of this article is to introduce the concept of centrally-extended Jordan epimorphisms and proving that if R is a non-commutative prime ring ($*$-ring) of characteristic not two, and G is a CE-Jordan epimorphism such that $[G(x), x] \in Z(R)$ ($[G(x), x^*] \in Z(R)$) for all $x \in R$, then R is an order in a central simple algebra of dimension at most 4 over its center or there is an element λ in the extended centroid of R such that $G(x) = \lambda x$ ($G(x) = \lambda x^*$) for all $x \in R$.

1. Introduction

Throughout this paper, R denotes an associative ring with center $Z(R)$, $Q_{mr}(R) = Q$ denotes the maximal right ring of quotients of R and the center of $Q_{mr}(R)$ is called the extended centroid of R and denoted by C. By a ring with involution "$*$", we mean a ring equipped with an involution "$*$", it is also called $*$-ring. For $x, y \in R$ we have $[x, y] = xy - yx$ and $x \circ y = xy + yx$.

In 2016, Bell and Daif [2] introduced the concept of centrally-extended derivations and centrally-extended endomorphisms on a ring R with center $Z(R)$. A map D on R is called a centrally-extended derivation (CE-derivation) if $D(x+y) - D(x) - D(y) \in Z(R)$ and $D(xy) - D(x)y - xD(y) \in Z(R)$ for all $x, y \in R$. A map T of R is said to be a centrally-extended endomorphism (CE-endomorphism) if for each $x, y \in R$, $T(x+y) - T(x) - T(y) \in Z(R)$ and $T(xy) - T(x)T(y) \in Z(R)$. They showed that if R is a semiprime ring with no nonzero central ideals, then every CE-derivation is a derivation and every CE-epimorphism is an epimorphism.

AMS (2020) Subject Classification: 16N60, 16W10, 16U10.
Keywords and phrases: derivations, centrally-extended derivations, endomorphisms, centrally-extended endomorphisms, centrally-extended Jordan endomorphisms, prime rings.
ISSN: 2081-545X, e-ISSN: 2300-133X.

Through the last five years, the authors in [7, 8, 14, 15, 9, 10, 11, 12, 13] introduced the notions of CE-generalized (θ,ϕ)-derivations, CE-generalized $*$-derivations, CE-reverse derivations, CE-generalized reverse derivations, CE-higher derivation, CE-higher $*$derivation, CE-homoderivation, CE-α-homoderivation, multiplicative CE-derivation and multiplicative generalized reverse $*$CE-derivation and proved similar results.

In [3] Bhushan et al. gave the notion of a CE-Jordan derivation to be a mapping D of R satisfying $D(x+y)-D(x)-D(y)\in Z(R)$ and $D(x\circ y)-D(x)\circ y-x\circ D(y)\in Z(R)$ for all $x,y\in R$. Also, they gave the notion of a CE- Jordan $*$-derivation to be a mapping D of a ring R with involution $*$ satisfying $D(x+y)-D(x)-D(y)\in Z(R)$ and $D(x\circ y)-D(x)y^*-xD(y)-D(y)x^*-yD(x)\in Z(R)$ for all $x,y\in R$. They proved the following. Let R be a non-commutative prime ring (with involution $*$) of characteristic not 2. If R admits a CE-Jordan derivation d of R such that $[d(x),x]\in Z(R)([d(x),x^*]\in Z(R))$ for all $x\in R$, then either $d=0$ or R is an order in a central simple algebra of dimension at most 4 over its center. Also they obtained the same results in the case of CE-Jordan $*$-derivations.

Bhushan et al. in [4] introduced the notion of CE-generalized derivation Jordan derivation is a mapping F of R satisfying $F(x+y)-F(x)-F(y)\in Z(R)$ and $F(x\circ y)-F(x)y-F(y)x-xD(y)-yD(x)\in Z(R)$ for all $x,y\in R$, where D is a CE-Jordan derivation of R. They proved the following:

(i) Let R be a non-commutative prime ring of characteristic not 2. If R admits a CE-generalized derivation F constrained with a CE-Jordan derivation d of R such that $[F(x),x]\in Z(R)$ for all $x\in R$, then R is an order in a central simple algebra of dimension at most 4 over its center or $F(x)=\lambda x$ for all $x\in R$, where $\lambda\in C$.

