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λ-statistical convergence of double sequences of functions via difference operators

Abstract. The paper aims to investigate λ-statistical convergence using modulus function and a generalized difference operator for double sequences of functions for order $\gamma \in (0, 1]$. Further, we prove that the statistical convergence in the newly formed sequence spaces is not well defined for $\gamma > 1$. Finally, we examine relevant inclusion relations concerning λ-statistical convergence and strongly λ-summable in the environment of the newly defined classes of double sequences of functions. Some interesting examples related to the examined results are also discussed in this paper.

1. Introduction and Preliminaries

The notion of statistical convergence, which is an extension of the idea of usual convergence was formerly given under the name "almost convergence" by Zygmund in the first edition of his celebrated monograph published in Warsaw in 1935 [45]. The concept was formally introduced by Fast [15] and later was reintroduced by Schoenberg [40], and also independently by Buck [3]. Although statistical convergence was introduced over nearly the last ninty years, it has become an active area of research for forty years with the contributions by several authors, Salat [39], Fridy [16], Di Maio and Kočinac [11], Çakalli and Khan [6]. For more work on statistical convergence, we refer the reader to [2], [4], [5], [6], [7], [22], [34] and [41]. The idea of statistical convergence was later extended to double sequences by Móricz [26], and Mursaleen and Edely [30]. Further, the notion of statistical

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convergence and its applications were studied by several authors ([17], [27], [28], [43], [44]). After that, statistical convergence of order α and strong $p-$Cesàro summability of order α were studied by Çolak in [9], and later Çolak and Altin [10] extended these results for double sequences. As an application, generalizations of statistical convergence appeared in the study of strong integral summability and the structure of ideals of bounded continuous functions on locally compact spaces. Statistical convergence and its generalizations were also connected with subsets of the Stone-Čech compactification of the natural numbers. Moreover, statistical convergence is closely related to the concept of convergence in probability. For more applications, one may refer to ([8], [18], [19], [21], [36], [37]). Recently, the notion of statistical convergence has been generalized through q-calculus (see [31], [32]).

A sequence $y = (y_{j,k})$ is convergent in Pringsheim sense if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ (set of all natural numbers) such that $|y_{j,k} - L| < \epsilon$ for all $j, k > N$.

Let $P \subseteq \mathbb{N} \times \mathbb{N}$ and $P(r, s) = \{(i, j) \in P : \ i \leq r, \ j \leq s\}$. The double natural density of P is defined by

$$\delta^2(P) = \lim_{r,s\to\infty} \frac{|P(r,s)|}{rs}, \text{ if the limit exists.}$$

A double sequence $y = (y_{j,k})$ is said to be statistically convergent to L (see [30]) if for every $\epsilon > 0$,

$$\lim_{r,s\to\infty} \frac{1}{rs} \left|\{(j,k) : j \leq r, \ k \leq s : \ |y_{j,k} - L| \geq \epsilon\}\right| = 0.$$

Note that every convergent sequence is statistical convergent but the converse does not have to be true. Consider the example

$$x_{jk} = \begin{cases} j^2, \ j,k = n^2, \\ 0, \ \text{ otherwise.} \end{cases}$$

Clearly, (x_{jk}) is statistically convergent to 0, but does not converge to 0.

A sequence $y = (y_{j,k})$ is said to be strongly Cesàro summable to l (see [30]) if

$$\lim_{r,s\to\infty} \frac{1}{rs} \sum_{j=1}^{r} \sum_{k=1}^{s} |y_{j,k} - l| = 0.$$

Throughout this paper, we will take $\gamma = (a, b)$ and $\chi = (c, d)$, where $a, b, c, d \in (0, 1]$, otherwise stated. Also, we define the following:

$$\begin{aligned} \gamma \leq \chi \quad &\text{iff} \quad a \leq c \text{ and } b \leq d, \\ \gamma < \chi \quad &\text{iff} \quad a < c \text{ and } b < d, \\ \gamma = \chi \quad &\text{iff} \quad a = c \text{ and } b = d, \\ 0 < \gamma \leq 1 \quad &\text{iff} \quad a, b \in (0, 1] \\ 0 < \chi \leq 1 \quad &\text{iff} \quad c, d \in (0, 1], \end{aligned}$$

$$\gamma = 1 \quad \text{iff} \quad a = b = 1,$$
$$\chi = 1 \quad \text{iff} \quad c = d = 1,$$
$$\gamma > 1 \text{ in case } a > 1 \text{ and } b > 1.$$

Now we define the γ-double natural density of the set $P \subseteq \mathbb{N} \times \mathbb{N}$ as

$$\delta^2_\gamma(P) = \lim_{r,s\to\infty} \frac{|P(r,s)|}{r^a s^b}, \text{ if the limit exists.}$$

A sequence $y = (y_{j,k})$ is said to be statistically convergent to L of order γ (see [10]) if for every $\epsilon > 0$,

$$\lim_{r,s\to\infty} \frac{1}{r^a s^b} \big|\{(j,k) : j \le r,\ k \le s :\ |y_{j,k} - L| \ge \epsilon\}\big| = 0.$$

For $\gamma = 1$, the statistical convergence of order γ coincides with statistical convergence of double sequences.

