sciendo

Ann. Univ. Paedagog. Crac. Stud. Math. 24 (2025), 7-47
DOI: 10.2478/aupcsm-2025-0001

FOLIA 422

Annales Universitatis Paedagogicae Cracoviensis
Studia Mathematica 24 (2025)

Alexandra Seceleanu

Lefschetz properties through a topological lens

Abstract. These notes were prepared for the *Lefschetz Preparatory School*, a graduate summer course held in Krakow, May 6–10, 2024. They present the story of the algebraic Lefschetz properties from their origin in algebraic geometry to some recent developments in commutative algebra. The common thread of the notes is a bias towards topics surrounding the algebraic Lefschetz properties that have a topological flavor. These range from the Hard Lefschetz Theorem for cohomology rings to commutative algebraic analogues of these rings, namely artinian Gorenstein rings, and topologically motivated operations among such rings.

1. Introduction

The topic of these notes is the algebraic Lefschetz properties, which are abstractions of the important Hard Lefschetz Theorem from complex geometry. Section 2 explains the topological context of this result. Section 4 introduces the algebraic Lefschetz properties and their relevance to commutative algebra. Section 5 establishes a correspondence between the strong Lefschetz property and an action of the Lie group $\mathfrak{sl}_2$. Section 6 focuses on the class of Gorenstein rings and their construction via Macaulay's inverse system. Sections 5.3 and 7 investigate how various constructions of new rings from old interact with the Lefschetz properties.

These notes are deeply influenced by the monograph [13] by T. Harima, T. Maeno, H. Morita, Y. Numata, A. Wachi and J. Watanabe. This reference contains

AMS (2010) Subject Classification: 13-XX.
Keywords and phrases: Lefschetz properties, Hard Lefschetz theorem, Hilbert function, Artinian ring.
ISSN: 2081-545X, e-ISSN: 2300-133X.

a parallel description of many topics in these notes except for section 7, which describes more recent developments based on [19]. The treatment of earlier chapters, while deeply influenced by [13], reflects the author's mathematical taste.

2. Cohomology rings and the Hard Lefschetz Theorem

This section gives an introduction to the origins of the algebraic Lefschetz properties. The motivation for this topic comes from algebraic topology, so we will spend some time looking at how the Lefschetz property arises there.

2.1. Cohomology rings

Let $\mathbb{F}$ be a vector space and let X be a topological space (such as projective space $\mathbb{P}^n$ or the n-dimensional sphere S^n). We recall the notion of cohomology of X with coefficients in $\mathbb{F}$.

First, one can think of X as being made out of simple cells (or at least one can approximate X in this manner). This endows X with a *cell complex (CW-complex)* structure.

Example 2.1 (CW structures on sphere)
The 2-dimensional sphere S^2 can be obtained by taking a point (0-dimensional cell) and gluing a 2-dimensional disc onto it along its entire boundary. So the CW-structure of S^2 is

$$S^2 = \text{pt} + \text{2-dimensional disc.}$$

More generally one can do the same for the n-dimensional sphere S^n:

$$S^n = \text{pt} + n\text{-dimensional disc.}$$

There is another, less economical way to give the sphere a CW-structure. For S^2 one takes two 0-dimensional cells, connects them using two line segments (1-dimensional cells) to form a circle S^1. Then one can glue two 2-dimensional discs via their boundaries to the circle to form S^2. Similarly, there is a CW-structure on S^n with two cells in each dimension summarized by

$$\begin{aligned} S^n =& 2 \times \text{pt} + 2 \times \text{1-dimensional disc} + 2 \times \text{2-dimensional disc} \\ &+ \cdots + 2 \times \text{n-dimensional disc.} \end{aligned}$$

Example 2.2 (CW structure on the real projective space)
Consider first $\mathbb{P}^n_{\mathbb{R}}$. It can be written as $S^n/\{\pm 1\}$. If we take a CW structure on S^n with two cells in each dimension, then the action of -1 swaps the cells, thus they become identified in the quotient. Due to this $\mathbb{P}^n_{\mathbb{R}}$ has a CW structure with one cell in each dimension.

$$\mathbb{P}^n_{\mathbb{R}} = \text{pt} + \text{1-dimensional cell} + \cdots + \text{n-dimensional cell.}$$

Next consider $\mathbb{P}^n_{\mathbb{C}}$. This has a cell in every even (real) dimension:

$$\mathbb{P}^n_{\mathbb{C}} = \text{pt} + \text{2-dimensional cell} + \cdots + \text{2n-dimensional cell.}$$

Proceeding towards homology, we define a *chain complex* $\mathbf{C}_\bullet(X)$ by letting $C_n(X)$ be the $\mathbb{F}$-vector space generated by the n-dimensional cells of X. There are so-called boundary maps[1], which fit into the following sequence

$$\mathbf{C}_\bullet(X) : 0 \leftarrow \mathbb{F}^{\#0\text{-cells}} \leftarrow \mathbb{F}^{\#1\text{-cells}} \leftarrow \cdots \leftarrow \mathbb{F}^{\#\dim(X)\text{-cells}} \leftarrow 0.$$

There is also a dual version called the *cochain complex* of X with coefficients in R

$$\mathbf{C}^\bullet(X) = \mathrm{Hom}(\mathbf{C}_\bullet(X), \mathbb{F}) : 0 \to \mathbb{F}^{\#0\text{-cells}} \xrightarrow{\partial_1} \mathbb{F}^{\#1\text{-cells}} \xrightarrow{\partial_2} \dots \xrightarrow{\partial_n} \mathbb{F}^{\#\dim(X)\text{-cells}} \to 0.$$

Definition 2.3
The *cohomology groups* of X are defined as

$$H^i(X, \mathbb{F}) = H^i\left(\mathbf{C}^\bullet(X)\right) = \operatorname{Ker} \partial_i / \operatorname{Im} \partial_{i-1}.$$

Example 2.4
Based on Example 2.1 we have the following chain complexes, which lead to easy computations of the corresponding cohomology groups.

$$\mathbf{C}^\bullet(S^n) : 0 \to \mathbb{F} \to 0 \to 0 \to \ldots \to \mathbb{F} \to 0$$

$$H^i(S^n, \mathbb{F}) = \begin{cases} \mathbb{F}, & i = 0, n, \\ 0, & \text{otherwise}, \end{cases}$$

$$\mathbf{C}^\bullet(\mathbb{P}^n_{\mathbb{C}}) : 0 \to \mathbb{F} \to 0 \to \mathbb{F} \to 0 \to \mathbb{F} \to \ldots \to \mathbb{F} \to 0$$

$$H^i(\mathbb{P}^n_{\mathbb{C}}, R) = \begin{cases} \mathbb{F}, & i = \text{even}, \\ 0, & i = \text{odd}. \end{cases}$$

The special property of these cohomology groups that allows us to study them using tools from ring theory is that they can be assembled into a graded ring.

Definition 2.5
The *cohomology ring* of X is

$$H^\bullet(X, \mathbb{F}) = \bigoplus_{i \geq 0} H^i(X, \mathbb{F}).$$

To study multiplication on this ring we need to define a map called the *cup product*

$$H^m(X, \mathbb{F}) \times H^n(X, \mathbb{F}) \to H^{m+n}(X, \mathbb{F}).$$

For this recall the Künneth isomorphism: for two topological spaces X and Y if one of X or Y has torsion-free homology (this holds when working over a field $\mathbb{F}$) and has finitely many cells in each dimension, there is an isomorphism $\mu : H^\bullet(X \times Y, \mathbb{F}) \cong H^\bullet(X, \mathbb{F}) \otimes_{\mathbb{F}} H^\bullet(Y, \mathbb{F})$. The composite with the diagonal map

$$H^\bullet(X, \mathbb{F}) \otimes_{\mathbb{F}} H^\bullet(X, \mathbb{F}) \xrightarrow{\cong} H^\bullet(X \times X, \mathbb{F}) \xrightarrow{\Delta^*} H^\bullet(X, \mathbb{F})$$

defines the cup product by $x \cup y = \Delta^* \mu(x \otimes y)$. The cup product is not commutative, but it is what we call *graded commutative*. This means that the ring is graded so that

[1] We will not describe the boundary maps here.

if $x \in H^m(X, \mathbb{F})$, $|x| = m$ denotes the *cohomological degree* of x,

and any elements x, y in this ring satisfy

$$x \cup y = (-1)^{|x||y|} y \cup x. \tag{2.1}$$

Note that in a graded commutative ring even degree elements commute with all other elements, while odd degree elements anti-commute with other odd degree elements.

EXAMPLE 2.6 (Cohomology ring of a sphere)
From Example 2.4 we have

$$H^\bullet(S^n, \mathbb{F}) = \mathbb{F} \oplus \mathbb{F}.$$

Set 1 and e to be the basis of $H^0(S^n, \mathbb{F})$ and $H^n(S^n, \mathbb{F})$ as $\mathbb{F}$-vector spaces, respectively. Then 1 is the multiplicative identity of the ring $H^\bullet(S^n, \mathbb{F})$ and $e^2 = e \cup e \in H^{2n}(S^n, \mathbb{F}) = 0$, so

$$H^\bullet(S^n, \mathbb{F}) = \mathbb{F}[e]/(e^2) \qquad \text{with } |e| = n.$$

EXAMPLE 2.7 (Cohomology ring of a torus)
Applying the Künneth formula to the torus $T^n = S^1 \times \cdots \times S^1$ gives for elements $e_1, \ldots, e_n$ with $|e_i| = 1$,

$$H^\bullet(T^n, \mathbb{F}) = \mathbb{F}[e_1]/(e_1^2) \otimes_{\mathbb{F}} \mathbb{F}[e_2]/(e_2^2) \otimes_{\mathbb{F}} \mathbb{F}[e_n]/(e_n^2) = \bigwedge_{\mathbb{F}} \langle e_1, \ldots, e_n \rangle.$$

Note that the tensor product above is taken in the category of graded-commutative algebras which implies that $e_i e_j = -e_j e_i$ as expected since $|e_i| = |e_j| = 1$. If the characteristic of $\mathbb{F}$ is not equal to 2 then this implies $e_i^2 = 0$ for all i. The ring above, denoted $\bigwedge_{\mathbb{F}} \langle e_1, \ldots, e_n \rangle$, is called an *exterior algebra*. As an $\mathbb{F}$-vector space, a basis of the exterior algebra is given by all the square-free monomials in the variables $e_1, \ldots, e_n$.

EXAMPLE 2.8 (Cohomology ring of projective space)
From Example 2.4 we have $H^\bullet(\mathbb{P}^n_{\mathbb{C}}, \mathbb{F}) = \mathbb{F} \oplus \mathbb{F} \oplus \cdots \oplus \mathbb{F}$ with n summands in degrees $0, 2, \ldots, 2n$. Set x to be the generator of $H^2(\mathbb{P}^n_{\mathbb{C}}, \mathbb{F})$. It turns out then that $x^i \neq 0 \in H^{2i}(\mathbb{P}^n_{\mathbb{C}}, \mathbb{F})$ for $1 \leq i \leq n$, so x^i generates $H^{2i}(\mathbb{P}^n_{\mathbb{C}}, \mathbb{F})$. Moreover, $x^{n+1} = 0$ since $H^{2n+2}(\mathbb{P}^n_{\mathbb{C}}, \mathbb{F}) = 0$. Thus we have

$$H^\bullet(\mathbb{P}^n_{\mathbb{C}}, \mathbb{F}) = \mathbb{F}[x]/(x^{n+1}) \qquad \text{with } |x| = 2.$$

We can apply the Künneth formula to compute

$$\begin{aligned} H^\bullet(\mathbb{P}^{d_1}_{\mathbb{C}} \times \mathbb{P}^{d_2}_{\mathbb{C}} \times \cdots \times \mathbb{P}^{d_n}_{\mathbb{C}}, \mathbb{F}) & \\ \cong & \mathbb{F}[x_1]/(x^{d_1+1}) \otimes_{\mathbb{F}} \mathbb{F}[x_2]/(x^{d_2+1}) \otimes_{\mathbb{F}} \cdots \otimes_{\mathbb{F}} \mathbb{F}[x_n]/(x^{d_n+1}) \\ \cong & \mathbb{F}[x_1, \ldots, x_n]/(x_1^{d_1+1}, \ldots, x_n^{d_n+1}) \qquad \text{with } |x_i| = 2. \end{aligned}$$

2.2. The Hard Lefschetz Theorem

Solomon Lefschetz (1888–1972) was a prominent mathematician who did fundamental work on algebraic topology, its applications to algebraic geometry, and the theory of non-linear ordinary differential equations. His career, including transitions from industry to mathematics and from working in Nebraska and Kansas to Princeton University is beautifully summarized in his own words in [22]. He was also a great supporter of mathematics in developing countries, helping train a great number of Mexican mathematicians. Lefschetz understood that cohomology rings can be used to study questions in algebraic geometry. Speaking about his work Lefschetz states:

> *"The harpoon of algebraic topology was planted in the body of the whale of algebraic geometry."*

We now come to the main result that we have been building up to. Let X be an algebraic subvariety of $P^n_{\mathbb{C}}$ and let H denote a general hyperplane in $P^n_{\mathbb{C}}$. Then $X \cap H$ is a subvariety of X of real codimension two and thus, by a, standard construction in algebraic geometry, represents a cohomology class $L \in H^2(X, \mathbb{R})$ called the class of a hyperplane section. In more detail, L is an $\mathbb{F}$-linear homomorphism that takes a dimension 2 subvariety of X (or a 2-dimensional cell of X if we view this as a CW-complex), intersects it with H and returns the number of points of intersection. This function is then extended $\mathbb{F}$-linearly.

Theorem 2.9 (Hard Lefschetz Theorem)
Let X be a smooth irreducible complex projective variety of complex dimension n (real dimension $2n$), $H^\bullet(X) = H^\bullet(X, \mathbb{R})$, and let $L \in H^2(X, \mathbb{R})$ be the class of a general hyperplane section. Then for $0 \leq i \leq n$ the following maps are isomorphisms

$$L^i : H^{n-i}(X) \to H^{n+i}(X), \qquad \text{where } L^i(x) = \underbrace{L \cup \cdots \cup L}_{L^i} \cup x.$$

Remark 2.10
The Hard Lefschetz theorem holds for $H^\bullet(X, \mathbb{F})$, where $\mathbb{F}$ is any field of characteristic zero (not just $\mathbb{R}$), but the conclusion of the theorem is false in positive characteristic.

The theorem above was first stated by Lefschetz in [21], but his proof was not entirely rigorous. The first complete proof of Theorem 2.9 was given by Hodge [16]. The standard proof given nowadays uses the representation theory of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ and is due to Chern [4]. Lefschetz's original proof was only recently made rigorous by Deligne [6], who extended it to positive characteristic.

Example 2.11 (The Hard Lefschetz theorem in action)
See Example 2.8 for context. For $H^\bullet(P^n_{\mathbb{C}}) = \mathbb{F}[x]/(x^{n+1})$ the class of a hyperplane is $L = x$ (recall that $|x| = 2$) and it gives whenever $i \equiv n \pmod 2$ isomorphisms

$$\begin{aligned} H^{n-i}(P^n_{\mathbb{C}}) = x^{\frac{n-i}{2}}\mathbb{F} &\xrightarrow{\times x^i} H^{n+i}(P^n_{\mathbb{C}}) = x^{\frac{n+i}{2}}\mathbb{F} \\ x^{\frac{n-i}{2}} y &\mapsto x^i (x^{\frac{n-i}{2}} y) = x^{\frac{n+i}{2}} y. \end{aligned}$$

Cohomology rings of n-dimensional complex projective varieties X with coefficients in a field $\mathbb{F}$ satisfy the following properties:

(1) $H^\bullet(X, \mathbb{F})$ is a graded commutative ring[2] in the sense of (2.1). Its even part $A := H^{2\bullet}(X, \mathbb{F}) = \bigoplus_{i \geq 0} H^{2i}(X, \mathbb{F})$ is a commutative graded ring as defined in the next chapter. We can re-grade this ring by setting $|x| = i$ if $x \in H^{2i}(X, \mathbb{F})$. With this convention we have $|L| = 1$.

