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Application of sine-Gordon model in description of Josephson junction – present state of affairs

Abstract. Josephson junctions find applications for construction of various devices in measuring, transmitting, receiving, and amplifying systems as well as classical and quantum computing devices. Some of the existing and future applications are related to shape engineering, which allows for adjustment of the parameters of the junction to the needs of various applications. The article includes review of both a description of the applications and theoretical considerations devoted to the Josephson junctions and their properties.

1. Superconductors – the base for Josephson junctions

Superconductivity was discovered in 1911 by H. Kamerlingh Onnes. Conventional superconductors usually have extremely low critical temperatures. Under normal pressure these temperatures take values up to a fraction of a Kelvin degree. The highest critical temperatures reach the value of several to several dozen Kelvin degrees. The highest critical temperature found for a conventional superconductor is on the level of 203 K for H_2S. This temperature was reached at extremely high pressures of up to 90 GPa [18].

At normal pressure, it is possible to achieve a state of superconductivity at higher temperatures in the so-called high temperature superconductors [3]. An example of such materials is cuprate-perovskite ceramic which has a critical temperature above 90 K. Exceeding the temperature of 77 K is significant because it allows the use of cheap cooling methods in experiments or superconducting devices.

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The temperature above 77 K allows the use of liquid nitrogen in experiments on superconducting devices. The physical mechanism responsible for the high critical temperature is still not clear.

In 1993, a high-temperature superconductor was found (a ceramic material consisting of mercury, barium, calcium, copper and oxygen $HgBa_2Ca_2Cu_3O_{8+\delta}$) with a critical temperature on the level of 133 – 138 K [61], [11].

On the other hand, if one applies extremely high pressure then superconductivity can be reached at room temperature [19], [63], [64]. The superconducting transition temperature 288 K was reached under pressure of about 270 GPa in carbonaceous sulfur hydride.

The tendency to obtain materials showing superconductivity at higher and higher temperatures and lower pressures results from the costs of potential technical applications of superconductors.

2. Josephson junctions – origins

Among the devices manufactured on the basis of superconductors, Josephson junctions occupy a prominent position. The effect of supercurrent flow without any voltage applied was predicted theoretically by Brian D. Josephson [37], [38]. A device known as a Josephson junction consists of two superconductors coupled by a weak link. The weak link can be made of thin insulating barrier (in S-I-S junctions), normal non-superconducting metal (in S-N-S junctions), or have a form of constriction that weakens the superconductivity at the point of contact (in S-s-S junctions). Experimental confirmation of the effect is due to Philip Anderson and John Rowell [2].

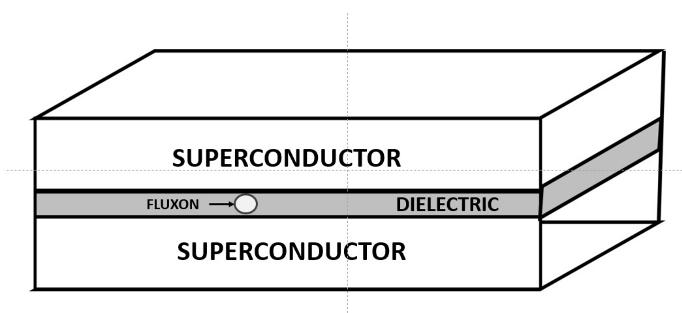

Fig. 1: A long Josephson junction consists of two superconducting electrodes separated by a thin insulator layer. In this layer, fluxons, i.e. quasiparticles carrying a quantum of magnetic flux, propagate. These quasiparticles are described by kink solutions of the sine-Gordon model.

3. Devices based on Josephson junction

Presently there are a variety of applications and devices based on superconducting elements, in particular Josephson junctions [62], [7]. The devices that contain Josephson junctions of different types in their construction can be divided into three groups. In the first group one can include antennas, filters, amplifies, bolometers, single photon detectors, transition edge detectors, superconductor nanostripe single photon detectors, magnetometers and many others. The second group consists of digital electronic appliances like rapid single flux quantum computing elements and digital-to-analogue and analogue-to-digital converters. The third group consists of quantum computing elements like processors.

