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δ_μ-connectedness in a μ-proximity space

Abstract. In this paper we introduce the notion of δ_μ-connectedness on a μ-proximity space. It has been proved that δ_μ-connectedness can be characterized by δ_μ-continuous functions. We initiate the idea of δ_μ-chain and establish some results related to this. The concepts of δ_μ-component and δ_μ-quasi component have been introduced and their interrelation has been studied.

1. Introduction and Preliminary Results

In topology, the notion of proximally continuous mapping is well-known in a proximity space. Császár introduced the concept of generalized topology in [1] and it was observed that many of the existing results for a topological space are still valid in this generalized premise. Generalized topology was defined by Császár as follows:
A collection μ of subsets of a set X is called a generalized topology (GT, in short) on X if

(i) $\emptyset \in \mu$,

(ii) for $U_\alpha \in \mu$, $\alpha \in \Lambda$ (Λ being an index set), $\bigcup_{\alpha\in\Lambda} U_\alpha \in \mu$.

The pair (X, μ) is called a generalized topological space (GTS, in short). The members of μ are called μ-open sets and their complements are μ-closed. For a subset A of X, the union of all μ-open sets contained in A is called the μ-interior

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of A and is denoted by $i_\mu A$. The intersection of all μ-closed sets of X containing A is known as the μ-closure of A and is denoted by $c_\mu A$. A GT is said to be *strong* if $X \in \mu$.

The notion of connectedness in a generalized topological space was studied by Császár in [2]. In [3], Dimitrijević and Kočinac have introduced the notion of connectedness in a proximity space. They carried on to infer about δ-component, δ-quasi component and local δ connectedness. Various relations between those concepts were developed and the use of proximally continuous function was a key in those results. We look to study similar conditions for a μ-proximity space. In this paper, we initiate a type of connectedness on a μ-proximity space [4]. After the introductory section, we have defined the δ_μ-connectedness and obtain certain results regarding it. In Section 3, the concept of δ_μ-component has been studied. In the last section, the notion of δ_μ-quasi component is introduced and a relation between δ_μ-component and δ_μ-quasi component is established.

Before going into the details, we first recall the definition of μ-proximity and some results related to it.

Definition 1.1 ([4])
A binary relation δ_μ on the power set $\mathcal{P}(X)$ of a set X is called a μ-proximity on X if δ_μ satisfies the following axioms:

(i) $A\ \delta_\mu\ B$ if and only if $B\ \delta_\mu\ A$ for all $A, B \in \mathcal{P}(X)$;

(ii) If $A\ \delta_\mu\ B$, $A \subseteq C$ and $B \subseteq D$, then $C\ \delta_\mu\ D$;

(iii) $\{x\}\ \delta_\mu\ \{x\}$ for all $x \in X$;

(iv) If $A\ \not\delta_\mu\ B$ then there exists $E(\subseteq X)$ such that $A\ \not\delta_\mu\ E$ and $(X \setminus E)\ \not\delta_\mu\ B$.

If a relation satisfies axioms (i)–(iii) then it is called a *basic μ-proximity* on X.

Proposition 1.2 ([4])
Let a subset A of a μ-proximity space (X, δ_μ) be defined to be δ_μ-closed if and only if

$$\{x\}\ \delta_\mu\ A \Rightarrow x \in A.$$

Then the collection of complements of all δ_μ-closed sets so defined yields a generalized topology $\mu = \tau(\delta_\mu)$ on X.

Proposition 1.3 ([4])
Let (X, δ_μ) be a μ-proximity space and $\mu = \tau(\delta_\mu)$. Then the μ-closure $c_\mu(A)$ of a set A in (X, μ) is given by $c_\mu(A) = \{x :\ \{x\}\ \delta_\mu\ A\}$.

Lemma 1.4 ([4])
For subsets A and B of a μ-proximity space (X, δ_μ),

$$A\ \delta_\mu\ B \Leftrightarrow c_\mu(A)\ \delta_\mu\ c_\mu(B),$$

where the μ-closures are taken with respect to $\tau(\delta_\mu)$.