(2) $H^\bullet(X, \mathbb{F})$ and A are finite dimensional $\mathbb{F}$-vector spaces (so A is an artinian ring cf. Definition 3.8).

(3) $H^\bullet(X, \mathbb{F})$ and A satisfy Poincaré duality (hence A is a Gorenstein ring cf. Proposition 3.11).

The main goal of these notes is to demonstrate how one may hope to extend the Hard Lefschetz theorem (and some weaker versions thereof) to arbitrary rings which may not necessarily be cohomology rings, but satisfy at least some of the properties above. Thus we are motivated by the following question.

Question 2.12
Which commutative graded rings A that are artinian or both artinian and Gorenstein also satisfy the conclusion of the Hard Lefschetz theorem?

3. Classes of graded rings

From now on all rings will be commutative unless specified otherwise.

3.1. Artinian algebras

Definition 3.1 (Graded ring)
A commutative ring A is an ($\mathbb{N}$-)*graded ring* provided it decomposes as

$$A = \bigoplus_{i \geq 0} A_i$$

with A_i abelian groups such that for all $i, j \in \mathbb{N}$ $A_i A_j \subseteq A_{i+j}$ ($a \in A_i, b \in A_j \Rightarrow ab \in A_{i+j}$).

From now on we restrict to graded rings A with $A_0 = \mathbb{F}$ a field. I will refer to these as $\mathbb{F}$-*algebras*. Note that in particular such a ring A and each of its homogeneous components A_i are $\mathbb{F}$ vector spaces.

An $\mathbb{N}$-graded ring A has a unique homogeneous maximal ideal, namely

$$\mathfrak{m} = \bigoplus_{i \geq 1} A_i.$$

[2] It is worth cautioning that the phrase graded commutative ring has a very different meaning from commutative graded ring. In the former odd degree elements anti-commute while in the latter all elements commute regardless of their degree.

EXAMPLE 3.2
$A = \mathbb{F}[x_1, \ldots, x_n]$ is the fundamental example of a graded ring with $A_i =$ the set of homogeneous polynomials of degree i. Note that the degree of x_i is allowed to be an arbitrary positive integer.

EXERCISE 3.3
Show that if A is a commutative, Noetherian, graded $\mathbb{F}$-algebra then $\dim_{\mathbb{F}} A_i$ is finite for each i.

DEFINITION 3.4 (Hilbert function)
The *Hilbert function* of a Noetherian graded $\mathbb{F}$-algebra A is the function

$$h_A : \mathbb{N} \to \mathbb{N}, \quad h_A(i) = \dim_{\mathbb{F}} A_i.$$

The *Hilbert series* of A is the power series $H_A(t) = \sum_{i \geq 0} h_A(i) t^i$.

EXERCISE 3.5
Prove that the Hilbert function of the polynomial ring $R = \mathbb{F}[x_1, \ldots, x_n]$ is given by

$$h_R(i) = \binom{n+i-1}{i} \qquad \text{for all } i \geq 0$$

and the Hilbert series is

$$H_R(t) = \frac{1}{(1-t)^n}.$$

EXAMPLE 3.6
The Hilbert function of the truncated polynomial ring $A = \frac{\mathbb{F}[x_1,\ldots,x_n]}{(x_1,\ldots,x_n)^d}$ is given by

$$h_A(i) = \begin{cases} \binom{n+i-1}{i}, & \text{if } 0 \leq i < d, \\ 0, & \text{if } i \geq d. \end{cases}$$

Thus $H_A(t) = \sum_{i=0}^{d-1} \binom{n+i-1}{i} t^i$.

EXAMPLE 3.7
Consider a field $\mathbb{F}$ and let $A = \mathbb{F}[x, y, z]/(x^2, y^2, z^2)$. Clearly, A is a finite dimensional $\mathbb{F}$-vector space with basis given by the monomials $\{1, x, y, z, xy, yz, xz, xyz\}$. We see that the elements of A have only four possible degrees $0, 1, 2, 3$ and moreover

$$\begin{aligned} A_0 &= \mathrm{Span}_{\mathbb{F}}\{1\} \cong \mathbb{F} \Rightarrow h_A(0) = 1, \\ A_1 &= \mathrm{Span}_{\mathbb{F}}\{x, y, z\} \cong \mathbb{F}^3 \Rightarrow h_A(1) = 3, \\ A_2 &= \mathrm{Span}_{\mathbb{F}}\{xy, yz, xz\} \cong \mathbb{F}^3 \Rightarrow h_A(2) = 3, \\ A_3 &= \mathrm{Span}_{\mathbb{F}}\{xyz\} \cong \mathbb{F} \Rightarrow h_A(3) = 1, \\ A_i &= 0 \text{ for all } i \geq 4 \Rightarrow h_A(i) = 0 \text{ for all } i \geq 4. \end{aligned}$$

Thus $H_A(t) = 1 + 3t + 3t^2 + t^3$.

In Examples 3.6 and 3.7 the Hilbert series was in fact a polynomial, equivalently the Hilbert function was eventually equal to zero. We now define a class of graded rings which satisfy this property.

Definition 3.8 (Artinian ring)
A (local or) graded $\mathbb{F}$-algebra A with (homogeneous) maximal ideal $\mathfrak{m}$ is *artinian* if any of the following equivalent conditions holds.

(a) A is finite dimensional as a $\mathbb{F}$-vector space.

(b) A has Krull dimension zero.

(c) $\mathfrak{m}^d = 0$ for some (hence all sufficiently large) integers $d \geq 1$. If A is graded this can be restated as $A_d = 0$ for sufficiently large integers $d \geq 1$.

(d) A satisfies the descending chain condition on ideals.

(e) There exists a descending sequence of ideals

$$A = \mathfrak{a}_0 \supseteq \mathfrak{a}_1 \supseteq \mathfrak{a}_2 \supseteq \cdots \supseteq \mathfrak{a}_\ell = 0 \text{ such that } \mathfrak{a}_{i-1}/\mathfrak{a}_i \cong \mathbb{F}.$$

Such a sequence of ideals is called a *composition series*.

Moreover, if $R = \mathbb{F}[x_1, \ldots, x_n]$ is a polynomial ring and $A = R/I$ for some homogeneous ideal I of R then the conditions above are also equivalent to

(f) For each $0 \leq i \leq n$ there is some integer p_i such that $x_i^{p_i} \in I$.

(g) If $\mathbb{F}$ is algebraically closed, another equivalent condition is that the vanishing locus of I in projective space is empty.

3.2. Artinian Gorenstein rings and complete intersections

Definition 3.9 (Socle)
For a graded artinian $\mathbb{F}$ algebra the maximal integer d such that $A_d \neq 0$ is called the *maximal socle degree* of A. The *socle* of A is the ideal

$$(0 :_A \mathfrak{m}) = \{x \in A \mid xy = 0 \text{ forall } y \in \mathfrak{m}\}.$$

There is always a containment $A_d \subseteq (0 :_A \mathfrak{m})$, where d denotes the maximal socle degree of A. When the converse holds, namely $(0 :_A \mathfrak{m}) \subseteq A_d$, then A is called a *level algebra*. This condition means that the socle is concentrated in a single degree.

Definition 3.10 (Artinian Gorenstein ring)
A graded $\mathbb{F}$-algebra is *artinian Gorenstein* (AG) if its socle is a one dimensional $\mathbb{F}$-vector space.

An equivalent characterization of AG algebras is given by the following proposition.

Proposition 3.11 (Poincaré duality)
A graded $\mathbb{F}$-algebra A of maximal socle degree d is AG if and only if for each nonzero element a_{soc} of A_d there exists an $\mathbb{F}$-vector space homomorphism $\int_A : A \to \mathbb{F}$ called an orientation*, satisfying the following properties:*

(1) *if $b \in A_i$ for some $i < d$ then $\int_A b = 0$,*

(2) *the orientation induces an isomorphism $A_d \cong \mathbb{F}$ such that $\int_A a_{soc} = 1$,*

(3) *for each* $0 \leq i \leq d$ *and each element* $a \in A_i$ *there exists a unique element* $b \in A_{d-i}$ *so that* $\int_A ab = 1$.

In Section 7.2 we will use the notation a_{soc} implicitly to mean fixing the unique orientation on A that satisfies $\int_A a_{soc} = 1$.

Example 3.12
Continuing with Example 3.7, the socle is $(0 :_A \mathfrak{m}) = \mathrm{Span}_{\mathbb{F}}\{xyz\}$, a 1-dimensional $\mathbb{F}$-vector space. This shows that A is Gorenstein. Take the orientation on A to be specified by $\int_A xyz = 1$. We see that the $\mathbb{F}$-basis elements $\{1, x, y, z, xy, yz, xz, xyz\}$ of A form pairs with respect to the given orientation in the following manner

$$\int_A 1 \cdot xyz = 1,$$
$$\int_A x \cdot yz = 1,$$
$$\int_A y \cdot xz = 1,$$
$$\int_A z \cdot xy = 1.$$

Exercise 3.13
Show that the following is an artinian Gorenstein ring

$$R = \frac{\mathbb{F}[x, y, z]}{(xy, xz, yz, x^2 - y^2, x^2 - z^2)}.$$

Exercise 3.14
Prove that if A is a graded AG algebra of maximum socle degree d then $h_A(i) = h_A(d - i)$ for each $0 \leq i \leq d$. This is usually stated by saying AG algebras have symmetric Hilbert function.

Definition 3.15
A graded *artinian* $\mathbb{F}$-algebra is a *complete intersection* (CI) if $A = R/I$, where $R = \mathbb{F}[x_1, \ldots, x_n]$ and $I = (f_1, \ldots, f_n)$, that is, I is a homogeneous ideal generated by as many elements as there are variables in R.

Example 3.16
The rings $A = \mathbb{F}[x_1, \ldots, x_n]/(x_1^{d_1}, \ldots, x_n^{d_n})$, where $d_1, \ldots, d_n \geq 1$ are integers, are called *monomial complete intersections*.

Exercise 3.17
Prove that the rings in Example 3.16 are the only artinian Gorenstein rings of the form R/I, where $R = \mathbb{F}[x_1, \ldots, x_n]$ and I is an ideal generated by monomials.

All CI rings are Gorenstein, but not all Gorenstein rings are CI, as exemplified by the ring in Exercise 3.13.

4. The Lefschetz properties

4.1. Weak Lefschetz property and consequences

DEFINITION 4.1 (Weak Lefschetz property)
Let $A = \bigoplus_{i=0}^{c} A_i$ be a graded artinian $\mathbb{F}$-algebra. We say that A has the *weak Lefschetz property* (WLP) if there exists an element $L \in A_1$ such that for $0 \leq i \leq c-1$ each of the multiplication maps

$$\times L : A_i \to A_{i+1}, x \mapsto Lx \quad \text{is injective or surjective.}$$

We call L with this property a *weak Lefschetz element*. The WLP says that $\times L$ has the maximum possible rank, which is referred to as *full rank*.

DEFINITION 4.2
The *non-weak Lefschetz locus* of a graded artinian $\mathbb{F}$-algebra A is the set (more accurately the algebraic set)

$$NLL_w(A) = \{(a_1, \ldots, a_n) \in \mathbb{F}^n \mid L = a_1x_1 + \cdots + a_nx_n \text{ not a weak Lefschetz element on } A\}.$$

EXERCISE 4.3 (Equivalent WLP statements)
Prove that for an artinian graded $\mathbb{F}$-algebra A the following are equivalent:

(1) $L \in A_1$ is a weak Lefschetz element for A.
(2) For all $0 \leq i \leq c-1$ the map $\times L : A_i \to A_{i+1}$ has rank $\min\{h_A(i), h_A(i+1)\}$.
(3) For all $0 \leq i \leq c-1$ $\dim_{\mathbb{F}}([(L)]_{i+1}) = \min\{h_A(i), h_A(i+1)\}$.
(4) For all $0 \leq i \leq c-1$ $\dim_{\mathbb{F}}([A/(L)]_{i+1}) = \max\{0, h_A(i+1) - h_A(i)\}$.
(5) For all $0 \leq i \leq c-1$ $\dim_{\mathbb{F}}([0 :_A L]_i) = \max\{0, h_A(i) - h_A(i+1)\}$.

EXERCISE 4.4
Show that the non-weak Lefschetz locus is a Zariski closed set.

EXAMPLE 4.5
Take $A = \mathbb{C}[x,y]/(x^2, y^2)$ with the standard grading $|x| = |y| = 1$ and $L = x + y$. Then the multiplication map $\times L$ gives the following matrices with respect to the monomial bases $\{1\}, \{x, y\}$ and $\{xy\}$:

map	matrix	rank	injective/surjective
$A_0 \to A_1$	$\begin{bmatrix}1\\1\end{bmatrix}$	1	injective
$A_1 \to A_2$	$\begin{bmatrix}1 & 1\end{bmatrix}$	1	surjective
$A_i \to A_{i+1},\ i \geq 2$	$\begin{bmatrix}0\end{bmatrix}$	0	surjective

We conclude that A has the WLP and $x + y$ is a Lefschetz element on A.

EXAMPLE 4.6 (Dependence on characteristic)
Take $A = \mathbb{F}[x,y,z]/(x^2,y^2,z^2)$ with the standard grading $|x| = |y| = 1$ and $L = ax+by+cz$. Then the multiplication map $\times L$ is represented by the following matrices with respect to the monomial bases 1 for A_0, $\{x,y,z\}$ for A_1, $\{xy,xz,yz\}$ for A_2, and xyz for A_3:

$$\times L : A_0 \to A_1 \quad M = \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \quad \text{injective unless } a=b=c=0,$$

$$\times L : A_1 \to A_2 \quad M = \begin{bmatrix} b & a & 0 \\ c & 0 & a \\ 0 & c & b \end{bmatrix}, \quad \det(M) = -2abc,$$

$$\times L : A_2 \to A_3 \quad M = \begin{bmatrix} a & b & c \end{bmatrix}, \quad \text{surjective unless } a=b=c=0.$$

The map $\times L : A_1 \to A_2$ has full rank if and only if $\mathrm{char}(\mathbb{F}) \neq 2$ and $a \neq 0$, $b \neq 0$, $c \neq 0$. We conclude that A has the WLP if and only if $\mathrm{char}(\mathbb{F}) \neq 2$ because in that case $L = x+y+z$ is a weak Lefschetz element.

The *non-(weak) Lefschetz locus* of A in this example is

$$\begin{aligned} &\mathrm{NLL}_w(A) \\ &\quad = \{(a,b,c) \in \mathbb{F}^3 \mid L = ax+by+cz \text{ is not a weak Lefschetz element on } A\} \\ &\quad = V(abc) = \{(a,b,c) \in \mathbb{F}^3 \mid a=0 \text{ or } b=0 \text{ or } c=0\} \\ &\quad = \text{the union of the three coordinate planes in } \mathbb{F}^3. \end{aligned}$$

DEFINITION 4.7
A sequence of numbers $h_1, \ldots, h_c$ is called *unimodal* if there is an integer j such that

$$h_1 \leq h_2 \leq \cdots \leq h_j \geq h_{j+1} \geq \cdots \geq h_c.$$

LEMMA 4.8
If B is a standard graded $\mathbb{F}$-algebra and $B_j = 0$ for some $j \in \mathbb{N}$ then $B_i = 0$ for all $i \geq j$.