(ii) Let R be a non-commutative prime ring with involution $*$ and characteristic not 2. If R admits a CE-generalized derivation F constrained with a CE-Jordan derivation d of R such that $[F(x),x^*]\in Z(R)$ for all $x\in R$, then R is an order in a central simple algebra of dimension at most 4 over its center or $F=0$.

Our objective in this article is to introduce the notion of CE-Jordan endomorphisms and CE-Jordan epimorphisms and prove the following results:

(i) Let R be a non-commutative prime ring of characteristic not two. If R admits a CE-Jordan epimorphism G such that $[G(x),x]\in Z(R)$ for all $x\in R$, then R is an order in a central simple algebra of dimension at most 4 over its center or $G(x)=\lambda x$ for all $x\in R$, where $\lambda\in C$.

(ii) Let R be a non-commutative prime ring of characteristic not two with involution $*$. If R admits a CE-Jordan epimorphism G such that $[G(x),x^*]\in Z(R)$ for all $x\in R$, then R is an order in a central simple algebra of dimension at most 4 over its center or $G(x)=\lambda x^*$ for all $x\in R$, where $\lambda\in C$.

2. Preliminary results

The following lemmas help us to prove our results.

Lemma 2.1 ([5, Theorem 3.2])
Let R be a prime ring. If $g\colon R \to R$ is an additive mapping such that $[g(x), x] = 0$ for all $x \in R$, then there exists $\lambda \in C$ and an additive mapping $\beta\colon R \to C$, such that $g(x) = \lambda x + \beta(x)$ for all $x \in R$.

Lemma 2.2 ([6, Lemma 1])
Let R be a prime ring with extended centroid C. Then the following statements are equivalent:

(i) *R satisfies s_4.*

(ii) *R is commutative or R embeds into $M_2(K)$, for a field K.*

(iii) *R is algebraic of bounded degree 2 over C.*

(iv) *R satisfies $[[x^2, y], [x, y]]$.*

In the following lemma, we generalize [5, Proposition 3.1] to use it in our results.

Lemma 2.3
Let R be a 2-torsion free semiprime ring with a nonzero map $\alpha\colon R \to R$ and $\alpha(x+y)-\alpha(x)-\alpha(y) \in Z(R)$. If $[\alpha(x), x] \in Z(R)$ for all $x \in R$, then $[\alpha(x), x] = 0$ for all $x \in R$.

Proof. By our hypothesis, we have

$$[\alpha(x), x] \in Z(R) \qquad \text{for all } x \in R. \tag{1}$$

Linearizing (1) implies that

$$[\alpha(x), y] + [\alpha(y), x] \in Z(R) \qquad \text{for all } x, y \in R. \tag{2}$$

Putting x^2 instead of y in (2) and using (1) yield

$$2[\alpha(x), x]x + [\alpha(x^2), x] \in Z(R) \qquad \text{for all } x \in R. \tag{3}$$

From (1) and (3), we obtain

$$\begin{aligned} 0 &= [\alpha(x), 2[\alpha(x), x]x + [\alpha(x^2), x]] \\ &= 2[\alpha(x), x]^2 + [\alpha(x), [\alpha(x^2), x]] \qquad \text{for all } x \in R. \end{aligned}$$

From (1), we have $[\alpha(x^2), x^2] = [\alpha(x^2), x]x + x[\alpha(x^2), x] \in Z(R)$ for all $x \in R$ Thus,

$$\begin{aligned} 0 &= [\alpha(x), [\alpha(x^2), x]x + x[\alpha(x^2), x]] \\ &= 2[\alpha(x), x][\alpha(x^2), x] + [\alpha(x), [\alpha(x^2), x]]x + x[\alpha(x), [\alpha(x^2), x]] \\ &= 2[\alpha(x), x][\alpha(x^2), x] - 4x[\alpha(x), x]^2 \qquad \text{for all } x \in R. \end{aligned}$$

So, we find

$$\begin{aligned} 0 &= [\alpha(x), 2[\alpha(x), x][\alpha(x^2), x] - 4x[\alpha(x), x]^2] \\ &= -8[\alpha(x), x]^3 \qquad \text{for all } x \in R. \end{aligned}$$

The two torsion freeness of R and the semiprimeness imply that $[\alpha(x), x] = 0$ for all $x \in R$.