3.1. Simple antenas and detectors

The simplest devices based on superconducting elements are antennas. The extremely low resistance of superconductors results in improved antenna efficiency. Electrically small passive antennas (ESA), superdirective small arrays of these, are often dedicated to superconducting receivers [32]. An instance of a superconducting antenna for low-frequency radiation is the coil of a SQUID magnetometer. An antenna of this type is used in the TEM geomagnetic exploration method. It detects very weak signals induced underground in deep-layers of mineral ores by electrical pulses. Similarly, superconducting quantum arrays (SQAs) or superconducting quantum interference filters (SQUIFs) [33], are effective amplifiers. They seems to be useful as a signal amplifying broadband antennas [43], [9]. Presently digital receivers can also use multipole superconducting filters [57].

Superconducting hot electron bolometers can be used both as bolometers and single photon detectors. At low temperatures they are highly efficient and extremely fast (a few picoseconds in the case of low temperature superconductor thin films [25]). These devices are suitable for very efficient detection of optical single photons with negligible dark counts. Moreover they enable long-distance optical quantum communication [51] while simultaneously extending frequency range of mixers up to 5 GHz [10]. A transition edge detector consists of a strip of low-Tc superconductor film, connected to a thermal sink. The current change resulting from a rise in temporary temperature is measured by SQUID [67].

Superconductor nanostripe single photon detectors are based on a localized hotspot formation in a two-dimensional superconducting nanostripe [44]. In this device an absorbed photon excites an energetic quasiparticle which creates many secondary excited electrons, initially through electron-electron fast scattering and later, by electron-phonon emission. This cascading quasiparticle multiplication continues until they approach the energy level of the upper edge of the energy gap of the superconductor. The quasiparticle thermalization process takes, for instance, 7 ps for NbN and even less than 1 ps for YBCO [65], [34].

Moreover, portable based on SQUID magnetometer systems were developed [66]. These devices are capable of operating in motion in the presence of electromagnetic fields with orders of magnitude larger than the measured signals. It turns out that these devices, due to their effectiveness, have a direct impact on the macroeconomic decisions of large mining concerns. These tools allow for noncontact geodetic measurements, both above-ground and in the air. The appliance produces strong pulse sequences from a transmitter and measures very weak electromagnetic field response signals due to currents induced deep in the ground. Similar equipment is applied in order to carry out non-destructive evaluation of materials and structures using SQUID magnetometers or gradiometers [45]. It uses solutions similar to those of geomagnetic SQUID applications, e.g. in requiring portability, operation while in motion, and immunity to much stronger environmental fields without using magnetic shielding of the superconducting pickup coil systems or antennas. Something interesting and still being developed is the application of SQUID in medicine and the biomagnetic research. Diagnostic techniques are still being improved in magnetoencephalography, which images and interprets the local distribution of magnetic fields due to neuronal currents [56]. Due to the fact that measurements are non-invasive they are used in magnetocardiography, in the imaging of local heart fields [35].

Nano-SQUIDs are widely applied in metrology [26], [21], [42]. The best known example of the application of SQUIDs in this field is the conventional Josephson voltage standard. The second Josephson relation connects the voltage and frequency by fundamental constants, i.e. the Planck constant and electron charge, that define the Josephson constant. Because the frequency can be measured with very high precision, it defines the accuracy of the volt measurement.