Definition 1.5 ([6])
If (X, δ_{μ_1}) and (Y, δ_{μ_2}) are two μ-proximity spaces, a mapping $f\colon X \to Y$ is said to be δ_μ-continuous if $A\ \delta_{\mu_1}\ B$ implies $f(A)\ \delta_{\mu_2}\ f(B)$ for $A, B \subseteq X$.

2. δ_μ-connectedness

Definition 2.1
A μ-proximity space (X, δ_μ) is said to be δ_μ-connected if it cannot be expressed as the union of two non-empty subsets of X that are not δ_μ-related. A subset Y of X is said to be a δ_μ-connected subset of X if it cannot be expressed as the union of two non-empty subsets of X that are not δ_μ-related.

We know that, by defining a proximity δ as

$$A \ \delta \ B \Leftrightarrow A \cap B \neq \emptyset,$$

where $A, B \subseteq X$, we get the discrete proximity on X, [5]. Of course, the corresponding topology generated by δ is the discrete topology on X. Since every proximity space is a μ-proximity space[details can be found in [4], Proposition 2.12], we have (X, δ) as the discrete μ-proximity space. As per our requirement, here we consider the discrete μ-proximity on the two-point set $\{0, 1\}$ and denote the discrete μ-proximity space $(\{0, 1\}, \delta)$ by X_d henceforth.

In a μ-proximity space (X, δ_μ), two non-empty subsets A and B of X are said to be δ_μ-separated, if A, B are not δ_μ related, i.e. $A \not\delta_\mu B$.

Theorem 2.2
A μ-proximity space (X, δ_μ) is δ_μ-connected if and only if every δ_μ-continuous function f on X to X_d is constant.

Proof. Let (X, δ_μ) be δ_μ-connected and $f\colon X \to X_d$ be a δ_μ-continuous function. If possible, let f be not constant. Then $f^{-1}(\{0\}) \neq \emptyset$ and $f^{-1}(\{1\}) \neq \emptyset$. Also $\{0\} \not\delta \{1\}$ which implies $f^{-1}(\{0\}) \not\delta_\mu f^{-1}(\{1\})$ [since f is δ_μ-continuous]. Again $X = f^{-1}(\{0\}) \cup f^{-1}(\{1\})$, which implies X is not δ_μ-connected, a contradiction. Therefore, f must be constant.

Conversely, if X is not δ_μ-connected then there exist two non-empty subsets A, B of X such that $X = A \cup B$ with $A \not\delta_\mu B$. Define $F\colon X \to X_d$ by

$$F(x) = \begin{cases} 0, & x \in A, \\ 1, & x \in B. \end{cases}$$

Since, $A \not\delta_\mu B$ implies $A \cap B = \emptyset$ [from (iii) and (ii) of Definition 1.1], therefore, F is well-defined. Let $C, D \subseteq X$ and $C \delta_\mu D$.

We claim that $F(C) \ \delta \ F(D)$. In fact, if $F(C) \not\delta F(D)$ then $F(C) \cap F(D) = \emptyset$ [since, X_d is discrete]. Therefore, without loss of generality, let $F(C) = \{0\}$ and $F(D) = \{1\}$ which implies $C \subseteq A$ and $D \subseteq B$. Since $C \ \delta_\mu \ D$, by (ii) of Definition 1.1, we get $A \ \delta_\mu \ B$, a contradiction.

Therefore, $C \ \delta_\mu \ D \Rightarrow F(C) \ \delta \ F(D)$, so F is δ_μ-continuous, but not constant. This gives the desired result.

Theorem 2.3
A μ-proximity space (X, δ_μ) is δ_μ-connected if and only if for any non-empty proper subset A of X, $A \ \delta_\mu \ (X \setminus A)$.

Proof. Let (X, δ_μ) be a δ_μ-connected μ-proximity space and $\emptyset \neq A \subset X$. It is evident that $A\ \delta_\mu\ (X \setminus A)$, otherwise $X = A \cup (X \setminus A)$ and $X \setminus A$ is also a non-empty proper subset of X which implies X is not δ_μ-connected, a contradiction.

Conversely, let (X, δ_μ) be not δ_μ-connected. So there exist non-empty subsets A, B of X such that $X = A \cup B$ and $A\ \not\delta_\mu\ B$. Since A and B are δ_μ-separated, $A \cap B = \emptyset$ which implies $B = X \setminus A$. Hence $A\ \not\delta_\mu\ (X \setminus A)$. This gives the desired result.