Proof. B standard graded means that $B = \mathbb{F}[B_1] = \mathbb{F}[x_1, \ldots, x_n]/I$, where $x_1, \ldots, x_n$ are an $\mathbb{F}$-basis for B_1, which means $|x_1| = \cdots = |x_n| = 1$, and I is a homogeneous ideal.

Then it follows that $B_i = \mathrm{Span}_{\mathbb{F}}\{B_{i-j}B_j\} = \mathrm{Span}_{\mathbb{F}}\{0\} = 0$ for any $i \geq j$.

PROPOSITION 4.9
Suppose that A is a standard graded artinian algebra over a field $\mathbb{F}$. If A has the weak Lefschetz property then A has a unimodal Hilbert function.

Proof. Let j be the smallest integer such that $\dim_{\mathbb{F}} A_j > \dim_{\mathbb{F}} A_{j+1}$ and let L be a Lefschetz element on A. Then $\times L : A_j \to A_{j+1}$ is surjective, i.e. $LA_j = A_{j+1}$. Now consider the cokernel $A/(L)$ of the map $A \xrightarrow{\times L} A$. We have that $(A/(L))_{j+1} =$

$A_{j+1}/LA_j = 0$, so by the previous Lemma $[A/(L)]_i = 0$ for $i \geq j+1$. This means that $\times L : A_i \to A_{i+1}$ is surjective for $i \geq j$ and so we have

$$h_0(A) \leq h_1(A) \leq \cdots \leq h_j(A) > h_{j+1}(A) \geq h_{j+2}(A) \geq \cdots \geq h_c(A).$$

The proof above yields:

COROLLARY 4.10
For a standard graded artinian algebra A with the weak Lefschetz property there exists $j \in \mathbb{N}$ such that the multiplication maps by a weak Lefschetz element $\times L : A_i \to A_{i+1}$ are injective for $i < j$ after which they become surjective for $i \geq j$.

EXAMPLE 4.11 (Dependence on grading)
We will see in Example 4.18 that the algebra $A = F[x,y]/(x^2, y^2)$ with $|x| = |y| = 1$ has WLP. The ring A is standard graded and has unimodal Hilbert function $1, 2, 1$.

Take $B = \mathbb{C}[x,y]/(x^2, y^2)$ with $|x| = 1$, $|y| = 3$. Then B is a graded algebra with nonunimodal Hilbert function $1, 1, 0, 1, 1$, but x is a weak Lefschetz element on B.

Take $C = \mathbb{C}[x,y]/(x^2, y^2)$ with $|x| = 1$, $|y| = 2$. Then C has a unimodal Hilbert function $1, 1, 1, 1$ but does not have the WLP because the only degree one elements are multiples of x, which do not have maximal rank with respect to multiplication from degree one to degree two.

4.2. Strong Lefschetz property and consequences

DEFINITION 4.12 (Strong Lefschetz property)
Let $A = \bigoplus_{i=1}^{c} A_i$ be a graded artinian $\mathbb{F}$-algebra. We say that A has the *strong Lefschetz property (SLP)* if there exists an element $L \in A_1$ such that for all $1 \leq d \leq c$ and $0 \leq i \leq c - d$ each of the multiplication maps

$$\times L^d : A_i \to A_{i+d}, x \mapsto L^d x \quad \text{is injective or surjective.}$$

We call L with this property a *strong Lefschetz element.*

REMARK 4.13
An element $L \in A_1$ is strong Lefschetz on A if and only if for all $1 \leq d \leq c$ and $0 \leq i \leq c - d$ the maps $\times L^d : A_i \to A_{i+d}$ have rank $\min\{h_A(i), h_A(d+i)\}$.

DEFINITION 4.14
The *non-strong Lefschetz locus* of a graded artinian $\mathbb{F}$-algebra A is the set (more accurately the algebraic set)

$$NLL_s(A) = \{(a_1, \ldots, a_n) \in \mathbb{F}^n \mid L = a_1x_1 + \cdots + a_nx_n \text{ not a strong Lefschetz element on } A\}.$$

REMARK 4.15
The non-strong Lefschetz locus is a Zariski closed set.

REMARK 4.16 (SLP ⇒ WLP)
If A satisfies SLP then A satisfies WLP (the $d = 1$ case).

The following exercise shows this implication is not reversible.

EXERCISE 4.17
Let $\mathbb{F}$ be a field of characteristic zero and let

$$A = \frac{\mathbb{F}[x, y, z]}{(x^3, y^3, z^3, (x + y + z)^3)}.$$

(1) Find the Hilbert function of A.
(2) Prove that A satisfies WLP but not SLP.

Hint: One can use a computer algebra system such as *Macaulay2* [12] to prove that an algebra satisfies WLP or SLP by finding a linear form that conforms to the respective definition. Usually such form is found by trial and picking at random from the set of linear forms. *Macaulay2* has a function `random()` which is helpful in this regard.

One can also prove that an algebra does or does not satisfy WLP or SLP by working over an extension of the coefficient field $\mathbb{F}$ of A. In the example above, one would work over the filed extension $K = \mathbb{F}(a, b, c)$, defined in *Macaulay2* as `K=frac(QQ[a,b,c])`. To compute the rank of multiplication by $L = ax + by + cz$ and its powers on the artinian algebra $A' = K \otimes_{\mathbb{F}} A$ in *Macaulay2* for $\mathbb{F} = \mathbb{Q}$ use the criteria in Exercise 4.3 to express the desired ranks in terms of the Hilbert function of cyclic modules $A'/(L^j)$.

EXAMPLE 4.18 (Dependence on characteristic)
Take $A = \mathbb{F}[x, y]/(x^2, y^2)$ with the standard grading $|x| = |y| = 1$ and $L = ax+by$. Then the multiplication map $\times L^2$ gives the following matrices with respect to the monomial bases $\{1\}, \{x, y\}$ and $\{xy\}$:

map	matrix	rank	injective/ surjective
$A_0 \to A_2$	$\begin{bmatrix} 2ab \end{bmatrix}$	$\begin{cases} 1, & \text{char}(\mathbb{F}) \neq 2, \\ 0, & \text{char}(\mathbb{F}) = 2, \end{cases}$	$\begin{cases} \text{bijective}, & \text{char}(\mathbb{F}) \neq 2, \\ \text{none}, & \text{char}(\mathbb{F}) = 2, \end{cases}$
$A_i \to A_{i+2},\ i \geq 1$	$\begin{bmatrix} 0 \end{bmatrix}$	0	surj

If $\text{char}(\mathbb{F}) \neq 2$ we conclude that A has the SLP and $ax + by$ where $a \neq 0, b \neq 0$ is a Lefschetz element on A. The *non-(strong) Lefschetz locus* is the union of the coordinate axes in $\mathbb{F}^2$,

$$NLL_s(A) = V(ab) = \{(a, b) \in \mathbb{F}^2 \mid a = 0 \text{ or } b = 0\}.$$

However A does not have the SLP if $\text{char}(\mathbb{F}) = 2$ so in that case $\text{NLL}_s(A) = \mathbb{F}^2$.

The SLP for artinian algebras with non-standard grading was studied in [14].

Proposition 4.19
Let A be a (not necessarily standard) graded artinian $\mathbb{F}$-algebra which satisfies the SLP. Then A has unimodal Hilbert function.

Proof. Suppose that the Hilbert function of A is not unimodal. Then there are integers $k < l < m$ such that $\dim_{\mathbb{F}} A_k > \dim_F A_l < \dim_{\mathbb{F}} A_m$. Recall the rank of a composite map is bounded above by the ranks of its components. Hence the multiplication map $\times L^{m-k} : A_k \to A_m$ cannot have full rank for any linear element $L \in A$ because it is the composition of $\times L^{m-l} : A_l \to A_m$ and $\times L^{l-k} : A_l \to A_k$, each of which have rank strictly less than $\min\{\dim_{\mathbb{F}} A_k, \dim_{\mathbb{F}} A_m\}$. Thus A cannot have the SLP.

Definition 4.20
Let $A = \bigoplus_{i=1}^{c} A_i$ be a graded artinian $\mathbb{F}$-algebra. We say that A has the *strong Lefschetz property in the narrow sense (SLPn)* if there exists an element $L \in A_1$ such that the multiplication maps $\times L^{c-2i} : A_i \to A_{c-i}, x \mapsto L^{c-2i}x$ are bijections for all $0 \leq i \leq \lceil c/2 \rceil$.

Remark 4.21
SLP in the narrow sense is the closest property to the conclusion of the Hard Lefschez Theorem 2.9.

Definition 4.22
We say that a graded artinian algebra $A = \bigoplus_{i=1}^{c} A_i$ of maximum socle degree c has a *symmetric Hilbert function* if $h_A(i) = h_A(c-i)$ for $0 \leq i \leq \lceil c/2 \rceil$.

Proposition 4.23
If a graded artinian $\mathbb{F}$-algebra A has the strong Lefschetz property in the narrow sense, then the Hilbert function of A is unimodal and symmetric. Moreover we have the equivalence: A has SLP and symmetric Hilbert function if and only if A has SLP in the narrow sense.

Proof. ($\Leftarrow$) The fact that SLP in the narrow sense implies symmetric Hilbert function follows from the definition because the bijections give $\dim_F A_i = \dim_F A_{c-i}$.

The fact that SLP in the narrow sense implies SLP can be noticed by considering $\times L^d : A_i \to A_{i+d}$. For each such d, i there exists $j = c - i - d$ such that:

- if $i \leq (c-d)/2$ then $j = c - i - d \geq i$ and $(\times L^d) \circ (\times L^{j-i}) = \times L^{c-2i}$ is a bijection implies that $\times L^d$ is surjective, hence has full rank,
- if $i > (c-d)/2$ then $c - 2i < d$ and $(\times L^{d-(c-2i)}) \circ (\times L^d) = \times L^{c-2i}$ is a bijection implies that $\times L^d$ is injective, hence full rank.

($\Rightarrow$) The fact that SLP + symmetric Hilbert function implies SLPn is clear from the definitions.

Example 4.24
The standard graded algebra $\mathbb{F}[x,y]/(x^2, xy, y^a)$ with $a > 3$ has non-symmetric Hilbert function $1, 2, 1^{a-2}$ (here 1^{a-2} means 1 repeated $a-2$ times). Notice that A has the SLP because $L = x + y$ is a strong Lefschetz element. However, A does not satisfy SLPn because its Hilbert function is not symmetric.

4.3. Stanley's Theorem

The most famous theorem in the area of investigation of the algebraic Lefschetz properties, and also the theorem which started this, is the following:

THEOREM 4.25 (Stanley's theorem)
If $\mathrm{char}(\mathbb{F}) = 0$*, then all monomial complete intersections, i.e.* $\mathbb{F}$*-algebras of the form*

$$A = \frac{\mathbb{F}[x_1, \dots, x_n]}{(x_1^{d_1}, \dots, x_n^{d_n})}$$

with $d_1, \dots, d_n \in \mathbb{N}$ *have the SLP.*

Proof. Recall that $H^\bullet(\mathbb{P}^{d-1}_{\mathbb{C}}, \mathbb{F}) = \mathbb{F}[x]/(x^d)$, so by Künneth we have

$$H^\bullet(\mathbb{P}^{d_1-1}_{\mathbb{C}} \times \mathbb{P}^{d_2-1}_{\mathbb{C}} \times \cdots \times \mathbb{P}^{d_n-1}_{\mathbb{C}}, \mathbb{F}) \\ = \mathbb{F}[x]_1/(x_1^{d_1}) \otimes_{\mathbb{F}} \mathbb{F}[x_2]/(x_2^{d_2}) \otimes_{\mathbb{F}} \cdots \otimes_{\mathbb{F}} \mathbb{F}[x_n]/(x_n^{d_n}) = A.$$

Since $X = \mathbb{P}^{d_1-1}_{\mathbb{C}} \times \mathbb{P}^{d_2-1}_{\mathbb{C}} \times \cdots \times \mathbb{P}^{d_n-1}_{\mathbb{C}}$ is an irreducible complex projective variety, the Hard Lefschetz theorem says that A has SLP in the narrow sense which implies that A has SLP.

Alternate proofs of this theorem have been given by Watanabe in [33] using representation theory, by Reid, Roberts and Roitman in [29] with algebraic methods, also by Lindsey [23] and by Herzog and Popescu [15]. We will give a different proof of Stanley's theorem later in these notes in Corollary 5.22.

EXERCISE 4.26
With help from a computer make conjectures regarding the WLP and SLP for monomial complete intersections in positive characteristics. A characterization is known for SLP, but not for WLP. See [5, 27] for related work.

4.4. Combinatorial applications

The following discussion of a spectacular application of SLP is taken from [32].

A *polytope* is a convex body in Euclidean space which is bounded and has finitely many vertices. Let $\mathcal{P}$ be a d-dimensional simplicial convex polytope with f_i i-dimensional faces, $0 < i < d-1$. We call the vector $f(\mathcal{P}) = (f_0, \dots, f_{d-1})$ the *f-vector* of $\mathcal{P}$.The problem of obtaining information about such vectors goes back to Descartes and Euler. In 1971 McMullen [26] gave a remarkable condition on a vector $(f_0, \dots, f_{d-1})$ which he conjectured was equivalent to being the f-vector of some polytope.

To describe this condition, define a new vector $h(\mathcal{P}) = (h_0, \dots, h_d)$, called the *$h$-vector* of $\mathcal{P}$, by

$$h_i = \sum_{j=0}^{i} \binom{d-j}{d-i} (-1)^{i-j} f_{j-1},$$

where we set $f_{-1} = 1$. McMullen's conditions, though he did not realize it, turn out to be equivalent to $h_i = h_{d-i}$ for all i together with the existence of a standard

graded commutative algebra A with $A_0 = \mathbb{F}$ and $h_A(i) = h_i - h_{i-1}$ for $1 \leq i < \lfloor d/2 \rfloor$.

Stanley [31] constructed from $\mathcal{P}$ a d-dimensional complex projective toric variety $X(\mathcal{P})$ for which $\dim_{\mathbb{C}} H^{2i}(X(\mathcal{P})) = h_i$. Although $X(\mathcal{P})$ need not be smooth, its singularities are sufficiently nice that the hard Lefschetz theorem continues to hold. Namely, $X(\mathcal{P})$ locally looks like $\mathbb{C}^n/G$, where G is a finite group of linear transformations. Taking $A = H^*(X(\mathcal{P}))/(L)$ with degrees scaled by $1/2$, where L is the class of a hyperplane section, the necessity of McMullen's condition follows from Exercise 4.3(4). Sufficiency was proved about the same time by Billera and Lee [3].

5. Lefschetz property via representation theory of $\mathfrak{sl}_2$

5.1. The Lie algebra $\mathfrak{sl}_2$ and its representations

Some of the exercises in this section are taken from [30].

Throughout this section let $\mathbb{F}$ be an algebraically closed field of characteristic zero.

Definition 5.1
A *Lie algebra* is a vector space $\mathfrak{g}$ equipped with a bilinear operator $[-,-] : \mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$ satisfying the following two conditions:

- $[x, y] = -[y, x]$,
- $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$.

The bilinear operator $[-,-]$ is called the bracket product, or simply the *bracket*. The second identity in the definition is called the *Jacobi identity*.

Remark 5.2
Any associative algebra has a Lie algebra structure with the bracket product defined by commutator $[x, y] = xy - yx$. Associativity implies the Jacobi identity.