3.2. RSFQ-electronics

In article [47] the concept of the rapid single flux quantum (RSFQ) logic was proposed. This idea relies on bit coding in the single magnetic flux quantum. This approach guarantees very low energy consumption and fast switching between binary states. In addition, complementary devices were proposed like digital-to-analogue and analogue-to-digital converters. These appliances have definitely superior performances than their semiconductor counterparts. On the other hand, the main shortcoming of this technology is the large size of the proposed elements. Presently, it is particularly unfavourable for random-access memory. In the case of RSFQ logic, the efforts concentrate on elimination of static power consumption [50]. Presently, there are 64-kbit working memories, and moreover various new concepts are under development [68], [55], [23], [24].

3.3. Quantum computing devices – perspectives

Experimental studies of the escape process to a finite voltage state in Josephson junction reveals at mK temperatures transition from thermal activation to macroscopic quantum tunnelling [70]. The transition can be observed because in contradiction to thermal activation, the quantum tunnelling is not temperature dependent. In such low temperatures the properly prepared junction behaves like macroscopic quantum two level system and therefore can be considered as realisation of the qubit. Applying an appropriate electromagnetic field one could bring it into controlled quantum superposition of their two states. The entanglement of such quantum systems can be used for quantum information processing. The significant point in constructing the qubit is its coherence time. If this time is too short then quantum correction algorithms are not sufficiently effective and it leads to low fidelity. To achieve the real benefit of using quantum processors over classical processors, they should contain at least 50 qubits [5]. Moreover, new research, both theoretical and experimental, suggests technologies based on

Majorana excitations because degenerate states associated with their zero modes should result in a topologically protected quantum memory [49], [59], [71], [69].

4. Theoretical studies based on sine-Gordon model

The sine-Gordon equation has historically appeared in the context of differential geometry, particularly in the study of surfaces with negative Gaussian curvature. It describes the local isometric deformation of pseudospherical surfaces, which are surfaces of constant negative curvature. In the case of a pseudospherical surface, the integrability condition (Gauss-Codazzi equation) reduce to the sine-Gordon equation [6].

4.1. Analytical techniques for studying the sine-Gordon equation

The sine-Gordon model is defined by a partial differential equation

$$\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} + \sin \phi = 0,$$

where $\phi = \phi(t, x)$ is a certain scalar function. Is a well-known nonlinear partial differential equation with applications in various fields, including physics, mathematics, and engineering. One of its remarkable properties is integrability, which allows for thorough understanding of the structure of the space of solutions to this equation.

The most well-known and general method found in this context is the Inverse Scattering Transform [1], [46]. It makes possible to obtain the formal solution of the equation on the ground of fixed initial data. In particular, for the nonreflective potentials, it allows to determine the analytical form of multi soliton solutions. The method involves mapping the nonlinear evolution problem to a linear scattering problem, which is then solved in order to reconstruct the original solution. The key points of the method are:

- Formulation of the scattering problem: A pair of auxiliary linear equations, known as the Lax pair, is introduced to define a scattering problem for a spectral parameter,

- Analysis of scattering data: The scattering coefficients, including reflection and transmission coefficients and discrete eigenvalues, are computed from the initial conditions,

- Inverse transformation: The solution is reconstructed from the scattering data.

A particularly significant approach to analysing the equation is its description by means of a lax pair. The Lax formulation allows for the derivation of an infinite sequence of conserved quantities, such as energy, momentum, and higher-order invariants. This formulation is essential for deriving the conserved quantities and understanding the integrality of the equation [20].

Important position among the methods for finding solutions to the sine-Gordon equation is occupied by methods based on the analysis of the symmetry of the equation. Lie symmetry analysis is a systematic method for identifying symmetries of differential equations, which can then be used to reduce the number of independent variables. For the sine-Gordon equation this enables the identification of symmetry generators corresponding to transformations that leave the equation invariant. These symmetries are applied to reduce the partial differential equation to ordinary differential equations, simplifying the analysis [58].

An interesting analytical technique for finding solutions to nonlinear equations is the Bäcklund transform. This transformation allows to convert a given nonlinear equation into another equation or into itself. In the case of the sine-Gordon equation, the auto-Bäcklund transformation is known. As a consequence, it allows transforming one solution into another. In particular, it allows to produce a non-trivial solution from a trivial one [60].