Proposition 2.4

If (X, δ_μ) is a μ-proximity space and C is a non-empty δ_μ-connected subset of X which is contained in the union of two δ_μ-separated subsets of X, then C is contained in one of the subsets.

Proof. Let C be a δ_μ-connected subset of X and $C \subseteq A \cup B$ with $A\ \not\delta_\mu\ B$, where $A, B \subseteq X$. If possible, let $A \cap C \neq \emptyset$ and $B \cap C \neq \emptyset$. Obviously, $(A \cap C)\ \not\delta_\mu\ (B \cap C)$ [otherwise by (ii) of Definition 1.1, $A\ \delta_\mu\ B$]. Also, $(A \cap C) \cup (B \cap C) = C$, which implies C is not δ_μ-connected, a contradiction.

Therefore, either $A \cap C = \emptyset$ or $B \cap C = \emptyset$, i.e. either $C \subseteq A$ or $C \subseteq B$.

Theorem 2.5

In a μ-proximity space (X, δ_μ), the μ-closure of a δ_μ-connected subset is δ_μ-connected.

Proof. Let A be a δ_μ-connected subset of X. Let $c_\mu(A) = P \cup Q$ and $P\ \not\delta_\mu\ Q$, where P, Q are non-empty subsets of X. Since $A \subseteq P \cup Q$, by Proposition 2.4, A is contained either in P or in Q. Without loss of generality, let $A \subseteq P$, which implies $c_\mu(A) \subseteq c_\mu(P)$.

Now $P\ \not\delta_\mu\ Q \Rightarrow c_\mu(P)\ \not\delta_\mu\ c_\mu(Q)$ [by Lemma 1.4] and so $c_\mu(P) \cap c_\mu(Q) = \emptyset$. Therefore, $c_\mu(A) \cap c_\mu(Q) = \emptyset$ which implies $c_\mu(A) \cap Q = \emptyset$ and so $Q = \emptyset$. Therefore, it is not possible to express $c_\mu(A)$ as the union of two δ_μ-separated sets. Hence $c_\mu(A)$ is δ_μ-connected.

Theorem 2.6

Let (X, δ_μ) be a δ_μ-connected μ-proximity space and let $f\colon X \to Y$ be an onto, δ_μ-continuous function to another μ-proximity space (Y, δ'_μ). Then Y is δ'_μ-connected.

Proof. If possible, let Y be not δ_μ-connected. So there exist non-empty subsets C and D of Y such that $Y = C \cup D$ and $C\ \not\delta'_\mu\ D$. Since f is δ_μ-continuous, $f^{-1}(C)\ \not\delta_\mu\ f^{-1}(D)$. Again, $X = f^{-1}(Y) = f^{-1}(C) \cup f^{-1}(D)$, with $f^{-1}(C) \neq \emptyset$ and $f^{-1}(D) \neq \emptyset$ [since f is onto and both C and D are non-empty], which implies X is not δ_μ-connected, a contradiction. Therefore, Y is δ'_μ-connected.

Remark 2.7

If (X, δ_μ) is a μ-proximity space and $Y \subseteq X$, then we define a relation δ_μ^Y on the subsets of Y in the following manner

$$A\ \delta_\mu^Y\ B \Leftrightarrow A\ \delta_\mu\ B, \quad \text{where } A, B \subseteq Y.$$

It can be easily checked that (Y, δ_μ^Y) is a μ-proximity space. Moreover, the generalized topology generated by δ_μ^Y, i.e. $\tau(\delta_\mu^Y)$, is the generalized subspace topology induced by $\tau(\delta_\mu)$ on Y.

Theorem 2.8
In a μ-proximity space (X, δ_μ), suppose $\{A_\lambda : \lambda \in \Lambda\}$ is a family of δ_μ-connected subspaces of X. If there exists a $\lambda' \in \Lambda$ such that $A_{\lambda'}\ \delta_\mu\ A_\lambda$ for all $\lambda \in \Lambda$, then $\bigcup_{\lambda\in\Lambda} A_\lambda$ is δ_μ-connected.