In 1953, the notion of a modulus function was introduced by Nakano [33]. This concept was later used by Ruckle [38] and Maddox [24] to construct some sequences spaces. A modulus g is a function from $[0,\infty)$ to $[0,\infty)$ such that

(i) $g(x) = 0$ if and only if $x = 0$,

(ii) $g(x + y) \le g(x) + g(y)$ for every $x, y \in \mathbb{R}^+$,

(iii) g is increasing,

(iv) g is continuous from the right at 0.

In 2012, Aizpuru et al. in [1] defined statistical convergence of double sequences using a modulus function. Torgut and Altin [42] further studied this work by using double sequences of order $\widetilde{\beta}$.

λ-statistical convergence of double sequences and the de la Vallée-Poussin mean for double sequences has been expressed by Mursaleen et al. in [29].

Let $\lambda = (\lambda_r)$ and $\mu = (\mu_s)$ be two sequences of real numbers such that $\lambda_r \le \lambda_{r+1},\ \mu_s \le \mu_{s+1}$ and $\lambda_r > 0,\ \mu_s > 0$ for all $r, s \in \mathbb{N}$ tending to ∞ with $\lambda_{r+1} \le \lambda_r + 1, \lambda_1 = 1$; $\mu_{s+1} \le \mu_s + 1, \mu_1 = 1$ be given. The de la Vallée-Poussin mean for a double sequence $x = (x_{j,k})$ is given as

$$t_{r,s}(x) = \frac{1}{\lambda_r \mu_s} \sum_{j\in J_r} \sum_{k\in K_s} x_{j,k},$$

where $J_r = [r - \lambda_r + 1, r]$ and $K_s = [s - \mu_s + 1, s]$. A sequence $y = (y_{j,k})$ is said to be strongly (V, λ, μ) summable to a number l if

$$\lim_{r,s\to\infty} \frac{1}{\lambda_r \mu_s} \sum_{j\in J_r} \sum_{k\in K_s} |y_{j,k} - l| = 0.$$

If $\lambda_r = r$ and $\mu_s = s$ then strongly (V, λ, μ) summability reduces to strongly Cesàro summability.

Isik and Altin [21] developed the concept of (λ, μ)-statistical convergence by a modulus function of order $\widetilde{\alpha}$ for double sequences. Let $P \subseteq \mathbb{N} \times \mathbb{N}$. Then $\delta_{\gamma}^{g2}(P)$-double density of P is defined as:

$$\delta_{\gamma}^{g2}(P) = \lim_{r,s\to\infty} \frac{1}{g(\lambda_r^a \mu_s^b)} g\big(\big|\{(j,k) \in J_r \times K_s : \ (j,k) \in P\}\big|\big), \text{ if the limit exists.}$$

A sequence $y = (y_{j,k})$ is said to be $g_{(\lambda,\mu)}$-statistically convergent to L of order γ if for every $\epsilon > 0$,

$$\lim_{r,s\to\infty} \frac{1}{g(\lambda_r^a \mu_s^b)} g\big(\big|\{(j,k) \in J_r \times K_s : \ |y_{j,k} - L| \geq \epsilon\}\big|\big) = 0.$$

For $g(x) = x$, $\lambda_r = r$, $\mu_s = s$ and $\gamma = 1$, $g_{(\lambda,\mu)}$-statistical convergence of order γ reduces to the statistical convergence of double sequences.

The study of sequence spaces through difference operators has become an emerging area of research in mathematical sciences providing a wide range of applications in several branches of science and engineering. The fundamental idea of difference operators is being often used in various fields of mathematics such as summability, operator theory, linear algebra, numerical analysis, calculus, and approximation theory. It has also been extensively used in the development of fractional calculus along with its original classical theory.