Finally, there exist a number of algebraic techniques based on elliptic Jacobi functions. These methods are applied, to explore periodic and quasiperiodic solutions which are often expressed in terms of special functions. Jacobi elliptic functions, generalize sinusoidal waves to nonlinear regimes [48].

The sine-Gordon equation, as a prototypical integrable system, offers a rich landscape of analytical methods. Techniques such as the inverse scattering transform, Lax pairs, Lie symmetry reductions, and Bäcklund transformations not only provide exact solutions but also deepen our understanding of nonlinear dynamics and integrability. These methods continue to be valuable tools in both theoretical research and practical applications.

4.2. Shape engineering – modifications of the sine-Gordon model

While the sine-Gordon equation is an integrable system, real-world phenomena often introduce effects that break integrability, such as damping, external forces, or discreteness. These modifications are present also in description of Josephson junctions. Effects of this type also include modifications related to the change of the shape of the junction. The main analytical techniques used to analyze these systems are perturbation techniques to study slightly modified systems, variational methods, multiple-scale analysis, stability analysis, homogenization and various forms of collective variables models that reduce the infinite-dimensional dynamics to a finite-dimensional problem [39].

Among many approaches directed at obtaining requested properties of Josephson junctions, shape engineering plays a significant role. In this approach, particular modifications of the junction shape are proposed in order to obtain their particular properties.

The first proposals of this type date back to the seventies of the previous century. For example in article [54] the authors discuss the logic design that uses long Josephson lines which are described by the sine-Gordon model with additional bias current and dissipation term. They propose interconnection of Josephson lines in ways that provide the complete logic capability. In this approach a single quanta of magnetic flux employed as information bits. The authors characterize collisions of flux quanta, features of terminus of the line, turning points, basic circuits, and finally design of a logic circuit. In particular construction of AND, OR, exclusive OR and NOT gates are discussed in detail.

On the other hand, in paper [4] the authors proposed a device that consists of a junction with an exponentially tapered width, decreasing toward the load. In this device the junction is preceded by an idle region, where the oxide layer is thicker, preventing the tunnelling of Cooper pairs. This region damps out the fluctuations of the bias current, when a bias current is applied to one end of the junction, and also ensures that it yields a uniform current density. This proposal also avoids the problem of trapped flux because, when the current is injected only at one end, the only static solution is a half fluxon. Moreover, due to the absence of an external magnetic field, perturbations due to the current inducing the magnetic field can be eliminated. Additionally, numerical studies show that an exponential shape helps to avoid chaotic regime.

In the heart-shaped annular junction two classically vortex states can be prepared, corresponding to two minima of the potential [40]. The bias current across the junction is used to tilt the potential. The external field is applied in the plane of the junction, although its strength and angle are treated as control parameters. Read out of the qubit state is achieved by manipulation of these parameters. The authors derive the effective potential in order to fix the range of parameters and geometries appropriate for experiments in both the thermal and quantum regime. They demonstrate that the structure of the effective potential for a vortex inside a shaped junction is influenced by the two in-plane field components and the external bias current. All these parameters can be used in experiments in order to prepare the bistable vortex states, modify the barrier height, change degeneracy in a controlled way and read out the state using a critical current measurement. The authors also fix the parameter range for the quantum regime.

The heart-shaped long Josephson junction placed in an in-plane external magnetic field was also considered in article [41]. Based on this geometry the authors designed and fabricated a classical vortex two-state system. At sufficiently low temperatures, this structure is expected to behave as a quantum two-state system and hence is a candidate for a qubit. The calculation scheme applied to description of the system relies on perturbation theory. In particular depinning currents obtained on the theoretical background agree with the values obtained in experiments. The authors highlight that the proposed analytical method can be used to study thermal activation and quantum tunnelling properties of vortices in sufficiently narrow long Josephson junctions. They tested a protocol to reliably prepare and read out the two vortex states. In the proposed approach the vortex state can be determined using a single-shot measurement of the depinning current. Manipulation on the vortex states is achieved by varying the magnetic field amplitude and its direction, and additionally by applying a bias current to the junction.