The set of $n \times n$ matrices $\mathcal{M}_n(\mathbb{F})$ forms a Lie algebra since it is associative. This Lie algebra is denoted by $\mathfrak{gl}_n(\mathbb{F})$.

Definition 5.3
Since the set of matrices of trace zero is closed under the bracket of $\mathfrak{gl}_n(\mathbb{F})$ (because $\operatorname{tr}(AB) = \operatorname{tr}(BA)$ for any matrices A, B), it forms a Lie subalgebra

$$\mathfrak{sl}_n(\mathbb{F}) = \{M \in \mathfrak{gl}_n(\mathbb{F}) \mid \operatorname{tr}(M) = 0\}.$$

Example 5.4 (The Lie algebra $\mathfrak{sl}_2(\mathbb{F})$)
In the case where $n = 2$, $\mathfrak{sl}_2(\mathbb{F})$ is three-dimensional, with basis

$$E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad F = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

The three elements E, H, F are called the $\mathfrak{sl}_2$*-triple*.

These elements satisfy the following three relations, which we call the fundamental relations:

$$[E, F] = H, \quad [H, E] = 2E, \quad [H, F] = -2F.$$

The algebra $\mathfrak{sl}_2(\mathbb{F})$ is completely determined by these relations.

We are interested in representations of $\mathfrak{sl}_2$.

Definition 5.5 (Lie algebra representation)
Let V be an $\mathbb{F}$-vector space. Then $\mathrm{End}(V)$ is a Lie algebra with the bracket defined by $[f, g] = f \circ g - g \circ f$. A *representation* of a Lie algebra $\mathfrak{g}$ is vector space V endowed with a Lie algebra homomorphism

$$\rho : \mathfrak{g} \to \mathrm{End}(V),$$

i.e. a vector space homomorphism which satisfies

$$\rho([x, y]) = [\rho(x), \rho(y)].$$

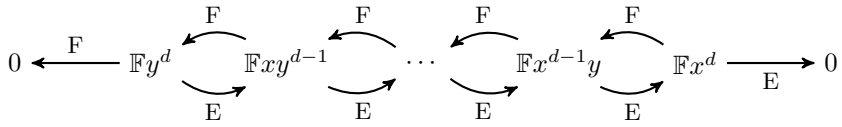
A representation is called *irreducible* if it contains no trivial (nonzero) subrepresentation, i.e. if $W \subsetneq V$ is such that $\rho(W) \subseteq W$ then $W = 0$.

In the case of $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{F})$, we abuse notation and call the set of elements $\rho(E), \rho(H), \rho(F)$ just E, H, F and say they form an $\mathfrak{sl}_2$*-triple.*

Exercise 5.6
Let $\mathbb{F}[x, y]_d$ be the vector space of homogeneous polynomials of degree d in $\mathbb{F}[x, y]$. Prove that

(1) $E = x\frac{\partial}{\partial y}, H = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}, F = y\frac{\partial}{\partial x}$ form an $\mathfrak{sl}_2$-triple.
(2) Prove that the monomial $x^a y^b$ is an eigenvector of H with eigenvalue $a-b \in \mathbb{Z}$. In particular the eigenvalues of H on $\mathbb{F}[x, y]_d$ are $d, d-2, d-4, \ldots, 4-d, 2-d, -d$.
(3) Prove that a basis of $\mathbb{F}[x, y]_d$ is $y^d, E(y^d), E^2(y^d), \ldots, E^d(y^d)$.

Pictorially this can be summarized as



We will soon see that the vector space in Exercise 5.6 is the basic building block of all other representations of $\mathfrak{sl}_2$.

An important result on Lie algebra representations is:

Theorem 5.7 (Weyl's Theorem)
Any Lie algebra representation decomposes uniquely up to isomorphism as a direct sum of irreducible representations.

Definition 5.8 (Weight vectors)
Let $\rho : \mathfrak{sl}_2(\mathbb{F}) \to \mathrm{End}(V)$ be a representation. The eigenvalues of H are called *weights* and the eigenvectors are called *weight vectors*. In particular an eigenvector u is called a *lowest weight vector* if $Fu = 0$ and is called a *highest weight vector* if $Eu = 0$.

Example 5.9
In the representation introduced in Exercise 5.6 the highest weight vectors are the elements of $\mathbb{F}x^d$ and the lowest weight vectors are the elements of $\mathbb{F}y^d$.

To justify the name of highest weight we state the following theorem:

Theorem 5.10 (Irreducible representations of $\mathfrak{sl}_2$)
Let $\rho : \mathfrak{sl}_2(\mathbb{F}) \to \mathrm{End}(V)$ be an irreducible representation with $\dim(V) = d + 1$. Then there exist a basis $\mathcal{B} = \{v_0, \ldots, v_d\}$ for V such that

1. *each v_i is an eigenvector for H with eigenvalue $-d + 2i$, i.e. $Hv_i = (-d + 2i)v_i$,*
2. *$Ev_i = v_{i+1}$ for $i < d$, $Ev_d = 0$,*
3. *$Fv_i = i(d - i + 1)v_{i-1}$ for $i > 0$, $Fv_0 = 0$.*

In particular, the elements $E, H, F \in M_{d+1}(\mathbb{F})$ are represented with respect to this basis by the matrices

$$[E]_{\mathcal{B}} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix},$$

$$[H]_{\mathcal{B}} = \begin{bmatrix} -d & 0 & \cdots & 0 \\ 0 & -d+2 & \cdots & 0 \\ \vdots & \ddots & & \vdots \\ 0 & 0 & \cdots & d \end{bmatrix}, \tag{5.1}$$

$$[F]_{\mathcal{B}} = \begin{bmatrix} 0 & 1 \cdot d & \cdots & 0 & 0 \\ 0 & 0 & 2(d-1) & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & d \cdot 1 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}. \tag{5.2}$$

Exercise 5.11
Find a basis that satisfies the properties given by Theorem 5.10 for the representation $\mathbb{F}[x, y]_d$ introduced in Exercise 5.6.

Theorem 5.10 above says in particular that there is only one representation of $\mathfrak{sl}_2(\mathbb{F})$ of dimension $d + 1$ (up to isomorphism). A representative for this isomorphism class can be chosen to be the representation $\mathbb{F}[x, y]_d$ in Exercise 5.6. Furthermore any representation of $\mathfrak{sl}_2(\mathbb{F})$ has a basis consisting of weight vectors.

This justifies the following:

Definition 5.12
Let V be a representation of $\mathfrak{sl}_2$ and let $W_\lambda(V) = \{v \in V \mid Hv = \lambda v\}$ be the eigenspace corresponding to a weight (eigenvalue) λ for H. Then there is a decomposition $V = \bigoplus_\lambda W_\lambda(V)$ called the *weight space decomposition* of V.

Remark 5.13
If V is an irreducible representation for $\mathfrak{sl}_2(\mathbb{F})$ and $\dim(V) = n+1$ then the weight spaces are the 1-dimensional spaces $W_{-n+2i}(V) = \mathbb{F}v_i$ with v_i as in Theorem 5.10.

Exercise 5.14

(1) Suppose that V is a representation of $\mathfrak{sl}_2$ and that the eigenvalues of H on V are $2, 1, 1, 0, -1, -1, -2$. Show that the irreducible decomposition of V is $V \cong \mathbb{F}[x,y]_2 \oplus \mathbb{F}[x,y]_1 \oplus \mathbb{F}[x,y]_1$.
(2) Prove that if V is any representation of $\mathfrak{sl}_2$ then its irreducible decomposition is determined by the eigenvalues of H.

Exercise 5.15
Let V be an $\mathfrak{sl}_2$ representation and set $W_k = \{v \in V \mid H(v) = kv\}$.

(1) Show that $\dim_{\mathbb{F}} W_k = \dim_{\mathbb{F}} W_{-k}$.
(2) Prove that $E^k : W_{-k} \to W_k$ is an isomorphism.
(3) Show that $\dim_{\mathbb{F}} W_{k+2} \leq \dim_{\mathbb{F}} W_k$ for all $k \geq 0$, that is, the two sequences

$$\ldots, \dim_{\mathbb{F}} W_4, \dim_{\mathbb{F}} W_2, \dim_{\mathbb{F}} W_0, \dim_{\mathbb{F}} W_{-2}, \dim_{\mathbb{F}} W_{-4}, \ldots$$

$$\ldots, \dim_{\mathbb{F}} W_3, \dim_{\mathbb{F}} W_1, \dim_{\mathbb{F}} W_{-1}, \dim_{\mathbb{F}} W_{-3}, \ldots$$

are unimodal.

5.2. Weight space decompositions and the narrow sense of SLP

We now show that there is a close connection between artinian algebras satisfying SLPn and the representations of $\mathfrak{sl}_2$.

Remark 5.16
If A is a graded artinian $\mathbb{F}$-algebra and L is a linear form, then we can view A as a $\mathbb{F}[L]$-module since by the universal mapping property of polynomial rings there exists a well defined ring homomorphism $\mathbb{F}[L] \to A$ which maps $L \mapsto L$. Since $\mathbb{F}[L]$ is a PID and A is a module over it, the structure theorem for modules over PIDs says that there is a module isomorphims

$$A \cong \mathbb{F}[L]/(p_1^{e_1}) \oplus \cdots \oplus F[L]/(p_k^{e_k}),$$

where each p_i is a prime element of $\mathbb{F}[L]$ (no free part since A is finite dimensional). Since A is furthermore graded the elementary divisors $p_i^{e_i}$ must be homogeneous elements of $\mathbb{F}[L]$, thus $p_i = L$ for all i. This implies that A decomposes as a direct sum

$$A \cong S^{(1)} \oplus \cdots \oplus S^{(k)} \qquad \text{with } S^{(i)} \cong \mathbb{F}[L]/(L^{e_i}).$$

The cyclic $\mathbb{F}[L]$ modules $S^{(i)}$ are the *strands* of multiplication by L on A which are essential in defining the notion of Jordan type, a finer invariant than the Lefschetz properties. The generic *Jordan type* of A is the multi-set of sizes of Jordan blocks $e_1, e_2, \ldots, e_s$ ordered decreasingly. This follows because the action of L on $S^{(i)}$ is given by a single Jordan block of size e_i.

Here is the connection between SLPn and the representations of $\mathfrak{sl}_2(\mathbb{F})$:

COROLLARY 5.17
The following are equivalent

(1) *S is a cyclic graded $\mathbb{F}[L]$ module, i.e. $S \cong \mathbb{F}[L]/(L^d)$ (not necessarily degree preserving isomorphism),*
(2) *$S \cong \mathbb{F}[x,y]_{d-1}$ as an irreducible representation of $\mathfrak{sl}_2$ with $Es = Ls$.*

Proof. This follows because both the action of L on S as well as the action of E on $\mathbb{F}[x,y]_{d-1}$ is given by a single Jordan block matrix. Once the basis of S has been fixed to be $1, L, L^2, \ldots, L^{d-1}$, the action of H and F can be simply defined to be the one given by the matrices displayed in Theorem 5.10.

If we put the $\mathfrak{sl}_2(\mathbb{F})$-module structures on the individual strands together we obtain:

THEOREM 5.18 (SLPn via weight decomposition)
Let A be a graded artinian algebra of socle degree c and let $L \in A_1$. The following are equivalent

(1) *L is a strong Lefschetz element on A in the narrow sense,*
(2) *A is an $\mathfrak{sl}_2(\mathbb{F})$-representation with $E = \times L$ and the weight space decomposition of A coincides with the grading decomposition via* $\operatorname{weight}(v) = 2\deg(v) - c$. *This means that*

$$A = \bigoplus_{i=0}^{c} A_i = \bigoplus_{i=0}^{c} W_{2i-c}(A), \qquad \text{where } A_i = W_{2i-c}(A).$$

Proof sketch. Suppose L is a strong Lefschetz element on A in the narrow sense. We construct an $\mathfrak{sl}_2(\mathbb{F})$ triple in $\operatorname{End}_{\mathbb{F}}(A)$ as follows: let $E = \times L : A \to A$. Consider the Jordan decomposition of A with respect to the endomorphism E written as $A = \bigoplus V_i$, that is, let the subspaces V_i be the eigenspaces of the operator E. For each V_i, let $F_i, H_i : V_i \to V_i$ to be the endomorphisms of V_i given with respect to the basis in which $E\big|_{V_i}$ is in Jordan form by the matrices in (5.1) and (5.2), respectively, where $d = \dim(V_i) - 1$. Setting $H = \bigoplus H_i$ and $F = \bigoplus F_i$, one can check E, H, F is an $\mathfrak{sl}_2(\mathbb{F})$ triple. Furthermore, from the properties of Jordan type one knows that the Jordan blocks of L are centered around the middle degree of A; see [17, Proposition 2.38]. It follows that if v is an eigenvector of weight $2k-d$ it is in degree $(c-d)/2+k$ (note that $c \equiv d \pmod 2$). Substituting $i = (c-d)/2 + k$ it follows that $W_{2i-c}(A) = A_i$.

Conversely, suppose A is an $\mathfrak{sl}_2(\mathbb{F})$-representation with $E = \times L$. Then one can use the information about the grading to verify that the Jordan blocks are

centered around degree $\lfloor c/2 \rfloor$. Thus the Jordan degree type is the transpose of the Hilbert function of A. By [13, Proposition 3.64 (2)] it follows that L is a strong Lefschetz element on A in the narrow sense.

5.3. Tensor products

From Theorem 5.18, we can deduce how SLPn behaves when we take tensor products. We need the following lemma.

LEMMA 5.19
If $\mathbb{F}$ is an algebraically closed field of characteristic zero and A, A' are associative algebras which are representations of $\mathfrak{sl}_2(\mathbb{F})$, then so is $A \otimes_{\mathbb{F}} A'$ with the action $g \cdot (v \otimes v') = (gv) \otimes v' + v \otimes (gv')$. If v, v' are weight vectors then $v \otimes v'$ is also a weight vector with $\operatorname{weight}(v \otimes v') = \operatorname{weight}(v) + \operatorname{weight}(v')$.

Proof. We show the statement about weights only: say $\operatorname{weight}(v) = \lambda$ and $\operatorname{weight}(v') = \lambda'$ so that $Hv = \lambda v$, $Hv' = \lambda v'$. Then

$$H(v \otimes v') = (Hv) \otimes v' + v \otimes (Hv') = \lambda v \otimes v' + v \otimes \lambda' v' = (\lambda + \lambda')v \otimes v'$$

shows that $v \otimes v'$ is a weight vector with weight $\lambda + \lambda'$.

THEOREM 5.20
Let $\mathbb{F}$ be an algebraically closed field of characteristic zero. If L is a strong Lefschetz element in the narrow sense on A and if L' is a strong Lefschetz element in the narrow sense on A' then $L \otimes 1 + 1 \otimes L'$ is a strong Lefschetz element in the narrow sense on $A \otimes_{\mathbb{F}} A'$.