Proof. Let $A, B \subseteq X$ and $\bigcup_{\lambda\in\Lambda} A_\lambda = A \cup B$ with $A \not\delta_\mu B$. We show that either $A = \emptyset$ or $B = \emptyset$. Clearly, $A_{\lambda'} \subseteq A \cup B$, so by Proposition 2.4, $A_{\lambda'} \subseteq A$ (without loss of generality).

Claim: for all $\lambda \in \Lambda$, $A_\lambda \subseteq A$. In fact, if for any $\lambda^* \in \Lambda$, $A_{\lambda^*} \subseteq B$ then since $A_{\lambda'}\ \delta_\mu\ A_{\lambda^*}$ we get $A\ \delta_\mu\ B$ [by (ii) of Definition 1.1]. Therefore, for all $\lambda \in \Lambda$, $A_\lambda \subseteq A$, which implies $B = \emptyset$ (as $A \cap B = \emptyset$). So, $\bigcup_{\lambda\in\Lambda} A_\lambda$ is δ_μ-connected.

Theorem 2.9
For a μ-proximity space (X, δ_μ) the following are equivalent

(i) *X is δ_μ-connected.*

(ii) *Every δ_μ-continuous function on X to X_d is constant.*

(iii) *For a non-empty proper subset A of X, $A\ \delta_\mu\ (X \setminus A)$.*

Proof. (i) $\Rightarrow$ (ii). Proved earlier.

(ii) $\Rightarrow$ (iii). Let $A \neq \emptyset$ and $A \subset X$. If possible, let $A \not\delta_\mu (X \setminus A)$. We define a function $f\colon X \to X_d$ by

$$f(x) = \begin{cases} 1 & \text{for } x \in A, \\ 0 & \text{for } x \notin A. \end{cases}$$

For $P, Q \subseteq X$ with $P\ \delta_\mu\ Q$ we claim that $f(P)\ \delta\ f(Q)$, where δ denotes the discrete proximity on X_d. If not, let $f(P) \not\delta f(Q)$ which implies, without loss of generality, $f(P) = \{1\}$ and $f(Q) = \{0\}$. Now, $P \subseteq A$ and $Q \subseteq (X \setminus A)$, so $A\delta(X \setminus A)$, a contradiction. Therefore, $f(P)\ \delta\ f(Q)$ and so f is δ_μ-continuous, which contradicts (ii). Hence $A\ \delta_\mu\ (X \setminus A)$.

(iii) $\Rightarrow$ (i). Straightforward.

Definition 2.10
A δ_μ-continuous function f from a μ-proximity space (X, δ_μ) to a μ-proximity space (Y, δ'_μ) is said to be δ_μ-monotone if for each $y \in Y$ the set $f^{-1}(\{y\})$ is δ_μ-connected in X.

We write the set $f^{-1}(\{y\})$ as $f^{-1}(y)$.

Definition 2.11
A δ_μ-continuous function f from a μ-proximity space (X, δ_μ) to a μ-proximity space (Y, δ'_μ) is called a δ_μ-quotient map if for each $C, D \subseteq Y$,

$$C\ \delta'_\mu\ D \Leftrightarrow f^{-1}(C)\ \delta_\mu\ f^{-1}(D).$$

Proposition 2.12
(X, δ_μ) and (Y, δ'_μ) are μ-proximity spaces and C is a δ_μ-connected set in Y. If $f\colon (X, \delta_\mu) \to (Y, \delta'_\mu)$ is a δ_μ-monotone and δ_μ-quotient function, then the set $f^{-1}(C)$ is δ_μ-connected in X.

Proof. If possible, let $f^{-1}(C)$ be not δ_μ-connected. So there exist non-empty δ_μ-separated subsets of X, say A and B, such that $f^{-1}(C) = A \cup B$. Since f is δ_μ-monotone, for each $y \in C$, $f^{-1}(y)$ is a δ_μ-connected subset of X. Therefore, for each $y \in C$, $f^{-1}(y)$ is contained either in A or in B, by Proposition 2.4. Consider the sets $C_A = \{y \in C : \ f^{-1}(y) \subseteq A\}$ and $C_B = \{y \in C : \ f^{-1}(y) \subseteq B\}$. Clearly, $f^{-1}(C_A) = A$ and $f^{-1}(C_B) = B$, also $C = C_A \cup C_B$. Again, f is a δ_μ-quotient map, so $f^{-1}(C_A) \not\delta_\mu f^{-1}(C_B) \Rightarrow C_A \not\delta'_\mu C_B \Rightarrow C$ is not δ_μ-connected, a contradiction. Therefore, $f^{-1}(C)$ is δ_μ-connected.