The notion of difference sequences was defined by Kizmaz[23] and later Et and Colak [14] generalized it as $\Delta^r(X) = \{x = (x_k) : \ \Delta^r x_k \in X\}$ for $X = l_\infty,\ c$ or c_0, where $r \in \mathbb{N}$, l_∞, c and c_0 are the spaces of bounded, convergent, and null sequences, respectively. $\Delta^0 x = (x_k)$, $\Delta^r x = \Delta^{r-1} x_k - \Delta^{r-1} x_{k+1}$ and so $\Delta^r x_k = \sum_{i=0}^{r} (-1)^i \binom{r}{i} x_{k+i}$. For a double sequence $y = (y_{j,k})$, we have generalized difference sequences as follows:

$$\Delta^r y_{j,k} = \sum_{u_1=0}^{r} \sum_{u_2=0}^{r} (-1)^{u_1+u_2} \binom{r}{u_1} \binom{r}{u_2} y_{j+u_1,k+u_2},$$

where $\Delta y_{j,k} = y_{j,k} - y_{j,k+1} - y_{j+1,k} + y_{j+1,k+1}$ for all $j,k,r \in \mathbb{N}$. For more results on difference sequence spaces, one may refer to [13], [14] and [35].

Pointwise convergence and uniform statistical convergence of sequences of real-valued functions were defined by Gókhan et al. ([18], [19], [20]) and independently by Duman and Orhan in [12]. Çinar and Et [8] introduced the concept of Δ^r-statistical convergence of order γ for double sequences of functions.

A sequence of real-valued functions $\{h_{jk}\}$ is said to be Δ^r-pointwise convergent to the function h on a set B if for every $\epsilon > 0$ and for every $x \in B$, there exists $N \in \mathbb{N}$ such that $|\Delta^r h_{jk}(x) - h(x)| < \epsilon$ for all $j,k > N$.

A sequence of real-valued functions $\{h_{jk}\}$ is said to be Δ^r-pointwise statistical convergent of order γ to the function h on a set B if for every $\epsilon > 0$ and for every $x \in B$,

$$\lim_{r,s\to\infty} \frac{1}{r^a s^b} \big|\{(j,k) : j \leq r,\ k \leq \ s : \ |\Delta^r h_{jk}(x) - h(x)| \geq \epsilon\}\big| = 0,$$

i.e. for every $x \in B$, $|\Delta^r h_{jk}(x) - h(x)| < \epsilon$ a.a (j,k) γ (see [8]).

Let p be a positive real number. A sequence of functions $\{h_{jk}\}$ is said to be strongly Δ_p^r-pointwise Cesàro summable of order γ to the function h on a set B (see [8]) if

$$\lim_{r,s\to\infty} \frac{1}{r^a s^b} \sum_{j=1}^{r} \sum_{k=1}^{s} |\Delta^r h_{jk}(x) - h(x)|^p = 0.$$

Motivated by the above discussion, here we investigate λ-statistical convergence by means of a modulus function and a generalized difference operator for double sequences of functions for order γ. Moreover, we show that every Δ^r-pointwise convergent sequence of functions is pointwise statistical convergent for our newly defined sequence space and give some compelling instances to show that the converse does not hold. Furthermore, we establish a relation among λ-statistical convergence and strongly λ-summable for our sequence spaces.

2. Main result

Definition 2.1

Let g be an unbounded modulus function. Let $\lambda = (\lambda_r)$ and $\mu = (\mu_s)$ be two sequences as defined above and $\gamma \in (0, 1]$. A sequence of real-valued functions $\{h_{jk}\}$ is said to be $(g_{(\lambda,\mu)}, \Delta^r)$-pointwise statistically convergent of order γ to the function h on a set B if for every $\epsilon > 0$ and every $x \in B$,

$$\lim_{r,s\to\infty} \frac{1}{g(\lambda_r^a \mu_s^b)} g\big(\big|\{(j,k) \in J_r \times K_s : \ |\Delta^r h_{jk}(x) - h(x)| \geq \epsilon\}\big|\big) = 0,$$

where $J_r = [r - \lambda_r + 1, r]$ and $K_s = [s - \mu_s + 1, s]$.

In this case, we write $\lim_{r,s\to\infty} \Delta^r h_{jk}(x) \overset{S^2}{=} h(x)$ on B. The set of all $(g_{(\lambda,\mu)}, \Delta^r)$-pointwise statistically convergent sequence of functions of order γ will be denoted by $S_\gamma^2(g_{(\lambda,\mu)}, \Delta^r)$. If we take $\gamma = 1$, that is, $a = 1$, $b = 1$, $S^2(g_{(\lambda,\mu)}, \Delta^r)$ denotes the set of all $(g_{(\lambda,\mu)}, \Delta^r)$-pointwise statistically convergent sequences of functions. Notice that if $g(x) = x$, $\lambda_r = r$, $\mu_s = s$, then $(g_{(\lambda,\mu)}, \Delta^r)$-pointwise statistical convergence of order γ coincides with Δ^r-pointwise statistical convergence of order γ.