By the following lemma, we say [1, Proposition 2.2] is true when we take $\alpha\colon R \to R$ such that $\alpha(x+y) - \alpha(x) - \alpha(y) \in Z(R)$ for all $x, y \in R$ instead of α is additive. Also, we use it in proving of our results.

Lemma 2.4
Let R be a 2-torsion free semiprime ring admitting an involution $*$. If $\alpha\colon R \to R$ a nonzero map, $\alpha(x+y)-\alpha(x)-\alpha(y) \in Z(R)$ for all $x, y \in R$ and $[\alpha(x), x^*] \in Z(R)$ for all $x \in R$, then $[\alpha(x), x^*] = 0$ for all $x \in R$.

Proof. By our assumption, we have $[\alpha(x), x^*] \in Z(R)$ for all $x \in R$. Putting x^* instead of x, we get $[\alpha(x^*), x] \in Z(R)$ for all $x \in R$. Define a map $\gamma\colon R \to R$ such that $\gamma(x) = \alpha(x^*)$ for all $x \in R$. We note $\gamma(x+y) - \gamma(x) - \gamma(y) \in Z(R)$ for all $x \in R$ and $[\gamma(x), x] \in Z(R)$ for all $x \in R$. Lemma 2.3 gives $[\gamma(x), x] = 0$ for all $x \in R$. This implies that $[\alpha(x^*), x] = 0$ for all $x \in R$, hence $[\alpha(x), x^*] = 0$ for all $x \in R$.

3. Centrally-extended Jordan epimorphisms

In the present section, we introduce the notion of centrally extended endomorphisms and centrally extended epimorphisms, and give some examples on both. Finally, we prove some results on centrally extended epimorphism.

Definition 3.1
A map $G\colon R \to R$ is called a centrally-extended Jordan endomorphism (CE-Jordan endomorphism) if $G(x+y)-G(x)-G(y) \in Z(R)$ and $G(x \circ y)-G(x) \circ G(y) \in Z(R)$ for all $x, y \in R$, and if G is also surjective, call it a CE-Jordan epimorphism.

In the following examples, we make sure the existance of CE-Jordan endomorphisms and CE-Jordan epimorphisms .

Example 3.2
Suppose R $= M_2(\mathbb{Z})$ where $\mathbb{Z}$ is the ring of integers. It is obvious that the mapping $G\colon R \to R$ defined by

$$G\left(\begin{bmatrix} a & b \\ c & t \end{bmatrix} \right) = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}$$

is a CE-Jordan endomorphism, but not a Jordan endomorphism.

Example 3.3

Let the ring R $= \left\{ \begin{bmatrix} 0 & t & w \\ 0 & 0 & n \\ 0 & 0 & 0 \end{bmatrix} : t, w, n \in \mathbb{R} \right\}$. It is obvious that the mapping $G\colon R \to R$ such that

$$G\left(\begin{bmatrix} 0 & t & w \\ 0 & 0 & n \\ 0 & 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & 2t & 2w \\ 0 & 0 & 2n \\ 0 & 0 & 0 \end{bmatrix}$$

is a CE-Jordan epimorphism but not a Jordan epimorphism.

Example 3.4

Let R $= M_2(\mathbb{Z})$. The mapping $G\colon R \to R$ defined by

$$G\left(\begin{bmatrix} a & b \\ c & t \end{bmatrix} \right) = \begin{bmatrix} t & -b \\ -c & a \end{bmatrix}$$

is a CE-Jordan epimorphism but not a CE-epimorphism.

Lemma 3.5

Let R be a 2-torsion free ring with no nonzero central ideals. Then every CE-Jordan epimorphism G of R is additive.