Opportunities resulting from T-shaped geometry were considered in the article [27]. The device under consideration consists of two perpendicular Josephson T-Lines forming a T-junction. The particular effect present in the device is the creation of a new vortex when a mother vortex, moving along the main Josephson T-line, is passing the T-junction. The new vortex created at the T-junction begins

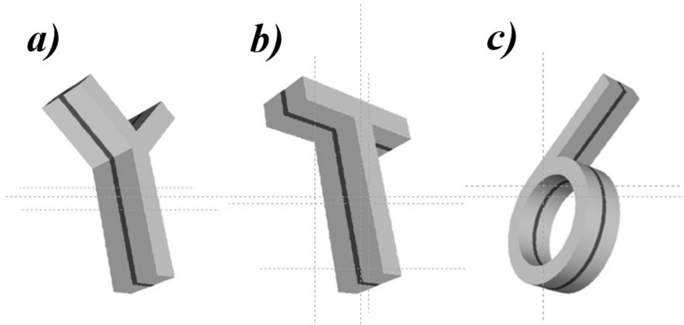

Fig. 2: Various junction geometries a) Y-junction, b) T-junction, c) sigma-junction.

its motion in the direction perpendicular to the main Josephson T-line. The creation of a new vortex is strongly dependent on the energy of the original vortex. If the kinetic energy of the original vortex is too small then the T-junction acts as a barrier and the original vortex is reflected without creation of a new vortex. In the opposite regime the mother vortex overcomes the barrier and the new vortex is created. In this case the original vortex continues its motion along the main Josephson T-line, while the new vortex starts its motion in the perpendicular direction along the additional Josephson T-line. The device proposed in the article is suitable for generation of periodic fluxon chains, fluxon-antifluxon pairs, and continuous breathers. The flux cloning effect has natural implementation in fluxon-based logic gates and logic networks. The effect described in this article may also find applications in quantum information processing.

The properties of the fluxon in the annular Josephson junction were studied in [28]. The main issue of the paper was switching distribution of the annular Josephson junction to a resistive state. The authors invoked the statistical field theory formulation of a string escape problem where the superconducting phase difference is the counterpart of a string. It was shown that the experimental results for the annular Josephson junction can be well described in terms of the thermal activation of a string crossing the potential barrier. Moreover, the agreement with experiments was obtained without any adjusting parameters.

The same authors propose a device that is sensitive to microwave radiation and externally applied magnetic fields [29]. This property makes this device suitable for use in radiation detectors and magnetic field sensors. The system has a form of a long annular Josephson junction attached to a thick Josephson T-line by means of a Y-junction that splits an incident fluxon into two. The device is used for generation and trapping of fluxon-antifluxon pairs. Moreover, multiple collisions of any number of fluxon-antifluxon pairs can be considered in the same way. The authors study multiple collisions of trapped fluxons and antifluxon pairs and their possible decay into plasmons. An analytical criterion for trapping of a fluxon-antifluxon pair in the fluxon collider was formulated and moreover numerical simulations made with the dissipative 2D sine-Gordon equation were executed. The agreement between analytical and numerical results is satisfying. The device is sensitive to an external magnetic field, in particular, application of a magnetic field shifts the critical velocity up. The impact of a magnetic field depends on its orientation with respect to the appliance. If the field is applied in the direction of the symmetry axis of the device then the velocities of a fluxon and an antifluxon are modified and thus it may lead to dissipative losses. In the case of the field applied in the direction perpendicular to the symmetry axis both solitons experience different potential which additionally causes a lag between the two solitons. In this situation both solitons no longer come simultaneously to the Y-junction and therefore do not have enough energy to combine and to leave the system. An interaction event is very quick and therefore devices that rely on the described arrangement can be used when the rapidity of measurements is mostly important. It is promising that the proposed device can be made of high temperature superconductors.