Definition 2.13

A finite collection of subsets $A_1, A_2, \ldots, A_n$ of a μ-proximity space X is said to be a δ_μ-chain if $A_i \ \delta_\mu \ A_{i+1}$ for each $i \in 1, 2, \ldots, n-1$.

A family $\mathcal{F}$ of subsets of X is said to be δ_μ-chained if for any two elements $A, B \in \mathcal{F}$, there exist finitely many elements of $C_1, C_2, \ldots, C_n$ in $\mathcal{F}$ such that $\mathcal{C} = \{A, C_1, C_2, \ldots, C_n, B\}$ is a δ_μ-chain. In such a case, we say that $\mathcal{C}$ joins A and B via the relation δ_μ.

Proposition 2.14

If $A_1, A_2, \ldots, A_n$ is a δ_μ-chain in a μ-proximity space X and each A_i is δ_μ-connected, where $i \in \{1, 2, \ldots, n\}$, then $\bigcup_{i=1}^n A_i$ is δ_μ-connected.

Proof. Let there exist two non-empty δ_μ-separated subsets C and D of X such that $\bigcup_{i=1}^n A_i = C \cup D$. Since each A_i is δ_μ-connected, each of those sets is contained either in C or in D, by Proposition 2.4.

We claim, without loss of generality, that $A_i \subseteq C$ for all $i \in \{1, 2, \ldots, n\}$. In fact, let $A_i \subseteq C$ and $A_j \subseteq D$ with $i \neq j$ and $i < j$. Since $A_i \ \delta_\mu \ A_{i+1}$ we must have $A_{i+1} \subseteq C$. Otherwise if $A_{i+1} \subseteq D$ then, by (ii) of Definition 1.1, $C \ \delta_\mu \ D$, a contradiction to the assumption. So for $A_i \subseteq C$ we have $A_{i+1} \subseteq C$. Continuing this process we get $A_j \subseteq C$. Therefore, $A_i \subseteq C$ for each $i \in \{1, 2, \ldots, n\}$ which implies $D = \emptyset$. Hence $\bigcup_{i=1}^n A_i$ is δ_μ-connected.

Proposition 2.15

Suppose (X, δ_μ) is a μ-proximity space and $\mathcal{F} = \{A_\lambda : \ \lambda \in \Lambda\}$ is a δ_μ-chained family of δ_μ-connected subsets of X. Then $\bigcup_{\lambda\in\Lambda} A_\lambda$ is δ_μ-connected.

Proof. Let U and V be two non-empty δ_μ-separated sets in (X, δ_μ) such that $\bigcup_{\lambda\in\Lambda} A_\lambda = U \cup V$.

We claim that for all $\lambda \in \Lambda$ either $A_\lambda \subseteq U$ or $A_\lambda \subseteq V$. If possible, let there exist $\lambda_1, \lambda_2 \in \Lambda$ such that $A_{\lambda_1} \subseteq U$ and $A_{\lambda_2} \subseteq V$. Since $\mathcal{F}$ is a δ_μ-chained family, there exists a δ_μ-chain, say $\{A_{\alpha_1}, A_{\alpha_2}, \ldots, A_{\alpha_n}\}$, where $A_{\alpha_i} \in \mathcal{F}, i \in \{1, 2, \ldots, n\}$ with $A_{\alpha_1} = A_{\lambda_1}$ and $A_{\alpha_n} = A_{\lambda_2}$, that joins A_{λ_1} and A_{λ_2} via the relation δ_μ.

We set $A = \bigcup_{i=1}^n A_{\alpha_i}$. Observe that by Proposition 2.14 A is δ_μ-connected. Again

$$A = \bigcup_{i=1}^{n} A_{\alpha_i} \subseteq \bigcup_{\lambda\in\Lambda} A_\lambda = U \cup V.$$

Therefore, by Proposition 2.4, either $A \subseteq U$ or $A \subseteq V$. Thus, A_{λ_1} and A_{λ_2} both are contained either in U or in V, a contradiction. Hence our claim is justified.