Proof. Since G is a CE-endomorphism, then we have for $x, y, w \in R$

$$G(x+y) = G(x) + G(y) + a_1, \qquad a_1 \in Z(R). \tag{4}$$

And,

$$G(w \circ (x+y)) = G(w)G(x+y) + G(x+y)G(w) + a_2, \qquad a_2 \in Z(R).$$

By (4), we get

$$G(w \circ (x+y)) = G(w)(G(x) + G(y) + a_1) + (G(x) + G(y) + a_1)G(w) + a_2. \tag{5}$$

Also, we have

$$\begin{aligned} G(w \circ (x+y)) = G((w \circ x) + (w \circ y)) &= G(w \circ x) + G(w \circ y) + a_3 \\ &= G(w)G(x) + G(x)G(w) + a_4 + G(w)G(y) \\ &\quad + G(y)G(w) + a_5 + a_3, \qquad a_3, a_4, a_5 \in Z(R). \end{aligned} \tag{6}$$

Equations (5) and (6) imply that $2a_1G(w) + a_2 = a_3 + a_4 + a_5$ i.e., $2a_1G(w) \in Z(R)$ for all $w \in R$. Since R is 2-torsion free, then $a_1G(w) \in Z(R)$ for all $w \in R$. Using G is surjective gives $a_1w \in Z(R)$ for all $w \in R$. Thus, a_1R is a central ideal, hence $a_1R = (0)$. Therefore, letting $A(R)$ be the two-sided annihilator of R, we have $a_1 \in A(R)$. But $A(R)$ is a central ideal, so $a_1 = 0$ and by (4), we get our proof.

Corollary 3.6

Let R be a non-commutative prime ring of characteristic not two. If G is a CE-Jordan epimorphism of R, then G is additive.

Now, we are ready to prove our first theorem.

Theorem 3.7
Let R be a non-commutative prime ring of characteristic not two. If R admits a CE-Jordan epimorphism G such that $[G(x), x] \in Z(R)$ for all $x \in R$, then R is an order in a central simple algebra of dimension at most 4 over its center or $G(x) = \lambda x$ for all $x \in R$, where $\lambda \in C$.

Proof. By our hypothesis and Lemma 2.3, we obtain

$$[G(x), x] = 0 \qquad \text{for all } x \in R. \tag{7}$$

Linearizing (7) gives

$$[G(x), y] + [G(y), x] = 0 \qquad \text{for all } x, y \in R. \tag{8}$$

Substituting $x \circ y$ for y in (8) gives

$$[G(x), x \circ y] + [G(x \circ y), x] = 0 \qquad \text{for all } x, y \in R.$$

Thus,

$$\begin{aligned} 0 &= [G(x), x \circ y] + [G(x)G(y) + G(y)G(x), x] \\ &= [G(x), x \circ y] + G(x)[G(y), x] + [G(y), x]G(x) \qquad \text{for all } x, y \in R. \end{aligned} \tag{9}$$

Corollary 3.6 implies that G is additive, then (7) and Lemma 2.1 give $G(x) = \lambda x + \beta(x)$ for all $x \in R$ where $\beta\colon R \to C$ is an additive mapping. By (9), we arrive at

$$\begin{aligned} 0 = [\lambda x + \beta(x), x \circ y] + (\lambda x + \beta(x))[\lambda y + \beta(y), x] \\ + [\lambda y + \beta(y), x](\lambda x + \beta(x)) \qquad \text{for all } x, y \in R. \end{aligned}$$

Since λ and $\beta(x) \in C$ for all $x \in R$, then

$$\begin{aligned} 0 &= \lambda[x, x \circ y] + \lambda^2 x[y, x] + \lambda^2[y, x]x + 2\lambda\beta(x)[y, x] \\ &= \lambda[x^2, y] + \lambda^2[y, x^2] + 2\lambda\beta(x)[y, x] \\ &= (\lambda^2 - \lambda)[y, x^2] + 2\lambda\beta(x)[y, x] \qquad \text{for all } x, y \in R. \end{aligned} \tag{10}$$

So, we find

$$0 = (\lambda^2 - \lambda)[[x^2, y], [y, x]] \qquad \text{for all } x, y \in R.$$

The primeness of R leads to the primeness of Q, then we get either $\lambda^2 = \lambda$ or $[[x^2, y], [x, y]] = 0$ for all $x, y \in R$. By Lemma 2.2, the second case is equivalent to the s_4 identity and R is assumed to be non-commutative, therefore R is an order in a central simple algebra of dimension at most 4 over $Z(R)$.

On the other hand, let us assume that $\lambda^2 = \lambda$. Then from (10), we have $2\lambda\beta(x)[y, x] = 0$ for all $x, y \in R$. Since characteristic R is not two, then characteristic Q is not two and so, we find $\lambda\beta(x)[y, x] = 0$ for all $x, y \in R$. The primeness of Q gives $\lambda = 0$ or $\beta(x)[y, x] = 0$ for all $x, y \in R$.