In article [30] the authors propose devices that seem to be more efficient for generation of fluxon chains. Moreover, the proposed appliances are less sensitive to current fluctuations and external noise. The last property causes smaller linewidth to be achieved. Additionally, in contradiction to other proposals no external magnetic field is needed for its operation. Due to their nature, these devices function as fluxon pumps. During utilization of this equipment the pulses of individual fluxons are generated by cloning single fluxons trapped inside a reservoir. If one applies an electric current, a flow of fluxons is generated in the long attachment connected to the reservoir of fluxons. A reservoir is made of Josephson junctions in the form of a loop where one or several fluxons are permanently confined. There are two proposals of fluxon pumps. The first has a form of a T-pump made of an annular Josephson junction with a T-junction connected to a Josephson transmission line. Depending on the orientation of the driving current it is suitable to generate a train of fluxons or antifluxons. The second proposal is sigma-pump, the main advantage of which is the lack of the barrier associated with the T-junction present in the T-pump. Instead, the Josephson transmission line is connected with the ring smoothly through the Y-junction. In this pump a nucleation barrier is absent. Instead, the nucleation energy is gathered by the trapped fluxon during its motion in the potential associated with increasing width. The proposed equipment can be implemented with Nb technology or even using high temperature superconductors. Moreover, test exemplar was prepared on the basis of BSCCO crystal. The measured I-V characteristics may provide indirect evidence of the flux cloning effect.

In the next article [31], exact solutions of the 2D sine-Gordon model are considered. The solutions describe the propagation of deformations, of an arbitrary shape, along a Josephson vortex line. The existence of such moving distortions can describe transmitting pulses of electromagnetic radiation along a Josephson vortex. This phenomenon can be employed to the transmission of information in various Josephson devices. In order to identify the presence of shape waves the authors propose application of a Y-junction. The evolution of the vortex in this system consists of three stages. Firstly, a straight vortex propagates along the system until it meets some region of local increase of the junction width. Because one end of the fluxon line is delayed with respect to the other one it results in a deformation of the vortex profile. In the second, a vortex with distortion propagates until it meets the Y shaped region of the junction. Finally, a vortex is split into two parts, which afterwards, propagate independently from each other in different branches of the Y-junction. The vortex with shape excitation moves in one of the branches. The authors derive an analytical formula for energy of shape excitations and discuss conditions for their creation. Moreover, they notice the similarity of this system to special relativity and propose an experiment to measure a time dilation effect.

The possibility of an effective one-dimensional description of kink and breather during its propagation in a Y- or T-junction is considered in paper [8]. The authors perform both theoretical and numerical investigations of the 2D sine-Gordon equation. In particular they find out that the angle of the junction practically plays no role in the dynamics for thin trees. This observation leads to the possibility of 1D effective description of this system with flux conservation. The predictions of the simplified model correctly reproduce the solutions of the original 2D equation. A critical velocity below which no crossing is possible is identified. Conservation of energy helps to obtain the relation for parameters of the kink that can cross the Y-junction. In the case of breather, crossing depends on the velocity and the frequency as well.