Without loss of generality, let $A_\lambda \subseteq U$ for all $\lambda \in \Lambda$, then $V = \emptyset$, as $U \cap V = \emptyset$. Hence $\bigcup_{\lambda\in\Lambda} A_\lambda$ is δ_μ-connected.

Definition 2.16
A cover $\mathcal{C}$ of a μ-proximity space X is said to be a δ_μ-cover if for $A, B \subseteq X$ with $A\ \delta_\mu\ B$ there exists a set $U \in \mathcal{C}$ such that $A \cap U \neq \emptyset$ and $B \cap U \neq \emptyset$.

Theorem 2.17
Every δ_μ-cover of a δ_μ-connected μ-proximity space is a δ_μ-chained family.

Proof. Let $\mathcal{C} = \{U_\alpha :\ \alpha \in \Lambda\}$ be a δ_μ-cover of a δ_μ-connected μ-proximity space (X, δ_μ). If possible, let there exist $U_{\alpha_1}, U_{\alpha_2} \in \mathcal{C}$ such that there does not exist any δ_μ-chain in $\mathcal{C}$ that joins U_{α_1} and U_{α_2}.

We define a set $U_\alpha \in \mathcal{C}$ to have property $\mathbf{P}$ if U_α and U_{α_1} are contained in some δ_μ-chain consisting of the elements of $\mathcal{C}$.

Let us consider two sets defined by $\mathcal{C}_1 = \{U_\alpha \in \mathcal{C} :\ U_\alpha$ has property $\mathbf{P}\}$ and $\mathcal{C}_2 = \{U_{\alpha'} \in \mathcal{C} :\ U_{\alpha'}$ does not have property $\mathbf{P}\}$. Clearly, the collection $\mathcal{C}_1$ is non-empty as $U_{\alpha_1} \in \mathcal{C}_1$.

We claim that $\mathcal{C}_2$ is empty. If possible, let $\mathcal{C}_2 \neq \emptyset$. We set $A = \bigcup_{U_\alpha\in\mathcal{C}_1} U_\alpha$ and $B = \bigcup_{U_{\alpha'}\in\mathcal{C}_2} U_{\alpha'}$. Clearly, $X = A \cup B$ and A, B are non-empty subsets of X. We assert that $A\ \not\delta_\mu\ B$. If not, let $A\ \delta_\mu\ B$, then there exists $U \in \mathcal{C}$ such that $A \cap U \neq \emptyset$ and $B \cap U \neq \emptyset$, which implies there exist $U_\alpha^A \in \mathcal{C}_1$ and $U_\alpha^B \in \mathcal{C}_2$ such that $U_\alpha^A \cap U \neq \emptyset$ and $U_\alpha^B \cap U \neq \emptyset$.

Now, U_α^A has property $\mathbf{P}$. So there exists a δ_μ-chain $\{U_1, U_2, \ldots, U_n\}$, where each $U_i \in \mathcal{C}$ for $i \in \{1, 2, \ldots, n\}$, that joins U_α^A with U_{α_1}. Again, $U_\alpha^A \cap U \neq \emptyset$ implies $U_\alpha^A\ \delta_\mu\ U$. Therefore the δ_μ-chain $\{U_1, U_2, \ldots, U_n, U_\alpha^A\}$ joins the sets U and U_{α_1}. Hence the set U has property $\mathbf{P}$.

Furthermore, $U_\alpha^B \cap U \neq \emptyset$ implies $U\ \delta_\mu\ U_\alpha^B$. Extending the δ_μ-chain as

$$\{U_1, U_2, \ldots, U_n, U_\alpha^A, U\}$$

we get that it joins the sets U_{α_1} and U_α^B. Therefore, U_α^B has property $\mathbf{P}$, a contradiction.

Hence the sets A and B are δ_μ-separated, but this implies X is not δ_μ-connected, a contradiction. Therefore, our claim is justified and $\mathcal{C}_2 = \emptyset$ which implies $\mathcal{C}_1 = \mathcal{C}$.