Remark 2.1

Observe that $(g_{(\lambda,\mu)}, \Delta^r)$-pointwise statistical convergence of order γ holds for $0 < \gamma \leq 1$. But it does not need to hold for $\gamma > 1$. For this, let g be an unbounded modulus such that $\lim_{t\to\infty} \frac{g(t)}{t} > 0$ and $\{h_{jk}\}$ be defined as follows:

$$h_{j,k}(x) = \begin{cases} 2, & j+k \text{ is even,} \\ x^j, & j+k \text{ is odd,} \end{cases} \qquad x \in [0, 1/2].$$

Then we calculate $\Delta h_{jk}(x)$ as follows

$$\Delta h_{j,k}(x) = \begin{cases} 4 - x^{j+1} - x^j, & j+k \text{ is even,} \\ x^j + x^{j+1} - 4, & j+k \text{ is odd.} \end{cases}$$

Thus for every $x \in [0, 1/2]$, we get

$$\lim_{r,s\to\infty} \frac{1}{g(\lambda_r^a \mu_s^b)} g\big(\big|\{(j,k) \in J_r \times K_s :\ |\Delta h_{jk}(x) - 4| \ge \epsilon\}\big|\big) \le \frac{g(\lfloor \lambda_r \mu_s \rfloor) + 1}{g(2\lambda_r^a \mu_s^b)} = 0$$

and

$$\lim_{r,s\to\infty} \frac{1}{g(\lambda_r^a \mu_s^b)} g\big(\big|\{(j,k) \in J_r \times K_s : |\Delta h_{jk}(x) - (-4)| \ge \epsilon\}\big|\big) \le \frac{g(\lfloor \lambda_r \mu_s \rfloor) + 1}{g(2\lambda_r^a \mu_s^b)} = 0$$

for $\gamma > 1$, where $\lfloor \cdot \rfloor$ denotes the floor function. Hence $\lim_{r,s\to\infty} \Delta h_{jk}(x) \overset{S^2}{=} 4$ and $\lim_{r,s\to\infty} \Delta h_{jk}(x) \overset{S^2}{=} -4$, which is impossible.

Remark 2.2

Every Δ^r-pointwise convergent sequence of functions is $(g_{(\lambda,\mu)}, \Delta^r)$-pointwise statistically convergent of order γ for each $\gamma \in (0, 1]$, but the converse does not hold. For this, let g be an unbounded modulus such that $\lim_{t\to\infty} \frac{g(t)}{t} > 0$ and $\{f_{jk}\}$ be defined as follows:

$$f_{jk}(x) = \begin{cases} 2, & j,k = n^3, \\ \frac{kx}{1+j^2k^2x^2}, & \text{otherwise.} \end{cases}$$

Then we calculate $\Delta f_{jk}(x)$,

$$\Delta f_{jk}(x) = \begin{cases} 2 - \frac{kx}{1+(j+1)^2k^2x^2} - \frac{(k+1)x}{1+j^2(k+1)^2x^2} + \frac{(k+1x)}{1+(j+1)^2(k+1)^2x^2}, & j,k = n^3, \\ \frac{kx}{1+j^2k^2x^2} - \frac{kx}{1+(j+1)^2k^2x^2} - \frac{(k+1)x}{1+j^2(k+1)^2x^2} + 2, & j,k = n^3 - 1, \\ \frac{kx}{1+j^2k^2x^2} - \frac{kx}{1+(j+1)^2k^2x^2} - \frac{(k+1)x}{1+j^2(k+1)^2x^2} + \frac{(k+1x)}{1+(j+1)^2(k+1)^2x^2}, & \text{otherwise,} \end{cases}$$

we get

$$\lim_{r,s\to\infty} \frac{1}{g(\lambda_r^a \mu_s^b)} g\big(\big|\{(j,k) \in J_r \times K_s :\ |\Delta f_{jk}(x)| \ge \epsilon\}\big|\big) \le \frac{g(\lfloor \lambda_r \mu_s \rfloor^{\frac{1}{3}}) + 1}{g(\lambda_r^a \mu_s^b)} = 0$$

for $a > \frac{1}{3}$, $b > \frac{1}{3}$, i.e. $\gamma > \frac{1}{3}$.