Proof. By 5.18 we have that if c, c' are the socle degrees of A, A', respectively, then $A_i = W_{2i-c}(A)$ and $A'_j = W_{2j-c'}(A')$, so

$$A = \bigoplus_{i=0}^{c} A_i = \bigoplus_{i=0}^{c} W_{2i-c}(A) \quad \text{and} \quad A' = \bigoplus_{j=0}^{c'} A'_j = \bigoplus_{j=0}^{c'} W_{2j-c'}(A') \tag{5.3}$$

imply

$$A \otimes_{\mathbb{F}} A' = \bigoplus_{i=0,j=0}^{c,c'} A_i \otimes_{\mathbb{F}} A'_j = \bigoplus_{i=0,j=0}^{c,c'} W_{2i-c}(A) \otimes_{\mathbb{F}} W_{2j-c'}(A'). \tag{5.4}$$

From the fact that $\deg(v \otimes v') = \deg(v) + \deg(v')$ and (5.3) we deduce that

$$(A \otimes_{\mathbb{F}} A')_k = \bigoplus_{i=0}^{c} A_i \otimes_F A'_{k-i}.$$

Note that the maximum socle degree of $A \otimes_{\mathbb{F}} A'$ is $c + c'$. From the identity

$$\operatorname{weight}(v \otimes v') = \operatorname{weight}(v) + \operatorname{weight}(v')$$

and (5.4) we deduce that

$$W_{2k-c-c'}(A \otimes_{\mathbb{F}} A') = \bigoplus_{i=0}^{c} W_{2i-c}(A) \otimes_{\mathbb{F}} W_{2(k-i)-c'}(A') = \bigoplus_{i=0}^{c} A_i \otimes_{\mathbb{F}} A'_{k-i}.$$

Comparing, we see that $(A \otimes_{\mathbb{F}} A')_k = W_{2k-c-c'}(A \otimes_{\mathbb{F}} A')$, where the weight spaces of $A \otimes_{\mathbb{F}} A'$ correspond to the action

$$E(v \otimes v') = Ev \otimes v' + v \otimes Ev' = Lv \otimes v' + v \otimes L'v' = (L \otimes 1 + 1 \otimes L')v \otimes v'.$$

Theorem 5.18 gives that $L \otimes 1 + 1 \otimes L'$ is a strong Lefschetz element on $A \otimes_{\mathbb{F}} A'$.

A corollary of Theorem 5.20 is the following

Corollary 5.21 (Tensor product preserves SLPn)
If $\mathbb{F}$ is a field of characteristic zero and A, A' are graded artinian $\mathbb{F}$-algebras which satisfy SLPn, then $A \otimes_{\mathbb{F}} A'$ also satisfies SLPn.

From the above corollary one can easily deduce Stanley's theorem applying induction on the embedding dimension n.

Corollary 5.22 (Stanley's Theorem - second proof)
If $\mathbb{F}$ has characteristic 0, then the algebra $A = \mathbb{F}[x_1, \dots, x_n]/(x_1^{d_1}, \dots, x_n^{d_n}) = \mathbb{F}[x_1]/(x_1^{d_1}) \otimes_{\mathbb{F}} \cdots \otimes_{\mathbb{F}} \mathbb{F}[x_n]/(x_n^{d_n})$ satisfies SLP in the narrow sense.

Remark 5.23

(1) While the symmetric unimodality of Hilbert functions is preserved under taking tensor product, just unimodality is not. For example for
$$A = \mathbb{F}[x, y, z]/(x^2, xy, y^2, xz, yz, z^5)$$
with Hilbert function $1, 3, 1, 1, 1$ we have that the Hilbert function of $A \otimes_{\mathbb{F}} A$ is $1, 6, 11, 8, 9, 8, 3, 2, 1$.

(2) While the SLPn is preserved under taking tensor product, the SLP (not in the narrow sense) is not preserved by tensor product. In the example above A has SLP but since its Hilbert function is not unimodal, $A \otimes_{\mathbb{F}} A$ cannot have the SLP.

The issue in part 2 of the remark is remedied by restricting to Gorenstein algebras, which have symmetric Hilbert function. Recall that for algebras with symmetric Hilbert function the SLP is equivalent to SLPn by Proposition 4.23. Thus we have:

Corollary 5.24 ([25, Theorem 6.1])
If $\mathbb{F}$ is a field of characteristic zero and A, A' are graded artinian Gorenstein $\mathbb{F}$-algebras which satisfy SLP, then $A \otimes_{\mathbb{F}} A'$ also satisfies SLP.

6. Gorenstein rings via Macaulay inverse systems

The description of the dual ring of the polynomial ring in Section 6.1 is taken from [7]. The material in Section 6.2 follows Eisenbud's Commutative Algebra book [8] and Geramita's lectures [9, Lecture 9]. The material on Hessians in Section 6.3 follows [25].

6.1. The graded dual of the polynomial ring

Recall the notion of a dual for an $\mathbb{F}$-vector space:

Definition 6.1
Let V be an $\mathbb{F}$-vector space. Its dual is

$$V^* = \mathrm{Hom}_{\mathbb{F}}(V, \mathbb{F}) = \{\varphi : V \to \mathbb{F} \mid \varphi \text{ is } \mathbb{F} - \text{linear}\},$$

the vector space of linear functionals on V.

Exercise 6.2
If V is a finite dimensional vector space, there is a natural isomorphism of vector spaces $V \cong V^{**}$.

We extend this idea to construct duals of rings and modules.

Definition 6.3 (Divided power algebra)
Say $R = \mathbb{F}[x_1, \ldots, x_n]$ is the polynomial ring. Let

$$R^* := \mathrm{Hom}^{\mathrm{gr}}_{\mathbb{F}}(R, \mathbb{F}) = \bigoplus_{i \geq 0} \mathrm{Hom}_{\mathbb{F}}(R_i, \mathbb{F}).$$

We use a standard shorthand for monomials: if $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{Z}^n_{\geq 0}$, then $x^{\mathbf{a}} = x_1^{a_1} \cdots x_n^{a_n}$ is the corresponding monomial in R. If $x^{\mathbf{a}}$ is in R_d, we write $X^{[\mathbf{a}]}$ for the functional (in R_d^*) on R_d which sends $x^{\mathbf{a}}$ to 1 and all other monomials in R_d to 0. We'll make the convention from now on to write X_i for the duals of the elements x_i in R_1^*. As a vector space, R^* is isomorphic to a polynomial ring in the n variables $X_1, \ldots, X_n$. However, as we recall shortly, R^* has the multiplicative structure of a *divided power algebra*. For this reason, we call $X^{[\mathbf{a}]}$ a *divided* monomial and we write $R^* = \mathbb{F}[X_1, \ldots, X_n]_{DP}$.

The ring R acts on R^* by *contraction*, which we denote by $\bullet$. That is, if $x^{\mathbf{a}}$ is a monomial in R and $X^{[\mathbf{b}]}$ is a divided monomial in R^*, then

$$x^{\mathbf{a}} \bullet X^{[\mathbf{b}]} = \begin{cases} X^{[\mathbf{b}-\mathbf{a}]}, & \text{if } \mathbf{b} \geq \mathbf{a}, \\ 0, & \text{otherwise.} \end{cases}$$

This action is extended linearly to all of R and R^*. This action of R on R^* gives a perfect pairing of vector spaces $R_d \times R_d^* \to \mathbb{F}$ for any degree $d \geq 0$. Suppose U is a subspace of R_d. We define

$$U^{\perp} = \{g \in R_d^* \mid f \bullet g = 0 \text{ for all } f \in U\}.$$

Macaulay [24] introduced the *inverse system* of an ideal I of R to be

$$I^{-1} := \mathrm{Ann}_{R^*}(I) = \{g \in R^* \mid f \bullet g = 0 \text{ for all } f \in I\}.$$

If I is a homogeneous ideal of R then the inverse system I^{-1} can be constructed degree by degree using the identification $(I^{-1})_d = I_d^{\perp}$. We return to this notion in Definition 6.11.

A priori, R^* is simply a graded R-module. However, R^* can be equipped with a multiplication which makes it into a ring. Suppose $\mathbf{a} = (a_1, \ldots, a_n)$, $\mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{Z}^n_{\geq 0}$. The multiplication in R^* is defined on monomials by

$$X^{[\mathbf{a}]}X^{[\mathbf{b}]} = \binom{\mathbf{a}+\mathbf{b}}{\mathbf{a}} X^{[\mathbf{a}+\mathbf{b}]},$$

where

$$\mathbf{a}! = \prod_{i=0}^{N} a_i! \quad \text{and} \quad \binom{\mathbf{a}+\mathbf{b}}{\mathbf{a}} = \prod_{i=1}^{n} \binom{a_i+b_i}{a_i}. \tag{6.1}$$

This multiplication is extended linearly to all of R^*. We see from the above definition that if $\mathbf{a} = (a_1, \ldots, a_n)$ then $X^{[\mathbf{a}]} = \prod_{i=1}^n X_i^{[a_i]}$.

Exercise 6.4
Now set $X^{\mathbf{a}} = \prod_{i=1}^n X_i^{a_i}$, where the multiplication occurs in the divided power algebra as defined above. Deduce from the above definition that

$$X^{\mathbf{a}} = \mathbf{a}!X^{[\mathbf{a}]}. \tag{6.2}$$

Remark 6.5
In characteristic zero, $\mathbf{a}!$ never vanishes and so, by (6.2), R^* is generated as an algebra by $X_0, \ldots, X_N$, just like the polynomial ring. However, in characteristic $p > 0$, R^* is infinitely generated by all the divided power monomials $X_j^{[p^{k_i}]}$ for all $j = 0, \ldots, N$ and $k_j \geq 0$. The exercise below justifies this last assertion.

Exercise 6.6
Prove that in characteristic p for any $\mathbf{a} = (a_1, \ldots, a_n)$ where $a_j = \sum a_{ij}p^i$, we have

$$X^{[\mathbf{a}]} = \prod_{j=1}^{n} \prod_{i} (X_j^{[p^i]})^{a_{ij}}.$$

Hint: Use Lucas' identity – given base p expansions $a = \sum a_i p^i$ and $b = \sum b_i p^i$ for $a, b \in \mathbb{N}$, then

$$\binom{b}{a} = \prod_{i=0}^{\infty} \binom{b_i}{a_i} \mod p.$$

We now revisit the characteristic zero case. Suppose $\mathbb{F}$ is a field of characteristic zero and let $S = \mathbb{F}[X_1, \ldots, X_n]$ be a polynomial ring. Consider the action of R on S by partial differentiation, which we represent by '$\circ$'. That is, if $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{Z}^n_{\geq 0}$, $x^{\mathbf{a}} = x_1^{a_0} \cdots x_n^{a_n}$ is a monomial in R, and $g \in S$, we write

$$x^{\mathbf{a}} \circ g = \frac{\partial^{\mathbf{a}} g}{\partial X^{\mathbf{a}}}$$

for the action of $x^{\mathbf{a}}$ on g (extended linearly to all of R). In particular, if $\mathbf{a} \leq \mathbf{b}$, then

$$x^{\mathbf{a}} \circ X^{\mathbf{b}} = \frac{\mathbf{b}!}{(\mathbf{b}-\mathbf{a})!} X^{\mathbf{b}-\mathbf{a}},$$

where we use the conventions in (6.1). This action gives a perfect pairing $R_d \times S_d \to \mathbb{F}$, and, given a homogeneous ideal $I \subset R$, we define $I_d^{\perp}$ and I^{-1} in the same way as we do for contraction.

Since we are in characteristic zero, the map of rings $\Phi : S \to R^*$ defined by $\Phi(X_i) = X_i$ extends to all monomials via (6.2) to give $\Phi(y^{\mathbf{a}}) = Y^{\mathbf{a}} = \mathbf{a}!Y^{[\mathbf{a}]}$. Thus S and R^* are isomorphic rings in view of Exercise 6.4. Moreover, if $F \in R$ and $g \in S$, then $\Phi(F \circ g) = F \bullet \Phi(g)$ [9, Theorem 9.5], so S and R^* are isomorphic as R-modules.

6.2. Macaulay inverse systems

Definition 6.7 (Dualizing functor)
Let M be a finitely generated R-module. Define the dual of M to be $D(M) = \mathrm{Hom}_R(M, R^*)$. Let $f : M \to N$ be an R-module homomorphism. Define $D(f)$ to be the induced R-module homomorphism

$$D(f) : D(N) = \mathrm{Hom}_R(N, R^*) \to D(M) = \mathrm{Hom}_R(M, R^*)$$

given by

$$D(f)(\varphi) = \varphi \circ f.$$

This makes D into a contravariant functor in the category of finitely generated R-modules.

Exercise 6.8
Let M be a finitely generated R-module. Recall that we defined $D(M) = \mathrm{Hom}_R(M, R^*)$. In this exercise we also consider the set $M^* = \mathrm{Hom}_{\mathbb{F}}(M, \mathbb{F})$ with its two structures induced from the R-module structure of M by setting

$$r\phi(x) = \phi(r \cdot x) \qquad \text{for all } r \in R,\ x \in M.$$

Show that $D(M) \cong \mathrm{Hom}_{\mathbb{F}}(M, \mathbb{F})$ as R-modules, so an equivalent way to define the dual module dual to M is M^* (with its R-module structure).

Hint: Hom-tensor adjointness may come in handy.

We now come to a form of duality that involves the above defined functor.

Theorem 6.9 (Matlis duality)
The functor D induces an anti-equivalence of categories between

$$\{\textit{noetherian } R\textit{-modules}\} \leftrightarrow \{\textit{artinian } R\textit{-submodules of } R^*\}$$

given by sending $M \mapsto D(M)$.

Next we wish to make the meaning of $D(M)$ more concrete in the special case when $M = R/I$ is a cyclic R-module.

Lemma 6.10
Suppose I is a homogeneous ideal of a polynomial ring R. We compute

$$\begin{aligned} D(R/I) &= \mathrm{Hom}_R(R/I, R^*) \cong \mathrm{Ann}_{R^*}(I) = (0 :_{R^*} I) \\ &= \{g \in R^* \mid f \bullet g = 0 \textit{ for all } f \in I\}. \end{aligned}$$

Definition 6.11 (Inverse system)
Suppose I is a homogeneous ideal of a polynomial ring R. The *inverse system* of I is the vector space

$$I^{-1} = \{g \in R^* \mid f \bullet g = 0 \text{ for all } f \in I\}.$$

Remark 6.12
Don't let the notation deceive you! If I is an ideal of R, it does not mean that $I^\perp$ is an ideal (or R^*-submodule) of R^*. It is just an R-module which happens to be a subset of R^*.

Example 6.13
Concretely, say

(1) $I = (x^2, y^3) \subseteq R = \mathbb{F}[x, y]$. Then

$$I^{-1} = (0 :_{R^*} I) = \text{Span}\{XY^2, XY, Y^2, X, Y, 1\} = R \bullet XY^2$$

is the R-submodule of R^* generated by XY^2.

(2) $I = (x^2, xy^2, y^3) \subseteq R = \mathbb{F}[x, y]$. Then

$$I^{-1} = (0 :_{R^*} I) = \text{Span}\{XY, Y^2, X, Y, 1\} = R \bullet XY + R \bullet Y^2$$

is an R-submodule of R^* with two generators.

Next, take

Example 6.14

(1) $I = (x) \subseteq R = \mathbb{F}[x, y]$. Then

$$I^{-1} = (0 :_{R^*} I) = \text{Span}\{Y^i \mid i \geq 0\}.$$

(2) $I = (x^d) \subseteq R = \mathbb{F}[x, y]$. Then

$$\begin{aligned} I^{-1} = (0 :_{R^*} I) &= \text{Span}\{X^i Y^j \mid 0 \leq i \leq d-1, j \geq 0\} \\ &= R_0^* \oplus R_1^* \oplus R_2^* \oplus \cdots \oplus R_{d-1}^* \oplus Y R_{d-1}^* \oplus Y^2 R_{d-1}^* \\ &\quad \oplus \cdots \oplus Y^k R_{d-1}^* \oplus \cdots . \end{aligned}$$

Both of the above $(0 :_{R^*} I)$ are non-finitely generated R-modules. Theorem 6.9 shows that this corresponds to R/I not being artinian.