Now for any two sets $U_\alpha, U_\beta \in \mathcal{C}$, both U_α and U_β have property $\mathbf{P}$. Let $\{A_1, A_2, \ldots, A_n\}$ and $\{B_1, B_2, \ldots, B_m\}$ be the δ_μ-chains that join U_α with U_{α_1} and U_{α_1} with U_β, where each $A_i, B_j \in \mathcal{C}$, $i \in \{1, 2, \ldots, n\}$, $j \in \{1, 2, \ldots, m\}$. Consider the δ_μ-chain $\{A_1, A_2, \ldots, A_n, U_{\alpha_1}, B_1, B_2, \ldots, B_m\}$, clearly this chain joins U_α with U_β. Hence, $\mathcal{C}$ is a δ_μ-chained family.

3. δ_μ-component

Definition 3.1
A maximal δ_μ-connected subset, i.e. a δ_μ-connected subset which is not properly contained in any larger δ_μ-connected subset of X of a μ-proximity space X, is called a δ_μ-component of X.

For a point $x \in X$, $C_{\delta_\mu}(x)$ denotes the δ_μ-component of X, containing x.

Proposition 3.2
In a μ-proximity space (X, δ_μ),

$$C_{\delta_\mu}(x) = \bigcup \{A_x : \ A_x \text{ is a } \delta_\mu\text{-connected set containing } x\}.$$

Proof. Follows from Definition 3.1 and Theorem 2.8.

Theorem 3.3
In a μ-proximity space (X, δ_μ), for two distinct points $x, y \in X$, either $C_{\delta_\mu}(x) = C_{\delta_\mu}(y)$ or $C_{\delta_\mu}(x) \not\delta_\mu C_{\delta_\mu}(y)$.

Proof. Case 1. If $C_{\delta_\mu}(x) \not\delta_\mu C_{\delta_\mu}(y)$, then there is nothing to prove.

Case 2. If $C_{\delta_\mu}(x)\ \delta_\mu\ C_{\delta_\mu}(y)$, then the set $C_{\delta_\mu}(x) \cup C_{\delta_\mu}(y)$ is δ_μ-connected [by Theorem 2.8]. Since $C_{\delta_\mu}(x)$ and $C_{\delta_\mu}(y)$ are the maximal δ_μ-connected sets containing x and y respectively, we have $C_{\delta_\mu}(x) = C_{\delta_\mu}(x) \cup C_{\delta_\mu}(y) = C_{\delta_\mu}(y)$. This gives the desired result.

Corollary 3.4
A μ-proximity space (X, δ_μ) is the union of its δ_μ-components which are δ_μ-separated.

Proof. Follows from Theorem 3.3.

Proposition 3.5
In a μ-proximity space (X, δ_μ), δ_μ-components are μ-closed.

Proof. Let A be a δ_μ-component of X. Since $c_\mu(A)$ is δ_μ-connected [by Theorem 2.5] and $A \subseteq c_\mu(A)$, we must have $A = c_\mu(A)$. Hence A is δ_μ-closed.

Proposition 3.6
Suppose (X, δ_μ) and (Y, δ'_μ) are two μ-proximity spaces. $f\colon X \to Y$ is a δ_μ-monotone function and f is δ_μ-quotient function onto Y. Then C is a δ'_μ-component in Y if and only if $f^{-1}(C)$ is a δ_μ-component in X.

Proof. Let C be a δ'_μ-component in Y. By Proposition 2.12, $f^{-1}(C)$ is δ_μ-connected in X. Let D be a δ_μ-connected subset of X such that $f^{-1}(C) \subseteq D$. Now, $f(f^{-1}(C)) \subseteq f(D) \Rightarrow C \subseteq f(D)$. Since f is a δ_μ-quotient map, f is δ_μ-continuous. Now, by Theorem 2.6, $f(D)$ is δ_μ-connected. Again, C is a δ_μ-component, so $C = f(D)$ which implies $f^{-1}(C) = f^{-1}(f(D)) \supseteq D$. Therefore, $D = f^{-1}(C)$, i.e. $f^{-1}(C)$ is a δ_μ-component.