The other possibility is an annular junction delimited by two closely spaced confocal ellipses that is characterized by a periodically modulated width [52]. This spatial dependence, in turn, produces a periodic potential that interchangeably attracts and repels the fluxons. In the article the minima of the potential energy occur at two opposite locations where the annulus is narrowest. In this particular junction double-well potential for an individual fluxon is produced by an intrinsic non-uniform width. In the system the heights of the barriers separating minima are determined by the confocal annular Josephson tunnel junction eccentricity. The strength of the in-plane magnetic field controls the height of the inter-well potential barrier and the distance between the potential wells. Moreover, application of the current ramp across the junction enables manipulation of the vortex states. Until a force exerted on the vortex by the bias current is smaller than the pinning force, a pinned vortex remains confined to one of the potential minima. In the opposite regime the vortex starts to move along the junction. There is a direct relationship between the vortex depinning current and the magnitude of the applied field. Measurement of the vortex depinning current in a small magnetic field enables read out of the state. Preparation of the vortex in a specific state is possible by application of a particular waveform of the bias current. If the temperature and dissipation in the junction are sufficiently low, the superposition of the macroscopically distinct states can be employed to implement a Josephson vortex qubit. The above system is properly modelled by a modified and perturbed one-dimensional sine-Gordon equation.

In the article [53], the authors describe vortex qubit design based on confocal an annular Josephson tunnel junction. Moreover the device was studied experimentally in the classical regime and bistable vortex states were observed on a high-quality Nb/Al-AlOx/Nb junction. The vortex was prepared in a fixed potential well by application of the external magnetic field in the barrier plane or in the transverse direction. In experiments performed at 4.2 K they measured the thermally activated process of fluxon escape from a well of the potential. As the temperature drops, it can be expected that the activation processes and dissipation decrease significantly. The authors demonstrated the possibility of reliable manipulation on the vortex state. The vortex is set in a given potential by application of an external magnetic field, while the state readout is made by measuring the vortex-depinning current. This experiment demonstrated that the vortex two-state system based on the confocal annular Josephson tunnel junction is solid and workable.

The impact of varying the thickness of the dielectric layer of the Josephson junction on the kink motion is studied in paper [17]. What is crucial for potential applications of the described system is the fact that regions with an increased thickness of the dielectric layer act on the fluxon as a potential barrier. It was shown that the details of the deformation of the dielectric layer have a negligible impact on the fluxon dynamics if the potential barrier has a clearly formed plateau. In this case the analytical formula for the critical velocity of the fluxon is solely determined by the parameters of deformation. Moreover, when external bias current and dissipation is included in the description then one obtains the critical current. The numerical simulations in this case are well described by an appropriate analytical formula. In these junctions, properly prepared dielectric layers make it possible to store fluxons on one side of a potential barrier. The crucial condition for the transition of a fluxon through such a barrier is the value of the applied bias current. Whenever the bias current exceeds the critical value, one can observe the transmission of the fluxon. In the opposite case, the fluxon is stored on one side of the deformed area. It is expected, for example, that the Josephson junction of this type can be potentially used in high-frequency appliances. In particular, the promising area of applications is technology of superconducting integrated receivers and generators of submillimetre waves. This technology is justified by the development of practical systems.

The influence of the curvature on the dynamics of the gauge invariant phase difference between two superconducting electrodes that comprise the Josephson junction was studied in [14]. A quasi-one-dimensional large area Josephson junction with one curved and one flat direction was considered. The central surface of the dielectric layer described in the article is internally flat but non-trivially embedded in the three-dimensional space. The proper choice of the magnetic field is a reason for the reduction of the dynamics of the phase difference to only one-dimension. The construction relies on local curved coordinates defined in the vicinity of the plane curve located on the central surface of the dielectric layer. This choice of coordinates causes this description to also be appropriate for the long Josephson junction. The equation that describes this system was obtained on the basis of field dynamics governed by Maxwell's equations in isolator and London's equations in superconducting electrodes with Landau Ginzburg current of Cooper pairs. The effective description agrees with the same result obtained on a purely geometrical background as a consequence of geometrical reduction of the sine-Gordon model to a lower dimensional curved subspace [12]. The generalization of this result to the situation when the magnetic field is oriented in an arbitrary direction was presented in article [36]. Moreover the paper describes the possibility of the storage of fluxons in the proposed device.