The sequence $\{f_{jk}\}$ is $(g_{(\lambda,\mu)}, \Delta)$-pointwise statistically convergent of order γ with $\lim \Delta f_{jk}(x) \overset{S^2}{=} 0$ for $\gamma > 1/3$, but it is not Δ-pointwise convergent.

Theorem 2.1

Let $\gamma \in (0, 1]$, $\{h_{jk}\}$ and $\{f_{jk}\}$ be two sequences of real-valued functions defined on a set B. Then the following statements are true:

(i) *If $\lim_{r,s\to\infty} \Delta^r h_{j,k}(x) \overset{S^2}{=} h(x)$ and c be any real number, then*

$$\lim_{r,s\to\infty} c\Delta^r h_{j,k}(x) \overset{S^2}{=} c\, h(x).$$

(ii) *If* $\lim_{r,s\to\infty} \Delta^r h_{j,k}(x) \overset{S^2}{=} h(x)$ *and* $\lim_{r,s\to\infty} \Delta^r f_{jk}(x) \overset{S^2}{=} f(x)$*, then*

$$\lim_{r,s\to\infty} (\Delta^r h_{jk}(x) + \Delta^r f_{jk}(x)) \overset{S^2}{=} h(x) + f(x).$$

Proof. For $c = 0$, the result holds trivially. Let $c \neq 0$, we have

$$\begin{aligned}&\frac{1}{g(\lambda_r^a \mu_s^b)} g\big(\big|\{(j,k) \in J_r \times K_s :\ |c\Delta^r h_{jk}(x) - ch(x)| \geq \epsilon\}\big|\big)\\ &\qquad = \frac{1}{g(\lambda_r^a \mu_s^b)} g\Big(\Big|\Big\{(j,k) \in J_r \times K_s :\ |\Delta^r h_{jk}(x) - h(x)| \geq \frac{\epsilon}{c}\Big\}\Big|\Big)\end{aligned}$$

and that of (ii) follows from

$$\begin{aligned}&\frac{1}{g(\lambda_r^a \mu_s^b)} g\big(\big|\{(j,k) \in J_r \times K_s :\ |\Delta^r h_{jk}(x) + \Delta^r f_{jk}(x) - (h(x) + f(x))| \geq \epsilon\}\big|\big)\\ &\qquad \leq \frac{1}{g(\lambda_r^a \mu_s^b)} g\Big(\Big|\Big\{(j,k) \in J_r \times K_s :\ |\Delta^r h_{jk}(x) - h(x)| \geq \frac{\epsilon}{2}\Big\}\Big|\Big)\\ &\qquad\quad + \frac{1}{g(\lambda_r^a \mu_s^b)} g\Big(\Big|\Big\{(j,k) \in J_r \times K_s :\ |\Delta^r f_{jk}(x) - f(x)| \geq \frac{\epsilon}{2}\Big\}\Big|\Big).\end{aligned}$$

Theorem 2.2

Suppose g is an unbounded modulus function and $\gamma, \chi \in (0,1]$ be such that $\gamma \leq \chi$. Then

$$S_\gamma^2(g_{(\lambda,\mu)}, \Delta^r) \subset S_\chi^2(g_{(\lambda,\mu)}, \Delta^r)$$

and the inclusion is strict.

Proof. Let $\gamma, \chi \in (0,1]$ be given such that $\gamma \leq \chi$. Since g is increasing, we have

$$\begin{aligned}&\frac{1}{g(\lambda_r^c \mu_s^d)} g\big(\big|\{(j,k) \in J_r \times K_s :\ |\Delta^r f_{jk}(x) - f(x)| \geq \epsilon\}\big|\big)\\ &\qquad \leq \frac{1}{g(\lambda_r^a \mu_s^b)} g\big(\big|\{(j,k) \in J_r \times K_s :\ |\Delta^r f_{jk}(x) - f(x)| \geq \epsilon\}\big|\big) \quad \text{for all } \epsilon > 0.\end{aligned}$$

Hence, $S_\gamma^2(g_{(\lambda,\mu)}, \Delta^r) \subset S_\chi^2(g_{(\lambda,\mu)}, \Delta^r)$.