Assume $\lambda = 0$. Then $G(x) = \beta(x) \in C$ for all $x \in R$. Since G is surjective, then $R \subseteq C$ i.e., R is a commutative ring, a contradiction.

If $\beta(x)[y, x] = 0$ for all $x, y \in R$, then the primeness of Q implies that for each $x \in R$ either $\beta(x) = 0$ or $[y, x] = 0$ for all $y \in R$. Write $E_1 = \{x \in R : \beta(x) = 0\}$ and $E_2 = \{x \in R : [y, x] = 0 \text{ for all } y \in R\}$. Therefore, we note that E_1 and E_2 are proper additive subgroups of R and its union equals R which is impossible. Thus, either $R = E_1$ or $R = E_2$. It implies that either $\beta(x) = 0$ for all $x \in R$ or $[y, x] = 0$ for all $x, y \in R$. If $[y, x] = 0$ for all $x, y \in R$, then R is commutative, a contradiction. If $\beta(x) = 0$ for all $x \in R$, then $G(x) = \lambda x$ for all $x \in R$.

Now, we prove the second theorem that deals with a ring with involution.

Theorem 3.8

Let R be a non-commutative prime ring of characteristic not two with involution $*$. If R admits a CE-Jordan epimorphism G such that $[G(x), x^*] \in Z(R)$ for all $x \in R$, then R is an order in a central simple algebra of dimension at most 4 over its center or $G(x) = cx^*$ for all $x \in R$, where $c \in C$.

Proof. Lemma 2.4 gives

$$[G(x), x^*] = 0 \qquad \text{for all } x \in R. \tag{11}$$

Applying involution in (11), we get

$$[G(x)^*, x] = 0 \qquad \text{for all } x \in R. \tag{12}$$

Linearizing (12) gives

$$[G(x)^*, y] + [G(y)^*, x] = 0 \qquad \text{for all } x, y \in R. \tag{13}$$

Substituting $x \circ y$ for y in (13) gives

$$[G(x)^*, x \circ y] + [G(x \circ y)^*, x] = 0 \qquad \text{for all } x, y \in R.$$

Thus,

$$\begin{aligned} 0 &= [G(x)^*, x \circ y] + [(G(x)G(y) + G(y)G(x))^*, x] \\ &= [G(x)^*, x \circ y] + G(x)^*[G(y)^*, x] \\ &\quad + [G(y)^*, x]G(x)^* \qquad \text{for all } x, y \in R. \end{aligned} \tag{14}$$

Since $*G$ is additive, then Lemma 2.1 gives $G(x)^* = \lambda x + \beta(x)$ for all $x \in R$, where $\beta \colon R \to C$ is an additive mapping. So, (14) becomes

$$\begin{aligned} 0 = [\lambda x + \beta(x), x \circ y] &+ (\lambda x + \beta(x))[\lambda y + \beta(y), x] \\ &+ [\lambda y + \beta(y), x](\lambda x + \beta(x)) \qquad \text{for all } x, y \in R. \end{aligned}$$

Since λ and $\beta(x)$ for all $x \in R$ belong to C, then

$$(\lambda^2 - \lambda)[y, x^2] + 2\lambda\beta(x)[y, x] = 0 \qquad \text{for all } x, y \in R. \tag{15}$$

So, we get $(\lambda^2-\lambda)[[x^2,y],[y,x]]=0$ for all $x,y\in R$. As above, we find either $\lambda^2=\lambda$ or R is an order in a central simple algebra of dimension at most 4 over its center. If $\lambda^2=\lambda$, then (15) implies that $\beta(x)[y,x]=0$ for all $x,y\in R$. By the same manner as in Theorem 3.7 we arrive at $\beta(x)=0$ for all $x\in R$. Therefore $G(x)^*=\lambda x$ for all $x\in R$. The involution leads to $G(x)=\lambda^* x^*=cx^*$ for all $x\in R$, where $c\in C$.

Acknowledgement. I would like to give my deep thanks to the referees and the handling editor for their valuable comments.

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Received: October 12, 2023; final version: January 9, 2024;
available online: March 22, 2024.