Exercise 6.15
Generalize Example 6.14 to find the inverse system of the ideal defining a point in projective n-space and the inverse systems of all of the powers of this ideal.

We now wish to study the inverse functor involved in the Matlis duality Theorem 6.9. In order to do this we define the inverse system of an $\mathbb{F}$-subspace of R^*.

Definition 6.16
Let V be an $\mathbb{F}$-vector subspace of the $\mathbb{F}$-algebra R^*. The *inverse system* of V is

$$V^\perp = \operatorname{Ann}_R(V) = \{f \in R \mid f \circ v = 0 \text{ for all } v \in V\}.$$

We will be most interested in the case when $V = \operatorname{Span}\{F\}$ is a 1-dimensional $\mathbb{F}$-vector space and thus

$$F^\perp = \operatorname{Ann}_R(F) = \{f \in R \mid f \circ F = 0\}.$$

Macaulay inverse system duality is a concrete version of Matlis duality Theorem 6.9 which can be stated in terms of the inverse systems defined above as follows:

Theorem 6.17 (Macaulay inverse system duality)
With notation as above, there are bijective correspondence between

$$\begin{aligned} \{R - \textit{modules } M \subseteq R^*\} &\leftrightarrow \{R/I \mid I \subseteq R \textit{ homogeneous ideal}\} \\ M &\mapsto R/\operatorname{Ann}_R(M) \\ I^\perp = D(R/I) &\leftarrow\!\shortmid R/I. \end{aligned}$$

Furthermore, we have the additional correspondences

(a)	M *finitely generated*	$\iff$	$R/\operatorname{Ann}_R(M)$ *artinian,*
(b)	$M = R \circ F$ *cyclic*	$\iff$	$R/\operatorname{Ann}_R(F)$ *artinian Gorenstein* $\deg(F) =$ *socle degree of* $R/\operatorname{Ann}_R(F)$.

The value of Theorem 6.17 often lies in producing examples of artinian Gorenstein rings.

Definition 6.18
In view of statement (b) in Theorem 6.17, the polynomial $F \in R^*$ is called a *Macaulay dual generator* for $R/\operatorname{Ann}_R(F)$.

Example 6.19
The artinian Gorenstein algebra with Macaulay dual generator

$$F = X^2 + Y^2 + Z^2$$

is the ring of Exercise 6.22,

$$\mathbb{F}[x,y,z]/\operatorname{Ann}_{\mathbb{F}[x,y,z]}(F) = \mathbb{F}[x,y,z]/(x^2 - y^2, y^2 - z^2, xy, xz, yz).$$

Example 6.20
The artinian Gorenstein algebra with Macaulay dual generator

$$F = X_1^{d_1} \cdots X_n^{d_n}$$

is the monomial complete intersection

$$\mathbb{F}[x_1,\ldots,x_n]/\operatorname{Ann}_{\mathbb{F}[x_1,\ldots,x_n]}(F) = \mathbb{F}[x_1,\ldots,x_n]/(x_1^{d_1+1},\ldots,x_n^{d_n+1}).$$

Definition 6.21
For a graded module M and an integer d, define $M(d)$ to be the module M with grading modified such that $M(d)_i = M_{d+i}$.

Exercise 6.22
For any homogeneous polynomial $F \in R^*$ of degree d, prove

(1) $\operatorname{Ann}_R(F)^\perp = R \circ F$. This statement is an instance of Macaulay's double annihilator theorem.
(2) the cyclic ring $A = R/\operatorname{Ann}_R(F)$ is artinian Gorenstein if and only if the function $A \to D(A)(-d)$, $a \mapsto [b \mapsto (ab) \circ F]$ is an isomorphism.

Hint for (1): Start by showing the equality is true in degree d, then use the R-module structure.
Hint for (2): Use Proposition 3.11. Prove that the function $a \mapsto (a \bullet F)(0)$ is an orientation on A and that A satisfies Poincaré duality with respect to this orientation.

In view of Exercise 6.22 we can state an alternate definition of graded Gorenstein rings.

Definition 6.23
An artinian graded ring A is *Gorenstein* of socle degree d if and only if $A \cong D(A)(-d)$ as graded A-modules (degree preserving isomorphism).

6.3. SLP for Gorenstein rings via Hessian matrices

For this section let $R = \mathbb{F}[x_1, \dots, x_n]$ be a polynomial ring and R^* its graded dual. We will further assume that $\operatorname{char}(\mathbb{F}) = 0$.

In this section we use that R^* is isomorphic to $\mathbb{F}[X_1, \dots, X_n]$ with R-action $x_i \circ F = \frac{\partial F}{\partial X_i}$. We will use this description for R^*.

Lemma 6.24
Let $F \in R^*_c$ and let $L = a_1x_1 + \cdots + a_nx_n \in R_1$. Then

$$L^c \circ F = c! \cdot F(a_1, \dots, a_n).$$

Proof.

$$L^c \circ F = \sum_{i_1+\cdots+i_n=c} \frac{c!}{i_1! \cdots i_n!} a_1^{i_1} \cdots a_n^{i_n} x_1^{i_1} \cdots x_n^{i_n} \circ F = c! \cdot F(a_1, \dots, a_n).$$

Definition 6.25 (Higher Hessians)
Let $F \in R^*$ be a homogeneous polynomial and let $B = \{b_1, \dots, b_s\} \subseteq R_d$ be a finite set of homogeneous polynomials of degree $d \geq 0$. We call

$$\operatorname{Hess}^d_B(F) = [b_i b_j \circ F]_{1 \leq i,j \leq s} \quad \text{and} \quad \operatorname{hess}^d_B(F) = \det \operatorname{Hess}^d_B(F)$$

the d-th *Hessian matrix* and the d-th *Hessian determinant* of F with respect to B, respectively.

REMARK 6.26
If $B = \{x_1, \dots x_n\}$ then

$$\mathrm{Hess}^1_B(F) = [x_i x_j F]_{1\le i,j\le n} = \left[\frac{\partial F}{\partial X_i \partial X_j}\right]_{1\le i,j\le n}$$

is the classical Hessian of F.

Hessians are useful in establishing the SLP for artinian Gorenstein rings.

THEOREM 6.27 (Hessian criterion for SLP [25])
Assume $\mathbb{F}$ is a field of characteristic zero. Let A be a graded artinian Gorenstein ring with Macaulay dual generator $F \in R^*_c$. Then A has the SLP if and only if

$$\mathrm{hess}^i_{B_i}(F) \neq 0 \qquad \text{for } 0 \le i \le \lfloor \frac{c}{2} \rfloor$$

where B_i is some (equivalently, any) basis of A_i.

Proof. From the hypothesis and Theorem 6.17 we have that $A = R/\operatorname{Ann}_R(F)$ has socle degree $d = \deg(F)$.

Since A is Gorenstein, A has symmetric Hilbert function, so by Proposition 4.23 A has SLP if and only if A has SLP in the narrow sense, i.e. there exists $L \in A_1$ such that for any $0 \le i \le \lfloor \frac{d}{2} \rfloor$ the multiplication maps $L^{c-2i} : A_i \to A_{c-i}$ are vector space isomorphisms. Say $L = a_1x_1 + \cdots + a_nx_n$.

Recall that the isomorphism $A \cong D(A)(-c) = (R \circ F)(-c), a \mapsto a \circ F$ induces vector space isomorphisms $A_{c-i} \cong A_i^*$ also defined by $a \mapsto [b \mapsto b \circ (a \circ F) = (ba) \circ F]$; see Exercise 6.22. This isomorphism is denoted $- \circ F$ in the sequence displayed below. The composite map

$$T_i : A_i \xrightarrow{\times L^{c-2i}} A_{c-i} \xrightarrow{-\circ F} A_i^*$$

is an isomorphism if and only if multiplication by L^{c-2i} is an isomorphism. Let B_i be a basis for A_i and let B_i^* be its dual, which is a basis for A_i^*. The matrix $[t^{(i)}_{jk}]$for T_i with respect to these bases is defined as follows

$$T_i(b_j) = \sum_{k=1}^{s} t^{(i)}_{jk} b^*_k.$$

Hence we compute

$$t^{(i)}_{jk} = T_i(b_j)(b_k) = (L^{c-2i}b_j)^*(b_k) = (L^{c-2i}b_jb_k) \circ F = L^{c-2i} \circ (b_jb_k \circ F).$$

Using Lemma 6.24 the formula above becomes

$$t^{(i)}_{jk} = (c-2i)!(b_jb_k \circ F)(a_1, \dots, a_n).$$

Thus T_i is an isomorphism for some $L \in R_1$ if and only if

$$\mathrm{hess}^i_{B_i} F(a_1, \dots, a_n) = \det\left[b_ib_j \circ F(a_1, \dots, a_n)\right]_{1\le i,j\le s} \neq 0.$$

Overall the SLP holds if and only if for $0 \le i \le \lfloor \frac{c}{2} \rfloor$ the hessian determinant $\mathrm{hess}^i_{B_i} F$ does not vanish identically.

Example 6.28
Say $F = X^2 + Y^2 + Z^2$. Then with respect to the standard monomial basis for each R_i

$$\begin{aligned}\operatorname{hess}^0(F) &= F,\\ \operatorname{hess}^1(F) &= \det \begin{bmatrix} 2 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2\end{bmatrix} = 8,\\ \operatorname{hess}^i(F) &= 0 \text{ for } i \geq 2.\end{aligned}$$

This shows the algebra in Exercise 3.13 has SLP in characteristic zero.

Example 6.29 (H. Ikeda [20])
Let $G = XYW^3 + X^3ZW + Y^3Z^2$. Then $A = R/\operatorname{Ann}_R(G)$ has Hilbert function $(1, 4, 10, 10, 4, 1)$ and a basis for A_1 is $B_1 = \{x, y, z, w\}$ whereas a basis for A_2 is $B_2 = \{x^2, xy, xz, xw, y^2, yz, yw, z^2, zw, w^2\}$. Furthermore

$$\operatorname{hess}^0(G) = G,$$

$$\operatorname{hess}^1_{B_1}(G) = \det \begin{pmatrix} 6XZW & W^3 & 3X^2W & 3X^2Z + 3YW^2\\ W^3 & 6YZ^2 & 6Y^2Z & 3XW^2\\ 3X^2W & 6Y^2Z & 2Y^3 & X^3\\ 3X^2Z + 3YW^2 & 3XW^2 & X^3 & 6XYW\end{pmatrix} \neq 0,$$

$$\operatorname{hess}^2_{B_2}(G) = \det \begin{pmatrix} 0 & 0 & 6W & 6z & 0 & 0 & 0 & 0 & 6X & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6W\\ 6W & 0 & 0 & 6X & 0 & 0 & 0 & 0 & 0 & 0\\ 6Z & 0 & 6X & 0 & 0 & 0 & 6W & 0 & 0 & 6Y\\ 0 & 0 & 0 & 0 & 0 & 12Z & 0 & 12Y & 0 & 0\\ 0 & 0 & 0 & 0 & 12Z & 12Y & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 6W & 0 & 0 & 0 & 0 & 0 & 6X\\ 0 & 0 & 0 & 0 & 12Y & 0 & 0 & 0 & 0 & 0\\ 6X & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 6W & 0 & 6Y & 0 & 0 & 6X & 0 & 0 & 0\end{pmatrix} = 0.$$

We conclude that the map $L : A_2 \to A_3$ fails to have maximum rank for all $L \in A_1$. However the map $L^3 : A_1 \to A_4$ does have maximum rank.

Exercise 6.30 (R. Gondim [10])
Let $x_1, \ldots, x_n$ and $u_1, \ldots, u_m$ be two sets of indeterminates with $n \geq m \geq 2$. Let $f_i \in \mathbb{F}[x_1, \ldots, x_n]_k$ and $g_i \in \mathbb{F}[u_1, \ldots, u_m]_e$ for $1 \leq i \leq s$ be linearly independent forms with $1 \leq k < e$. If $s > \binom{m-1+k}{k}$, then

$$F = f_1g_1 + \cdots + f_sg_s$$

is called a Perazzo form and $A = \mathbb{F}[x_1, \ldots, x_n, u_1, \ldots, u_m]$ is called a Perazzo algebra.

(1) Show that $\operatorname{hess}_k(F) = 0$ and so A does not have SLP.

(2) Make conjectures regarding the Hilbert functions of Perazzo algebras.
(3) Make conjectures regarding the WLP for Perazzo algebras.
(4) Do there exist two Perazzo algebras A and B having the same Hilbert function so that A has WLP and B does not?

Some answers to (2) and (3) can be found in [1]. Part (4) is an open problem suggested by Lisa Nicklasson.

COROLLARY 6.31
Let $F \in \mathbb{F}[X_1, \dots, X_n], G \in \mathbb{F}[Y_1, \dots, Y_m]$ be homogeneous polynomials of the same degree. Then $A = \mathbb{F}[x_1, \dots, x_n]/\operatorname{Ann}_R(F)$ and $B = \mathbb{F}[y_1, \dots, y_m]/\operatorname{Ann}_R(G)$ have SLP if and only if

$$C = \mathbb{F}[x_1, \dots, x_n, y_1, \dots, y_m]/\operatorname{Ann}_R(F+G) \text{ satisfies SLP.}$$

Proof. It turns out that for $1 \leq i < \deg(F)$ a basis β of C_i is given by the union of a basis β' of A_i and a basis β'' of B_i (for a proof of this refer to Proposition 7.8 and (7.1)) and hence the hessians of $F+G$ look like

$$\operatorname{Hess}^i(F+G) = \left[b_i b_j(F+G)\right]_{b_i, b_j \in \beta} = \begin{bmatrix} b_i' b_j'(F) & 0 \\ 0 & b_i'' b_j''(F) \end{bmatrix}_{b_i', b_j' \in \beta', b_i'', b_j'' \in \beta''}$$

$$= \begin{bmatrix} \operatorname{Hess}^i(F) & 0 \\ 0 & \operatorname{Hess}^i(G) \end{bmatrix}$$

$$\operatorname{hess}^i(F+G) = \operatorname{hess}^i(F) \operatorname{hess}^i(G).$$

Now we see that $\operatorname{hess}^i(F+G) \neq 0$ if and only if $\operatorname{hess}^i(F) \neq 0$ and $\operatorname{hess}^i(G) \neq 0$, which gives the desired conclusion.

We take the preceding example up again in the following section, generalizing the construction that produces C from A and B in Definition 7.7.

7. Topological ring constructions and the Lefschetz properties

We have seen in Section 2 that the Lefschetz properties emerged from algebraic topology. Now we return to this idea implementing some constructions that originate in topology at the ring level. The material in Section 7.1 is taken from [18] and the material in Section 7.2 is taken from [19].

7.1. Fiber products and connected sums

We first consider the operation termed connected sum. A connected sum of manifolds along a disc is obtained by identifying a disk in each (with opposite orientations). One can more generally take connected sums by identifying two homeomorphic sub-manifolds, one from each summand. If the cohomology rings of the two summands are A and B and the cohomology ring of the common submanifold is T, then it turns out that the cohomology ring of the connected

sum is $A\#_T B$, a ring that we term the connected sum of A and B over T in Definition 7.7.

To define a connected sum of rings we need a preliminary construction. Recall that an *oriented* AG algebra is a pair $(A, \int_A)$ with A an AG algebra and $\int_A$ an orientation as in Proposition 3.11. A choice of orientation on A also corresponds to a choice of Macaulay dual generator.

Exercise 7.1
Every orientation on A can be written as the function $\int_A : A \to K$ defined by $\int_A g = (g \circ F)(0)$ for some Macaulay dual generator F of A. The notation $(g \circ F)(0)$ refers to evaluating the element $g \circ F$ of R' at $X_1 = \cdots = X_n = 0$.