To show that the inclusion is strict, let g be an unbounded modulus such that $\lim_{t\to\infty} \frac{g(t)}{t} > 0$ and $\{f_{jk}\}$ be defined as follows:

$$f_{jk}(x) = \begin{cases} 2, & j,k = n^3,\\ \frac{kx}{1+j^2k^2x^2}, & \text{otherwise.}\end{cases}$$

Then

$$\begin{aligned}&\Delta f_{jk}(x)\\ &= \begin{cases} 2 - \frac{kx}{1+(j+1)^2k^2x^2} - \frac{(k+1)x}{1+j^2(k+1)^2x^2} + \frac{(k+1x)}{1+(j+1)^2(k+1)^2x^2}, & j,k = n^3,\\ \frac{kx}{1+j^2k^2x^2} - \frac{kx}{1+(j+1)^2k^2x^2} - \frac{(k+1)x}{1+j^2(k+1)^2x^2} + 2, & j,k = n^3 - 1,\\ \frac{kx}{1+j^2k^2x^2} - \frac{kx}{1+(j+1)^2k^2x^2} - \frac{(k+1)x}{1+j^2(k+1)^2x^2} + \frac{(k+1x)}{1+(j+1)^2(k+1)^2x^2}, & \text{otherwise.}\end{cases}\end{aligned}$$

Further,

$$\lim_{r,s\to\infty} \frac{1}{g(\lambda_r^c \mu_s^d)} g\big(\big|\{(j,k) \in J_r \times K_s : |\Delta f_{jk}(x)| \geq \epsilon\}\big|\big) \leq \lim_{r,s\to\infty} \frac{g(\lfloor \lambda_r \mu_s \rfloor^{\frac{1}{3}}) + 1}{g(\lambda_r^c \mu_s^d)} = 0$$

for $\chi > \frac{1}{3}$. Hence, $\{f_{jk}\} \in S_\chi^2(g_{(\lambda,\mu)}, \Delta)$ for $\chi \in (1/3, 1]$ but $\{f_{jk}\} \notin S_\gamma^2(g_{(\lambda,\mu)}, \Delta)$ for $\gamma \in (0, 1/3]$.

If we take $\chi = 1$ in the Theorem 2.2, we get the following result.

Corollary 2.1
If a sequence of functions $\{f_{jk}\}$ is $(g_{(\lambda,\mu)}, \Delta^r)$-pointwise statistical convergent of order γ to the function f, then it is $(g_{(\lambda,\mu)}, \Delta^r)$-pointwise statistical convergent to the function f, that is, $S_\gamma^2(g_{(\lambda,\mu)}, \Delta^r) \subset S^2(g_{(\lambda,\mu)}, \Delta^r)$ for each $\gamma \in (0,1]$, and the inclusion is strict.

Corollary 2.2
Let $\gamma, \chi \in (0,1]$ be given. Then the following hold:

(i) $S_\gamma^2(g_{(\lambda,\mu)}, \Delta^r) = S_\chi^2(g_{(\lambda,\mu)}, \Delta^r)$ *iff* $\gamma = \chi$,

(ii) $S_\gamma^2(g_{(\lambda,\mu)}, \Delta^r) = S^2(g_{(\lambda,\mu)}, \Delta^r)$ *iff* $\gamma = 1$.

Definition 2.2
Let g be an unbounded modulus function. Let $\lambda = (\lambda_r)$ and $\mu = (\mu_s)$ be two sequences as defined above and $\gamma \in (0,1]$. A sequence of real-valued functions $\{h_{jk}\}$ is strongly $(V, \lambda, \mu, g)\, \Delta_p^r$-pointwise summable of order γ (or $[w_p^2]_\gamma^{g,\lambda,\mu}(\Delta^r)$-summable) on a set B if there is a function h such that

$$\lim_{r,s\to\infty} \frac{1}{\lambda_r^a \mu_s^b} \sum_{j\in J_r} \sum_{k\in K_s} g(|\Delta^r h_{jk}(x) - h(x)|^p) = 0.$$

Thus, we write $[w_p^2]_\gamma^{g,\lambda,\mu} - \lim \Delta^r h_{j,k}(x) = h(x)$ on B. The set of all strongly $(V, \lambda, \mu, g)\, \Delta_p^r$-pointwise summable double sequence of functions of order γ will be denoted by $[w_p^2]_\gamma^{g,\lambda,\mu}(\Delta^r)$.

Here note that, if $g(x) = x$. $\lambda_r = r$ and $\mu_s = s$ then strongly $(V, \lambda, \mu, g)\, \Delta_p^r$-pointwise summable of order γ reduces to strongly Δ_p^r-pointwise Cesàro summable of order γ.