Conversely, let $f^{-1}(C)$ is a δ_μ-component in X, where $C \subseteq Y$. By Theorem 2.6, $f(f^{-1}(C)) = C$ is a δ_μ-connected set in Y. Let K be a δ_μ-connected subset in Y such that $C \subseteq K$. Now $f^{-1}(C) \subseteq f^{-1}(K)$. By Proposition 2.12, $f^{-1}(K)$ is δ_μ-connected. But $f^{-1}(C)$ is a δ_μ-component which implies $f^{-1}(C) = f^{-1}(K) \Rightarrow C = K$. Therefore, C is a δ'_μ-component in Y.

4. δ_μ-quasi component

Let (X, δ_μ) be a μ-proximity space. We define a relation ρ on X in the following way: for $x, y \in X$,

$$x \ \rho \ y \Leftrightarrow \text{there do not exist any } \delta_\mu\text{-separated sets } A \text{ and } B \text{ such that } x \in A, \\ y \in B \text{ and } X = A \cup B.$$

Theorem 4.1
ρ is an equivalence relation on X.

Proof. It is evident that ρ is reflexive and symmetric.

Let for $x, y, z \in X$, $x \ \rho \ y$ and $y \ \rho \ z$, we shall show that $x \ \rho \ z$ in order to show that ρ is transitive. If possible, let $x \not\rho \ z$ which implies there exist δ_μ-separated sets C and D such that $x \in C$, $z \in D$ with $X = C \cup D$. Observe that $y \in C \cup D$ implies $y \in C$ or $y \in D$. If $y \in C$ then $y \not\rho \ z$, which is not possible. Again, if $y \in D$, then $x \not\rho \ y$, another impossibility. Therefore ρ is transitive. This ends the proof.

From the above theorem we observe that ρ induces a partition of X and X can be expressed as the union of equivalence classes of ρ.

Definition 4.2
For $x \in X$, the set $\{y \in X : \ x \ \rho \ y\}$ is said to be the δ_μ-quasi component at the point x and is denoted by $Q_{\delta_\mu}(x)$.

Theorem 4.3
In a μ-proximity space (X, δ_μ), δ_μ-quasi components are μ-closed sets in $\tau(\delta_\mu)$.

Proof. Let $Q_{\delta_\mu}(x)$ be the δ_μ-quasi component at the point $x \in X$. Let $y \notin Q_{\delta_\mu}(x)$ which implies $x \not\rho \ y$, so there exist $C, D \subseteq X$ such that $x \in C$, $y \in D$ with $X = C \cup D$ and $C \not\delta_\mu \ D$. Now

$$z \in D \Rightarrow x \not\rho \ z \Rightarrow z \notin Q_{\delta_\mu}(x) \Rightarrow D \cap Q_{\delta_\mu}(x) = \emptyset.$$

Since $X = C \cup D$ we have $Q_{\delta_\mu}(x) \subseteq C$, which implies $Q_{\delta_\mu}(x) \not\delta_\mu \ D$ [otherwise by (ii) of Definition 1.1 we get $C \ \delta_\mu \ D$, a contradiction] and so $\{y\} \not\delta_\mu \ Q_{\delta_\mu}(x)$. Hence $y \notin c_\mu(Q_{\delta_\mu}(x))$ [from Proposition 1.3], therefore $Q_{\delta_\mu}(x)$ is μ-closed.

Theorem 4.4
In a μ-proximity space (X, δ_μ), $C_{\delta_\mu}(x) \subseteq Q_{\delta_\mu}(x)$, where $x \in X$.

Proof. Let $y \notin Q_{\delta_\mu}(x)$ then $x \not\rho \ y$ which implies there exist $A, B \subseteq X$ such that $x \in A$, $y \in B$ with $X = A \cup B$ and $A \not\delta_\mu \ B$. Now $x \in C_{\delta_\mu}(x)$ and

$$C_{\delta_\mu}(x) \subseteq A \cup B \Rightarrow C_{\delta_\mu}(x) \subseteq A \Rightarrow y \notin C_{\delta_\mu}(x).$$

Therefore, $y \notin Q_{\delta_\mu}(x) \Rightarrow y \notin C_{\delta_\mu}(x)$, hence $C_{\delta_\mu}(x) \subseteq Q_{\delta_\mu}(x)$.

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