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A general Hardy-Hilbert-type integral inequality theorem

Abstract. This article presents a unified framework that extends the scope of two existing theorems on multivariate Hardy-Hilbert-type integral inequalities. Key to this extension is the use of two additional adjustable parameters that increase flexibility and generality. The framework also has the originality of including the incomplete lower gamma function in the integral definitions governed by a parameter. Detailed proofs are given, mainly based on the Laplace transform, the generalized Young inequality, the generalized Hölder integral inequality and changes of variables. This article thus provides a new comprehensive foundation for future research in generalized multivariate integral inequalities.

1. Introduction

A classic bivariate result in analysis is the Hardy-Hilbert integral inequality. It provides a sharp upper bound for a double integral with integrands that depend on two functions through a particular ratio product form. The upper bound is defined by the multiplication of a constant and the norms of the two functions. A possible statement of this inequality is given below. Let $p, q > 1$ such that $1/p+1/q = 1$ and $f, g\colon (0, +\infty) \to (0, +\infty)$ be two functions under some integrability assumptions.

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Then the following applies:

$$\begin{aligned}\int_{(0,+\infty)^2} &\frac{f(t_1)g(t_2)}{t_1+t_2}dt_1dt_2 \\ &\leq \left[\frac{\pi}{\sin(\pi/p)}\right]\left[\int_{(0,+\infty)} f^p(t)dt\right]^{1/p}\left[\int_{(0,+\infty)} g^q(t)dt\right]^{1/q},\end{aligned} \tag{1.1}$$

provided that all the integrals involved converge. The classical Hilbert integral inequality corresponds to the special case $p = 2$, with the peculiarity of having π as a constant factor. See [6, 24]. In the context of bivariate analysis, the Hardy-Hilbert (or Hilbert) integral inequality has provided the basis for numerous variants. A selection of key references on this topic are [1, 2, 5, 10, 12, 17, 19, 20, 21, 22, 23, 25, 26, 27]. Among them, we emphasize the variant in [12], which is related to the main result of this study. Let $p, q > 1$ such that $1/p + 1/q = 1$, $\alpha > \max(p-1, q-1)$ and $f, g : (0,+\infty) \to (0,+\infty)$ be two functions under some integrability assumptions. Then the following applies:

$$\begin{aligned}\int_{(0,+\infty)^2} \frac{f(t_1)g(t_2)}{(t_1+t_2)^\alpha}dt_1dt_2 \leq \frac{1}{\Gamma(\alpha)}&\left[\frac{\Gamma(\alpha-p+1)}{p}\int_{(0,+\infty)} t^{p-\alpha-1}f^p(t)dt\right. \\ &\left. + \frac{\Gamma(\alpha-q+1)}{q}\int_{(0,+\infty)} t^{q-\alpha-1}g^q(t)dt\right]^{1/q},\end{aligned} \tag{1.2}$$

where $\Gamma(\epsilon) = \int_{(0,+\infty)} t^{\epsilon-1}e^{-t}dt$ with $\epsilon > 0$ denotes the standard gamma function, provided that all the integrals involved converge. The originality of this result lies in the presence of the adjustable parameter α, the sum integral expression of the upper bound, and the consideration of special weighted integral norms of f and g. It is extended to the multivariate case in [12]. Other multivariate variants of the Hardy-Hilbert integral inequality can be found in [4, 7, 8, 9, 11, 13, 14, 15, 28, 29, 30].

For the purposes of this article, we present two such multivariate variants, which are [12, Theorem 3] and [13, Theorem 2]. Possible statements are presented below.

Theorem 1.1 ([12, Theorem 3])
Let $n \in \mathbb{N}\backslash\{0\}$, $p_1, \ldots, p_n$ be such that $\min_{i=1,\ldots,n} p_i > 1$ and $\sum_{i=1}^n (1/p_i) = 1$, $\alpha > \max_{i=1,\ldots,n}(p_i - 1)$ and $f_1, \ldots, f_n : (0,+\infty) \to (0,+\infty)$ be n functions under some integrability assumptions. Then the following applies:

$$\int_{(0,+\infty)^n} \frac{\prod_{i=1}^n f_i(t_i)}{(\sum_{i=1}^n t_i)^\alpha}\prod_{i=1}^n dt_i \leq \frac{1}{\Gamma(\alpha)}\sum_{i=1}^n \frac{\Gamma(\alpha-p_i+1)}{p_i}\int_{(0,+\infty)} t^{p_i-\alpha-1}f_i^{p_i}(t)dt,$$

provided that all the integrals involved converge. If we set $n = 2$, then it is reduced to the inequality in (1.2).