The method of constructing a low-dimensional model based on a more-dimensional model was proposed in paper [12], where the reduction of the sine-Gordon model from a three-dimensional space to a curved one-dimensional and quasi-one-dimensional subspace was carried out. In the low-dimensional models a further reduction based on collective coordinate method was performed. In the one-dimensional case it was shown that curvature of lower dimensional subspace corresponds to the existence of the energy barrier that affects the motion of the kink. Moreover the existence of the direct relation between effective potential and curvature was shown. The one-dimensional model provides the natural description of the fluxon motion in the long Josephson junction. The same formalism allows for the description of the quasi-one-dimensional junction (which is a two-dimensional system with a flat binormal direction). In this case the effective potential is determined by curvature and torsion of the central curve. Additionally, it was noticed that the motion of the kink in the junction can be controlled, even in the straight junction by modulating its width. If the width of the junction grows, then one has to provide some energy connected with increasing the length of the kink profile. This is another way of producing a potential well in the junction. The existence of effective potential barriers is a base for applications in a variety of devices.

The analytical tool for construction of the approximate solutions in the sine-Gordon model in a curved space was formulated in article [16]. The complete perturbation consists of an infinite number of equations. The first of these equations is the only nonlinear equation in the scheme and its solution coincides with the soliton solution of the sine-Gordon model in a flat space. The rest of the equations in the scheme are linear equations defined by the same linear operator. Having knowledge about the Green function for this operator one can construct, with arbitrary precision the soliton solution of the considered model in a curved space. The results of this scheme were successfully compared with numerical solutions for some static configurations based on the relaxation method.

The paper [13] contains the construction of a two-dimensional model defined on a curved surface on the basis of a three-dimensional model. The article describes the reduction of the sine-Gordon model from a three-dimensional space to a curved two-dimensional subspace. The method concerns arbitrary surfaces (surfaces for which external and internal curvatures may be non-trivial). The main advantage of the article is possibility of obtaining the effective potential for an arbitrary deformed junction. The formalism allows geometrical forms of the junctions completely different from the forms described in other papers. The junctions described in other articles have an insulator layer located on the plane (although they have curved boundaries), while the dielectric layer in [13] is located on the arbitrary curved surface. The application of this formalism to describe the Josephson junction with one compact direction is presented in article [15]. In this paper, the propagation of the kink through a curved surface with one compact dimension is considered. Moreover, different geometries were studied. The first realizes the potential barrier. The second one causes the kink to accelerate and the last to slow it down.

Theoretical work on possible modifications of Josephson junctions, the impact of thermal noise [22] and the possible practical use of these modifications is ongoing and will probably contribute to the development of computing and measurement technologies in the future. In fact, the literature on the subject is much broader and includes thousands of items on both the practical and theoretical aspects of the subject. The items proposed in this article, however, provide a fairly comprehensive introduction to the issues discussed here.

5. Summary

Certainly, with modified sine-Gordon models, a suitable construction of models that reduce the description to a finite number of degrees of freedom is one way to learn about the processes that occur in a variety of physical, biological and other systems. Such models not only simplify the mathematical description but also allow us to better understand the processes that occur in them.

Each modification describing, for example, the various types of dissipation occurring in the system brings slightly different challenges in describing such a system. Moreover, modifications related to changing the geometry of the system have similar consequences. The presence of the latter is most often associated with technological challenges faced by engineers designing various devices. Also, the presence of various types of external influences on the system requires additional study in many cases. In various types of devices, it is these external influences that allow the system to be controlled.

An element that often cannot be ignored in the description of such systems is thermal noise, which introduces random system disturbances. The behaviour of systems in the presence of noise certainly also still requires much research. A particularly under-researched aspect of the behaviour of systems described by modified sine-Gordon equations is the influence of noise, which has temporal and spatial correlations that cannot be neglected.

Another area of generalization is the question of system behaviour in more than one-dimension. In particular, much work remains to be done in the case of three spatial dimensions.

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