Theorem 2.3
Let g be an unbounded modulus function and $\gamma, \chi \in (0,1]$ be given such that $\gamma \leq \chi$ then

$$[w_p^2]_\gamma^{g,\lambda,\mu}(\Delta^r) \subset [w_p^2]_\chi^{g,\lambda,\mu}(\Delta^r)$$

and the inclusion is strict for some $\gamma < \chi$.

Proof. The inclusion part of the proof is easy. To show that the inclusion is strict, define a double sequence $\{f_{jk}\}$ by

$$f_{jk}(x) = \begin{cases} \frac{jkx}{1+j^2k^2x^2}, & j,k = n^2, \\ 1, & \text{otherwise.} \end{cases}$$

Then

$$\Delta f_{jk}(x) = \begin{cases} \frac{jkx}{1+j^2k^2x^2} - 1, & j,k = n^2, \\ 1 + \frac{(j+1)(k+1)x}{1+(j+1)^2(k+1)^2x^2}, & j,k = n^2-1, \\ 0, & \text{otherwise} \end{cases}$$

and taking $g(x) = x$ we get

$$\frac{1}{\lambda_r^c \mu_s^d} \sum_{j\in J_r} \sum_{k\in K_s} |\Delta f_{jk}(x) - f(x)|^p \le \frac{(\lambda_r)^{\frac{1}{2}}(\mu_s)^{\frac{1}{2}}}{\lambda_r^c \mu_s^d} \le \frac{1}{\lambda_r^{c-\frac{1}{2}} \mu_s^{d-\frac{1}{2}}} \to 0$$

as $r,s\to\infty$ for $c>\frac{1}{2}$, $d>\frac{1}{2}$ and so the sequence $\{f_{jk}\}$ is strongly $(V,\lambda,\mu,g)\,\Delta_p$-pointwise summable of order χ for $\chi\in(\frac{1}{2},1]$. Also,

$$\frac{1}{\lambda_r^a \mu_s^b} \sum_{j\in J_r} \sum_{k\in K_s} |\Delta f_{jk}(x) - f(x)|^p \ge \frac{((\lambda_r)^{\frac{1}{2}}-1)((\mu_s)^{\frac{1}{2}}-1)}{\lambda_r^a \mu_s^b} \to \infty$$

as $r,s\to\infty$ for $a<\frac{1}{2}$, $b<\frac{1}{2}$.

Hence the sequence $\{f_{jk}\}$ is not strongly $(V,\lambda,\mu,g)\,\Delta_p$-pointwise summable of order γ for $\gamma\in(0,\frac{1}{2}]$.

If we take $\chi=1$ in the Theorem 2.3, then we obtain the following result.

Corollary 2.3

If $\{f_{jk}\}$ is strongly $(V,\lambda,\mu,g)\,\Delta_p^r$-pointwise summable of order γ then it is strongly $(V,\lambda,\mu,g)\,\Delta_p^r$-pointwise summable, that is, $[w_p^2]_\gamma^{g,\lambda,\mu}(\Delta^r)\subset[w_p^2]^{g,\lambda,\mu}(\Delta^r)$ for each $\gamma\in(0,1]$ and $0<p<\infty$ and the strict inclusion holds.

The following result is a consequence of Theorem 2.3.

Corollary 2.4

Let $\gamma,\chi\in(0,1]$ and $0<p<\infty$. Then the following statements are true

(i) $[w_p^2]_\gamma^{g,\lambda,\mu}(\Delta^r,f)=[w_p^2]_\chi^{g,\lambda,\mu}(\Delta^r,f)$ *iff* $\gamma=\chi$,

(ii) $[w_p^2]_\gamma^{g,\lambda,\mu}(\Delta^r,f)=[w_p^2]^{g,\lambda,\mu}(\Delta^r,f)$ *iff* $\gamma=1$.

Before proceeding to the proof of our next theorem, we first need to recall an important result by Maddox, which plays a crucial role in the analysis of modulus functions and their behavior. This result establishes a key inequality for unbounded modulus functions, which will help us derive important relationship in our proof. The lemma, as stated by Maddox [25], is presented below:

Lemma 2.1 ([25])

Suppose that g is an unbounded function. Then there exists an $l>0$ such that $g(xy)\ge lg(x)g(y)$ for every non negative real number x,y.

Theorem 2.4

Suppose that $\gamma,\chi\in(0,1]$ with $\gamma\le\chi$, $0<p<\infty$ and g be an unbounded modulus function with $\lim_{t\to\infty}\frac{g(t)}{t}>0$. If $\{h_{jk}\}$ is strongly $(V,\lambda,\mu,g)\,\Delta_p^r$-pointwise summable of order γ to $h(x)$, then it is $(g_{(\lambda,\mu)},\Delta^r)$-pointwise statistical convergent of order χ to $h(x)$, that is $[w_p^2]_\gamma^{g,\lambda,\mu}(\Delta^r)\subseteq S_\chi^2(g_{(\lambda,\mu)},\Delta^r)$.