Next we discuss how the orientations of two AG algebras relate.

Definition 7.2 (Thom class)
Let $(A, \int_A)$ and $(T, \int_T)$ be two oriented AG K-algebras with socle degree d for A and k for T, respectively, with $d \geq k$. Let $\pi : A \to T$ be a graded map. By [18, Lemma 2.1], there exists a unique homogeneous element $\tau_A \in A_{d-k}$ such that $\int_A(\tau_A a) = \int_T(\pi(a))$ for all $a \in A$; we call it the *Thom class* for $\pi : A \to T$.

Note that the Thom class for $\pi : A \to T$ depends not only on the map π, but also on the orientations chosen for A and T.

Example 7.3
Let $(A, \int_A)$ be an oriented AG K-algebra with socle degree d. Consider $(K, \int_K)$ where $f_K : K \to K$ is the identity map. Then the Thom class for the canonical projection $\pi : A \to K$ is the unique element $a_{soc} \in A_d$ such that $\int_A a_{soc} = 1$.

Exercise 7.4
Given a homomorphism $\pi : A \to T$ of AG algebras having dual generators F, H of degrees d and k, respectively, with $d \geq k$, show that the Thom class of Definition 7.2 is the unique element τ of A_{d-k} such that $\tau \circ F = H$.

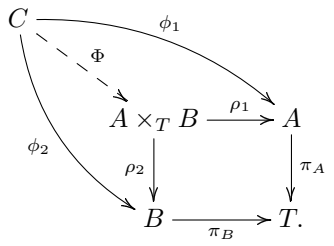
Definition 7.5
Given graded $\mathbb{F}$-algebras A, B, and T, and graded $\mathbb{F}$-algebra maps $\pi_A \colon A \to T$ and $\pi_B \colon B \to T$, the *fiber product* of A and B over T (with respect to π_A and π_B) is the graded $\mathbb{F}$-subalgebra of $A \oplus B$,

$$A \times_T B = \{(a, b) \in A \oplus B \mid \ \pi_A(a) = \pi_B(b)\}.$$

Let $\rho_1 \colon A \times_T B \to A$ and $\rho_2 \colon A \times_T B \to B$ be the natural projection maps. It is well known that fiber products are pullbacks in the category of $\mathbb{F}$ algebras and hence they satisfy the following universal property.

Lemma 7.6
The fiber product $A \times_T B$ satisfies the following universal property: If C is another $\mathbb{F}$-algebra with maps $\phi_1 \colon C \to A$ and $\phi_2 \colon C \to B$ such that $\pi_A \circ \phi_1(c) = \pi_B \circ \phi_2(c)$ for all $c \in C$, then there is a unique $\mathbb{F}$-algebra homomorphism $\Phi \colon C \to A \times_T B$

which makes the diagram below commute:



By [18, Lemma 3.7] the fiber product is characterized by the following exact sequence of vector spaces:

$$0 \to A \times_T B \to A \oplus B \xrightarrow{\pi_A - \pi_B} T \to 0, \tag{7.1}$$

whence the Hilbert function of the fiber product satisfies

$$H_{A \times_T B} = H_A + H_B - H_T.$$

Henceforth we assume that $\pi_A(\tau_A) = \pi_B(\tau_B)$, so that $(\tau_A, \tau_B) \in A \times_T B$.

Definition 7.7
The *connected sum* of the oriented AG K-algebras A and B over T is the quotient ring of the fiber product

$$A \times_T B := \{(a, b) \in A \oplus B \mid \pi_A(a) = \pi_B(b)\}$$

by the principal ideal generated by the pair of Thom classes (τ_A, τ_B), i.e.

$$A \#_T B = (A \times_T B)/\langle(\tau_A, \tau_B)\rangle.$$

By [18, Lemma 3.7] the connected sum is characterized by the following exact sequence of vector spaces:

$$0 \to T(k-d) \to A \times_T B \to A \#_T B \to 0.$$

Therefore, the Hilbert series of the connected sum satisfies

$$HF_{A \#_T B}(t) = HF_A(t) + HF_B(t) - (1 + t^{d-k}) HF_T(t). \tag{7.2}$$

When $T = \mathbb{F}$ we have an easy description of the fiber product and connected sum.

Proposition 7.8
Let $R = \mathbb{F}[x_1, \ldots, x_n]$, $R' = \mathbb{F}[y_1, \ldots, y_m]$ be polynomial rings. Let $\left(A = R/I, \int_A\right)$ and $\left(B = R'/I', \int_B\right)$ be oriented AG algebras each with socle degree d, and let $\pi_A \colon A \to \mathbb{F}$ and $\pi_B \colon B \to \mathbb{F}$ be the natural projection maps with Thom classes $\tau_A \in A_d$ and $\tau_B \in B_d$. Then the fiber product $A \times_{\mathbb{F}} B$ has a presentation

$$A \times_{\mathbb{F}} B \cong \frac{\mathbb{F}[x_1, \ldots, x_n, y_1, \ldots, y_m]}{(x_i y_j \mid 1 \le i \le n, 1 \le j \le m) + I + I'}$$

and the connected sum $A\#_{\mathbb{F}}B$ has a presentation

$$A\#_{\mathbb{F}}B \cong \frac{\mathbb{F}[x_1,\dots,x_n,y_1,\dots,y_m]}{(x_iy_j \mid 1\leq i\leq n, 1\leq j\leq m)+I+I'+(\tau_A+\tau_B)}.$$

In particular, if A and B are standard graded then so are $A\times_{\mathbb{F}} B$ and $A\#_{\mathbb{F}}B$.

Example 7.9 (Standard graded fiber product and connected sum)
Let $A=\mathbb{F}[x,y]/(x^2,y^4)$ and $B=\mathbb{F}[u,v]/(u^3,v^3)$ each with the standard grading $\deg(x)=\deg(y)=\deg(u)=\deg(v)=1$. Let $T=\mathbb{F}[z]/(z^2)$, and define maps $\pi_A: A\to T$, $\pi_A(x)=z$, $\pi_A(y)=0$ and $\pi_B\colon B\to T$, $\pi_B(u)=z$, $\pi_B(v)=0$. Then the fiber product $A\times_T B$ is generated as an algebra by elements $z_1=(y,0)$, $z_2=(x,u)$, and $z_3=(0,v)$, all having degree one. One can check that it has the following presentation:

$$A\times_T B=\frac{\mathbb{F}[z_1,z_2,z_3]}{\langle z_1^4,z_2^3,z_3^3,z_1z_3,z_1z_2^2\rangle}. \tag{7.3}$$

The Hilbert function of the fiber product is

$$\begin{aligned} H(A\times_T B)=&(1,3,5,4,2)\\ =&(1,2,2,2,1)+(1,2,3,2,1)-(1,1,0,0,0)\\ =&H(A)+H(B)-H(T). \end{aligned}$$

Fix orientations on A, B, and T by $\int_A\colon xy^3\mapsto 1$, $\int_B\colon u^2v^2\mapsto 1$, and $\int_T\colon z\mapsto 1$, respectively. Then the Thom classes for $\pi_A\colon A\to T$ and $\pi_B\colon B\to T$ are, respectively, $\tau_A=y^3$, $\tau_B=uv^2$. Note that $\pi_A(\tau_A)=0=\pi_B(\tau_B)$, hence $(\tau_A,\tau_B)\in A\times_T B$, and in terms of (7.3) we have $(\tau_A,\tau_B)=z_1^3+z_2z_3^2$. Therefore we see that

$$A\#_T B=\frac{\mathbb{F}[z_1,z_2,z_3]}{\langle z_1^4,z_2^3,z_3^3,z_1z_3,z_1z_2^2,z_1^3+z_2z_3^2\rangle}.$$

The Hilbert function of the connected sum is

$$\begin{aligned} H(A\#_T B)=&(1,3,5,3,1)\\ =&(1,2,2,2,1)+(1,2,3,2,1)-(1,1,0,0,0)-(0,0,0,1,1)\\ =&H(A)+H(B)-H(T)-H(T)[3], \end{aligned}$$

where $H(T)[3]$ is the Hilbert function of $T(-3)$.

However, if $T\neq\mathbb{F}$ the presentation of the connected sum and fiber product can be complicated and they need not be standard graded.

Example 7.10 (Non-standard graded fiber product and connected sum)
Let

$$A=\mathbb{F}[x]/(x^4),\quad B=\mathbb{F}[u,v]/(u^3,v^2),\quad T=\mathbb{F}[z]/(z^2),$$

have Hilbert functions $H(A)=(1,1,1,1)$ and $H(B)=(1,2,2,1)$. Define maps $\pi_A\colon A\to T$, $\pi_A(x)=z$ and $\pi_B\colon B\to T$, $\pi_B(u)=z$, $\pi_B(v)=0$. Then the fibered

product has the presentation

$$A \times_T B = \frac{\mathbb{F}[z_1, z_2, z_3]}{(z_1^4, z_2^2, z_3^2, z_1z_3, z_1^2z_2 - z_2z_3)}, \qquad \text{where} \begin{cases} z_1 = & (x, u), \\ z_2 = & (0, v), \\ z_3 = & (0, u^2). \end{cases}$$

Here z_1, z_2 have degree one, and z_3 has degree two. We then have a presentation for the connected sum $C = A\#_T B = A \times_T B/(\tau)$, whence

$$A\#_T B \cong \frac{\mathbb{F}[z_1, z_2, z_3]}{(z_1^4, z_2^2, z_3^2, z_1z_3, z_1^2z_2 - z_2z_3, (z_1^2 - z_3) + z_1z_2)} \cong \frac{\mathbb{F}[z_1, z_2]}{(z_1^3 + z_1^2z_2, z_2^2)}.$$

It has Hilbert function $H(C) = (1, 2, 2, 1) = H(A) + H(B) - H(T) - H(T)[1]$ as in (7.2). It is interesting to note that the connected sum $A\#_T B$ has a standard grading whereas the fibered product $A \times_T B$ does not.

Finally, we have the following result which shows how the Lefschetz properties of the components influence the Lefschetz property of the fiber product and connected sum.

THEOREM 7.11

(1) *If A and B are artinian Gorenstein algebras of the same socle degree that each have the SLP, then the fiber product $D = A \times_{\mathbb{F}} B$ over a field $\mathbb{F}$ also has the SLP. If A and B have the standard grading, then the converse holds as well.*

(2) *If A and B both have the SLP, then the connected sum $C = A\#_{\mathbb{F}}B$ over a field $\mathbb{F}$ also has the SLP. If A and B have the standard grading, then the converse holds as well.*

(3) *Let A, T be artinian Gorenstein algebras with socle degrees d, k respectively and let $\pi_A \colon A \to T$ be a surjective ring homomorphism such that its Thom class τ_A satisfies $\pi_A(\tau_A) = 0$. Let x be an indeterminate of degree one, set $B = T[x]/(x^{d-k+1})$, and define $\pi_B \colon B \to T$ to be the natural projection map satisfying $\pi_B(t) = t$ and $\pi_B(x) = 0$. In this setup, if A and T both satisfy the SLP, then the fiber product $A \times_T B$ also satisfies the SLP. Moreover if the field $\mathbb{F}$ is algebraically closed, then the connected sum $A\#_T B$ also satisfies the SLP.*

(4) *Let A and B be standard graded artinian Gorenstein algebras of socle degree d satisfying the SLP, and let T be a graded artinian Gorenstein algebra of socle degree k, with $k < \lfloor \frac{d-1}{2} \rfloor$, endowed with surjective $\mathbb{F}$-algebra homomorphisms $\pi_A : A \to T$ and $\pi_B : B \to T$. Then the resulting fiber product $A \times_T B$ and the connected sum $A\#_T B$ both satisfy the WLP.*

EXAMPLE 7.12
Take

$$F = XY(XZ - YT) \in K[X, Y, Z, T]$$

and set $A = K[x, y, z, t]/\operatorname{Ann}(F)$. Then

$$\operatorname{Ann}(F) = (zt, xz + yt, x^2t, y^2z, x^2y^2, x^3, y^3, z^2, t^2),$$

A is a connected sum

$$A = K[x,y,z]/\operatorname{Ann}(X^2YZ)\#_{K[x,y]/\operatorname{Ann}(XY)}K[x,y,t]/\operatorname{Ann}(XY^2T),$$

and the Hilbert function of A is $(1,4,6,4,1)$. By Theorem 7.11 (4), since the summands of A are monomial complete intersections, A has WLP if the characteristic of $\mathbb{F}$ is 0.

EXAMPLE 7.13 ([2])
Take

$$F = X^3YZ - XY^3T = XY(X^2Z - Y^2T) \in K[X,Y,Z,T]$$

and set $A = K[x,y,z,t]/\operatorname{Ann}(F)$. Then

$$\operatorname{Ann}(F) = (z^2, t^2, tz, x^2t, y^2z, x^2z + y^2t, y^4, x^2y^2, x^4),$$

A is a connected sum

$$A = K[x,y,z]/\operatorname{Ann}(X^3YZ)\#_{K[x,y]/\operatorname{Ann}(XY)}K[x,y,t]/\operatorname{Ann}(XY^3T),$$

and the Hilbert function of A is $(1,4,7,7,4,1)$.

The Hessian matrix of F of order two is of the following form

$$\operatorname{Hess}^2(F) = 6\begin{pmatrix} 0 & y & x & z & 0 & 0 & 0 \\ y & 0 & 0 & x & 0 & 0 & 0 \\ x & 0 & 0 & 0 & 0 & 0 & 0 \\ z & x & 0 & 0 & 0 & -y & -t \\ 0 & 0 & 0 & 0 & 0 & 0 & -y \\ 0 & 0 & 0 & -y & 0 & 0 & -x \\ 0 & 0 & 0 & -t & -y & -x & 0 \end{pmatrix}$$

and it has vanishing determinant. According to the Hessian criteria Theorem 6.27 A does not have WLP because in this case the second Hessian corresponds to the multiplication map from degree 2 to degree 3. Note that the socle degrees don't satisfy the condition in Theorem 7.11 since $2 = k = \lfloor \frac{d-1}{2} \rfloor = \frac{5-1}{2}$.

7.2. Cohomological blowups

The second construction is inspired by the geometric operation of blowing up a smooth projective algebraic variety. The blow-up of such a space at a point replaces the point with the set of all directions through the point, that is, a projective space. More generally one can blow up a subset and replace it with another space called an *exceptional divisor*. The cohomology ring of the blow-up can be determined based on the cohomology ring of the original variety (called A below), that of the subvariety being blown up (called T below) and the way the latter sits inside the former, specifically captured via the cohomology class of the *normal bundle* of the subvariety, encoded via a polynomial $f_A(\xi)$ below.

We now explain the algebraic construction for the cohomology ring of a blowup.