Proof. For any $\epsilon > 0$ we have

$$\begin{aligned}\sum_{j\in J_r}\sum_{k\in K_s} g(|\Delta^r h_{jk}(x)-h(x)|^p) \\ &\geq g(\sum_{j\in J_r}\sum_{k\in K_s}|\Delta^r h_{jk}(x)-h(x)|^p) \\ &\geq g\big(\big|\{(j,k)\in J_r\times K_s:\ |\Delta^r h_{jk}(x)-h(x)|\geq\epsilon\}\big|\cdot\epsilon^p\big) \\ &\geq l\,g\big(\big|\{(j,k)\in J_r\times K_s:\ |\Delta^r h_{jk}(x)-h(x)|\geq\epsilon\}\big|\big)\cdot g(\epsilon^p)\end{aligned}$$

and since $\gamma \leq \chi$,

$$\begin{aligned}&\frac{1}{\lambda_r^a\mu_s^b}\sum_{j\in J_r}\sum_{k\in K_s} g(|\Delta^r h_{jk}(x)-h(x)|^p) \\ &\geq \frac{1}{\lambda_r^a\mu_s^b} l\,g\big(\big|\{(j,k)\in J_r\times K_s:\ |\Delta^r h_{jk}(x)-h(x)|\geq\epsilon\}\big|\big)\cdot g(\epsilon^p) \\ &\geq \frac{1}{\lambda_r^c\mu_s^d} l\,g\big(\big|\{(j,k)\in J_r\times K_s:\ |\Delta^r h_{jk}(x)-h(x)|\geq\epsilon\}\big|\big)\cdot g(\epsilon^p) \\ &= \frac{1}{\lambda_r^c\mu_s^d\, g(\lambda_r^c\mu_s^d)} l\,g\big(\big|\{(j,k)\in J_r\times K_s:\ |\Delta^r h_{jk}(x)-h(x)|\geq\epsilon\}\big|\big)\cdot g(\epsilon^p)\, g(\lambda_r^c\mu_s^d),\end{aligned}$$

where using the fact that $\lim_{t\to\infty}\frac{g(t)}{t} > 0$ and $\{h_{jk}\} \in [w_p^2]_\gamma^{g,\lambda,\mu}(\Delta^r)$, it follows that $\{h_{jk}\} \in S_\gamma^2(g_{(\lambda,\mu)}, \Delta^r)$.

Corollary 2.5
Suppose that $0 < \gamma \leq 1$, $0 < p < \infty$. If $\{f_{jk}\}$ is strongly $(V, \lambda, \mu, g)\,\Delta_p^r$-pointwise summable of order γ to $f(x)$, then it is $(g_{(\lambda,\mu)}, \Delta^r)$-pointwise statistical convergent of order γ to $f(x)$, that is, $[w_p^2]_\gamma^{g,\lambda,\mu}(\Delta^r) \subseteq S_\gamma^2(g_{(\lambda,\mu)}, \Delta^r)$.

Corollary 2.6
Suppose that $0 < \gamma \leq 1$, $0 < p < \infty$. If $\{f_{jk}\}$ is strongly $(V, \lambda, \mu, g)\,\Delta_p^r$-pointwise summable of order γ to $f(x)$, then it is $(g_{(\lambda,\mu)}, \Delta^r)$-pointwise statistical convergent to $f(x)$, that is $[w_p^2]_\gamma^{g,\lambda,\mu}(\Delta^r) \subseteq S^2(g_{(\lambda,\mu)}, \Delta^r)$.

Proof. From Corollary 2.1 and Corollary 2.5, we have

$$[w_p^2]_\gamma^{g,\lambda,\mu}(\Delta^r, f) \subseteq S^2(g_{(\lambda,\mu)}, \Delta^r, f).$$

3. Conclusion

In the present paper, we explore $(g_{(\lambda,\mu)}, \Delta^r)$-pointwise statistical convergence for double sequences of real-valued functions of order γ, where g is an unbounded modulus function. In addition, we also introduce strongly $(V, \lambda, \mu, g)\,\Delta_p^r$-pointwise summable for double sequences of real-valued functions of order γ and give some inclusion relations. In future work, one can obtain the corresponding results by using uniform statistical convergence for the sequence of fuzzy valued functions.

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