Definition 7.14 (Cohomological Blow-Up)
For oriented artinian Gorenstein algebras A and T of socle degrees $d > k$, respectively, and surjective degree-preserving algebra map $\pi\colon A \to T$ with Thom class $\tau \in A_n$ where $n = d-k$, set $K = \mathrm{Ker}(\pi)$. Given a homogeneous monic polynomial $f_A(\xi) = \xi^n + a_1\xi^{n-1} + \cdots + a_n \in A[\xi]$ of degree n with homogeneous elements $a_i \in A_i$ for $1 \leq i \leq n$ and with $a_n = \lambda \cdot \tau$ for some non-zero constant λ, we call the artinian Gorenstein algebra $\tilde{A}$ below a *cohomological blow up of A along π* or BUG for short

$$\tilde{A} = \frac{A[\xi]}{(\xi \cdot K, \underbrace{\xi^n + a_1\xi^{n-1} + \cdots + \lambda \cdot \tau}_{f_A(\xi)})}.$$

Setting $t_i = \pi(a_i)$ for $1 \leq i \leq n-1$, the AG algebra

$$\tilde{T} = \frac{T[\xi]}{(\underbrace{\xi^n + t_1\xi^{n-1} + \cdots + \lambda \cdot \pi(\tau)}_{f_T(\xi)})}$$

is called the *exceptional divisor of T with parameters* $(t_1, \ldots, t_{n-1}, \lambda)$. These algebras fit in the following commutative diagram, where we refer to A as the *cohomological blow down of $\tilde{A}$ along $\hat{\pi}$*.

$$\begin{array}{ccc} A & \longrightarrow & \tilde{A} \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\hat{\pi}} \\ T & \longrightarrow & \tilde{T}. \end{array}$$

Since $\tilde{T}$ is a quotient of a 1-dimensional Gorenstein ring by a non zero-divisor, it is clear that $\tilde{T}$ is artinian Gorenstein. It is shown in [19] that the condition that the last term of $f_A(\xi)$ be a scalar multiple of the Thom class τ is precisely equivalent to $\tilde{A}$ being AG.

Example 7.15
Let

$$A = \frac{\mathbb{F}[x,y]}{(x^3,y^3)} \xrightarrow{\pi} T = \frac{\mathbb{F}[x,y]}{(x^2,y)},$$

where $\pi(x) = x$ and $\pi(y) = 0$. Note $K = \mathrm{Ker}(\pi) = (x^2, y)$. Orient A and T with socle generators $a_{soc} = x^2y^2$ and $t_{soc} = x$; then the Thom class of π is $\tau = xy^2 \in A_3$. Set $f_T(\xi) = \xi^3 + x\xi^2 \in T[\xi]$ and let $\tilde{T}$ be the associated exceptional divisor algebra:

$$\tilde{T} = \frac{T[\xi]}{(f_T(\xi))} = \frac{\mathbb{F}[x,y,\xi]}{(x^2, y, \xi^3 + x\xi^2)}.$$

Consider $f_A(\xi) = \xi^3 + x\xi^2 + xy^2 \in A[\xi]$. This gives rise to the BUG

$$\tilde{A} = \frac{A[\xi]}{(\xi \cdot K, f_A(\xi))} = \frac{\mathbb{F}[x,y,\xi]}{(x^3, y^3, x^2\xi, y\xi, \xi^3 + x\xi^2 + xy^2)}$$

which has basis

$$\{1, x, y, \xi, x^2, xy, y^2, x\xi, \xi^2, x^2y, xy^2, x\xi^2, x^2y^2\}$$

and Hilbert function $H(\tilde{A}) = (1, 3, 5, 3, 1)$. Here the socle of $\tilde{A}$ is generated by $\tilde{a}_{soc} = a_{soc} = x^2y^2$, hence $\tilde{A}$ is Gorenstein, as expected.

We are now ready to discuss the Lefschetz properties for cohomological blow-up algebras.

THEOREM 7.16 ([19, Theorem 8.5])
Let $\mathbb{F}$ be an infinite field and let $\pi : A \to T$ be a surjective homomorphism of graded AG $\mathbb{F}$-algebras of socle degrees $d > k$ respectively such that both A and T have SLP. Assume that characteristic $\mathbb{F}$ is zero or characteristic F is $p > d$. Then every cohomological blow-up algebra of A along T satisfies SLP.

The following example shows that the converse of Theorem 7.16 is not true: if the cohomological blowup $\tilde{A}$ has SLP it does not follow that A has SLP. In other words, while the process of blowing up preserves SLP, the process of blowing down does not preserve SLP, nor even WLP.

EXAMPLE 7.17
As in Exercise 6.30, the following example, originally due to U. Perazzo [28], but re-examined more recently by R. Gondim and F. Russo [11], is an artinian Gorenstein algebra with unimodal Hilbert function which does not have SLP or WLP:

$$\begin{aligned} A &= \frac{\mathbb{F}[x, y, z, u, v]}{\mathrm{Ann}(XU^2 + YUV + ZV^2)} \\ &= \frac{\mathbb{F}[x, y, z, u, v]}{(x^2, xy, y^2, xz, yz, z^2, u^3, u^2v, uv^2, v^3, xv, zu, xu - yv, zv - yu)}. \end{aligned}$$

Taking the quotient T of A given by the Thom class $\tau = u^2$ yields

$$T = \frac{\mathbb{F}[x, y, z, u, v]}{\mathrm{Ann}(X)} = \frac{\mathbb{F}[x, y, z, u, v]}{(x^2, y, z, u, v)} \cong \frac{\mathbb{F}[x]}{(x^2)}.$$

Fix a parameter $\lambda \in \mathbb{F}$ and define polynomials $f_T(\xi) \in T[\xi]$ and $f_A(\xi) \in A[\xi]$ by

$$f_T(\xi) = \xi^2 - \lambda x\xi \quad \text{and} \quad f_A(\xi) = \xi^2 - \lambda x\xi + u^2.$$

Denoting the ideal of relations of A by I we obtain the cohomological blowup

$$\tilde{A} = \frac{\mathbb{F}[x, y, z, u, v, \xi]}{I + \xi(y, z, u, v) + (f_A(\xi))},$$

which has Hilbert function $H(\tilde{A}) = H(A) + H(T)[1] = (1, 6, 6, 1)$. Fix $\mathbb{F}$-bases

$$\tilde{A}_1 = \mathrm{span}_{\mathbb{F}}\{x, y, z, u, v, \xi\} \quad \text{and} \quad \tilde{A}_2 = \mathrm{span}_{\mathbb{F}}\{u^2, uv, v^2, yv, yu, -x\xi\}$$

and let $\ell \in \tilde{A}_1$ be a general linear form

$$\ell = ax + by + cz + du + ev + f\xi.$$

Then the matrix for the Lefschetz map $\times\ell\colon \tilde{A}_1 \to \tilde{A}_2$ and its determinant are given by

$$M = \begin{pmatrix} 0 & 0\,0\,d\,0 & -f \\ 0 & 0\,0\,e\,d & 0 \\ 0 & 0\,0\,0\,e & 0 \\ d & e\,0\,a\,b & 0 \\ 0 & d\,e\,b\,c & 0 \\ -f & 0\,0\,0\,0 & -(a+\lambda f) \end{pmatrix} \Rightarrow \det(M) = f^2e^4.$$

Thus ℓ is a strong Lefschetz element for $\tilde{A}$ if and only if $e \cdot f \neq 0$. In particular $\tilde{A}$ satisfies SLP and also WLP.

Surprisingly, the analogous result to Theorem 7.16 does not hold for the weak Lefschetz property. We now give an example illustrating that blowing up does not preserve WLP.

Exercise 7.18
Consider the following algebra

$$\begin{aligned} A &= \frac{\mathbb{F}[x,y,z,u,v]}{\mathrm{Ann}(XU^6 + YU^4V^2 + ZU^5V)} \\ &= \frac{\mathbb{F}[x,y,z,u,v]}{(yz, xz, xy, vy-uz, vx, ux-vz, u^5y, u^5v^2, u^6v, u^7, v^3, x^2, y^2, z^2)} \end{aligned}$$

and its quotient corresponding to the Thom class $\tau = u^3$,

$$\begin{aligned} T &= \frac{\mathbb{F}[x,y,z,u,v]}{\mathrm{Ann}(XU^3 + YUV^2 + ZU^2V)} \\ &= \frac{\mathbb{F}[x,y,z,u,v]}{(z^2, yz, xz, y^2, xy, vy-uz, x^2, vx, ux-vz, u^2y, v^3, u^2v^2, u^3v, u^4)}. \end{aligned}$$

Consider also the cohomological blowup

$$\tilde{A} = \frac{\mathbb{F}[x,y,z,u,v,\xi]}{I + \xi \cdot K + (\xi^3 - u^3)}, \qquad \text{where } K = \mathrm{Ker}(A \twoheadrightarrow T).$$

(1) Compute the Hilbert functions of A and T respectively.
(2) Show that both A and T satisfy WLP, but not SLP.
(3) Show that the BUG $\tilde{A}$ does not satisfy WLP.

In Exercise 7.18 the Thom class of the map $A \to T$ has degree 3. This is the minimal possible value for such an example based on the following result.

Theorem 7.19 ([19, Theorem 8.9])
Let $\mathbb{F}$ be an infinite field and let $\pi : A \to T$ be a surjective homomorphism of graded AG $\mathbb{F}$-algebras such that the difference between the socle degrees of A and T is at most 2 and A and T both satisfy WLP. Then every cohomological blow-up algebra of A along π satisfies WLP.

References

[1] Abdallah, N., et al. "Hilbert functions and Jordan type of Perazzo Artinian algebras." In: *Lefschetz Properties*, 59-80. Vol. 59 of *Springer INdAM Ser.*. Singapore: Springer, 2024. Cited on 37.

[2] Altafi, N., et al. "Artinian Gorenstein algebras having binomial Macaulay dual generator." In preparation. Cited on 42.

[3] Billera, L. J., and C. W. Lee. "A proof of the Sufficiency of McMullen's Conditions for f-Vectors of Simplicial Convex Polytopes." *J. Combin. Theory Ser. A* 31 (1981): 237-255. Cited on 22.

[4] Chern, S.-S. "On a Generalization of Kähler Geometry." In: R. H. Fox, et al. (eds.), *Algebraic Geometry and Topology*, 103-124. Princeton, NJ: Princeton Univ. Press, 1951. Cited on 11.

[5] Cook, D. II. "The Lefschetz properties of monomial complete intersections in positive characteristic." *J. Algebra* 369 (2012): 42-58. Cited on 21.

[6] Deligne, P. "La conjecture de Weil. II." *Inst. Hautes Études Sci. Publ. Math.* no. 52 (1980): 137-252. Cited on 11.

[7] DiPasquale, M., T. Nguyễn, and A. Seceleanu. "Duality for asymptotic invariants of graded families." *Adv. Math.* 430 (2023): paper no. 109208. Cited on 28.

[8] Eisenbud, D. *Commutative Algebra: With a View Toward Algebraic Geometry*. Graduate Texts in Mathematics, Vol. 150. New York: Springer-Verlag, 1995. Cited on 28.

[9] Geramita, A. V. "Inverse systems of fat points: Waring's problem, secant varieties of Veronese varieties and parameter spaces for Gorenstein ideals." In: *The Curves Seminar at Queen's*, 2-114. Vol. 102 of Queen's Papers in Pure and Appl. Math. Kingston, ON: Queen's Univ., 1996. Cited on 28 and 31.

[10] Gondim, R. "On higher Hessians and the Lefschetz properties." *J. Algebra* 489 (2017): 241-263. Cited on 36.

[11] Gondim, R., and F. Russo. "Cubic hypersurfaces with vanishing Hessian." *J. Pure Appl. Algebra* 219 (2015): 779-806. Cited on 44.

[12] Grayson, D., and M. Stillman. *Macaulay2*, a software system for research in algebraic geometry. Available at: `http://www2.macaulay2.com`. Cited on 19.

[13] Harima, T., et al. *The Lefschetz Properties*. Vol. 2080 of *Lecture Notes in Mathematics*. Heidelberg: Springer, 2013. Cited on 7, 8 and 27.

[14] Harima, T., and J. Watanabe. "The strong Lefschetz property for Artinian algebras with non-standard grading." *J. Algebra* 311 (2007): 511-537. Cited on 19.

[15] Herzog, J., and D. Popescu. "The strong Lefschetz property and simple extensions." `arXiv:math/05065537`. Cited on 21.

[16] Hodge, W. V. D. *The Theory and Applications of Harmonic Integrals*. 2nd ed. London: Cambridge Univ. Press, 1952. Cited on 11.

[17] Iarrobino, A., P. Macias Marques, and C. McDaniel. "Artinian algebras and Jordan type." *J. Commut. Algebra* 14, no. 3 (2022): 365-414. Cited on 26.

[18] Iarrobino, A., C. McDaniel, and A. Seceleanu. "Connected sums of graded Artinian Gorenstein algebras and Lefschetz properties." *J. Pure Appl. Algebra* 226, no. 1 (2022): 106787. Cited on 37, 38 and 39.

[19] Iarrobino, A., P. Macias Marques, C. McDaniel, A. Seceleanu, and J. Watanabe. "Cohomological blow ups of graded Artinian Gorenstein algebras along surjective maps." *Int. Math. Res. Not. IMRN* 2023, no. 7 (2023): 5816-5886. Cited on 8, 37, 43, 44 and 45.

[20] Ikeda, H. "Results on Dilworth and Rees numbers of Artinian local rings." *Jpn. J. Math. (N.S.)* 22, no. 1 (1996): 147-158. Cited on 36.

[21] Lefschetz, S. *L'Analysis Situs et la Géometrie Algébrique*. Paris: Gauthier-Villars, 1924. Reprinted in: *Selected Papers*. New York: Chelsea, 1971. Cited on 11.

[22] Lefschetz, S. "Reminiscences of a mathematical immigrant in the United States." *Amer. Math. Monthly* 77 (1970): 344-350. Cited on 11.

[23] Lindsey, M. "A class of Hilbert series and the strong Lefschetz property." *Proc. Amer. Math. Soc.* 139 (2011): 79-92. Cited on 21.

[24] Macaulay, F. S. *The Algebraic Theory of Modular Systems*. Cambridge Mathematical Library. Cambridge: Cambridge Univ. Press, 1994. Cited on 29.

[25] Maeno, T., and J. Watanabe. "Lefschetz elements of Artinian Gorenstein algebras and Hessians of homogeneous polynomials." *Illinois J. Math.* 53, no. 2 (2009): 591-603. Cited on 28 and 35.

[26] McMullen, P. "The numbers of faces of simplicial polytopes." *Israel J. Math.* 9 (1971): 559-570. Cited on 21.

[27] Nicklasson, L. "The strong Lefschetz property of monomial complete intersections in two variables." *Collect. Math.* 69, no. 3 (2018): 359-375. Cited on 21.

[28] Perazzo, U. "Sulle varietà cubiche la cui hessiana svanisce identicamente." *Giorn. Mat. Battaglini* 38 (1900): 337-354. Cited on 44.

[29] Reid, L., L. Roberts, and M. Roitman. "On complete intersections and their Hilbert functions." *Canad. Math. Bull.* 34 (1991): 525-535. Cited on 21.

[30] Robles, C. "Linear structures of Hodge theory." Available at: `https://sites.math.duke.edu/~robles/21.02-ICERM.pdf`. Cited on 22.

[31] Stanley, R. "The number of faces of a simplicial convex polytope." *Adv. Math.* 35 (1980): 236-238. Cited on 22.

[32] Stanley, R. "Combinatorial applications of the hard Lefschetz theorem." In: *Proc. Int. Congr. Math.*, Warsaw, Aug. 16-24, 1983, 447-453. Warsaw: PWN-Polish Scientific Publishers, 1984. Cited on 21.

[33] Watanabe, J. "The Dilworth number of Artinian rings and finite posets with rank function." In: *Commutative Algebra and Combinatorics*, Vol. 11 of *Adv. Stud. Pure Math.* Amsterdam: North-Holland, 1987. Cited on 21.

Alexandra Seceleanu
University of Nebraska-Lincoln
Department of Mathematics
203 Avery hall
Lincoln, NE 68503
United States
E-mail: aseceleanu@unl.edu

Received: August 24, 2024; final version: December 12, 2024;
available online: June 24, 2025.