Theorem 1.2 ([13, Theorem 2])
Let $n \in \mathbb{N}\backslash\{0\}$*,* $p_1, \ldots, p_n$ *be such that* $\min_{i=1,\ldots,n} p_i > 1$ *and* $\sum_{i=1}^{n}(1/p_i) = 1$*,* $\alpha > \max[0, n(1 - 2/\max_{i=1,\ldots,n} p_i) + 1]$ *and* $f_1, \ldots, f_n \colon (0, +\infty) \to (0, +\infty)$ *be* n *functions under some integrability assumptions. Then the following applies:*

$$\int_{(0,+\infty)^n} \frac{\prod_{i=1}^{n} f_i(t_i)}{(\sum_{i=1}^{n} t_i)^{\alpha}} \prod_{i=1}^{n} dt_i \le \frac{1}{\Gamma(\alpha)} \prod_{i=1}^{n} \left[\Gamma\left(\left(\frac{\alpha-1}{n} - 1\right)p_i + 2\right)\right]^{1/p_i} \times \left[\int_{(0,+\infty)} t^{[1-(\alpha-1)/n]p_i - 2} f_i^{p_i}(t) dt\right]^{1/p_i},$$

provided that all the integrals involved converge. If we set $n = 2$ *and* $\alpha = 1$*, then we get the classical Hardy-Hilbert integral inequality, as presented in* (1.1)*.*

These results are particularly interesting because they include several Hardy-Hilbert-type integral inequalities found in the literature. They are very flexible due to the presence of the adjustable parameter α and the n functions $f_1, \ldots, f_n$. Furthermore, they allow for n dimensions, which are often encountered in multivariate analysis and related applications.

In this article, we make some contributions to Theorem 1.1 and Theorem 1.2. Specifically, we first propose an unified framework in the form of a single theorem that extends the scope of these two results. This is done by introducing two additional adjustable parameters, thus achieving a higher degree of flexibility. A notable consequence of these extensions is the inclusion of the incomplete lower gamma function in the definition of the integrals involved. One of the parameters also modulates this particular function through a direct mathematical link. The tunable effect of the incomplete lower gamma function is an important innovative aspect of our extensions. The proof makes use of the generalized Young inequality, the generalized Hölder integral inequality, the Laplace transform, suitable changes of variables, and technical manipulations of the incomplete lower gamma function. This result is then complemented by another theorem, which can be seen as a functional generalization. Again for the sake of clarity and completeness, a detailed proof is given. It is mainly based on the first theorem and suitable changes of variables. We thus provide a comprehensive framework for generalizing existing multivariate integral inequalities, which may inspire future studies in this area.

The rest of the article consists of three sections: Section 2 presents the main results, along with some discussion. Section 3 is devoted to the corresponding proofs. A conclusion is given in Section 4.

2. Results

2.1. Main theorem

The main theorem of the article is given in the statement below.

Theorem 2.1
Let $n \in \mathbb{N}\backslash\{0\}$*,* $p_1, \ldots, p_n$ *be such that* $\min_{i=1,\ldots,n} p_i > 1$ *and* $\sum_{i=1}^{n}(1/p_i) = 1$*,* $\beta \geq 0$*,* $\theta > 0$ *or "*$\theta = +\infty$*", and* $f_1, \ldots, f_n \colon (0, +\infty) \to (0, +\infty)$ *be* n *functions under some integrability assumptions (clarification of this aspect is described below).*

(a) *Let* α *such that* $\alpha > \max_{i=1,\ldots,n}(p_i - 1)$. *Then the following applies:*

$$\begin{aligned}\int_{(0,+\infty)^n} & \frac{\prod_{i=1}^{n} f_i(t_i)}{(\beta+\sum_{i=1}^{n} t_i)^{\alpha}} \gamma\left(\alpha, \theta\left(\beta + \sum_{i=1}^{n} t_i\right)\right) \prod_{i=1}^{n} dt_i \\ & \leq \sum_{i=1}^{n} \frac{1}{p_i} \int_{(0,+\infty)} (\beta+t)^{p_i-\alpha-1} f_i^{p_i}(t) \gamma\left(\alpha - p_i + 1, \theta(\beta+t)\right) dt,\end{aligned} \tag{2.1}$$

where $\gamma(\epsilon, \zeta) = \int_0^{\zeta} t^{\epsilon-1} e^{-t} dt$ *with* $\epsilon > 0$ *and* $\zeta > 0$ *denotes the standard lower incomplete gamma function, provided that all the integrals involved converge.*

(b) *Let* α *such that* $\alpha > \max\left[0, n(1 - 2/\max_{i=1,\ldots,n} p_i) + 1\right]$. *Then the following applies:*

$$\begin{aligned}\int_{(0,+\infty)^n} & \frac{\prod_{i=1}^{n} f_i(t_i)}{(\beta+\sum_{i=1}^{n} t_i)^{\alpha}} \gamma\left(\alpha, \theta\left(\beta + \sum_{i=1}^{n} t_i\right)\right) \prod_{i=1}^{n} dt_i \\ & \leq \prod_{i=1}^{n} \left[\int_{(0,+\infty)} \left(\frac{\beta}{n} p_i + t\right)^{[1-(\alpha-1)/n]p_i - 2} f_i^{p_i}(t) \right. \\ & \qquad \left. \times \gamma\left(\left(\frac{\alpha-1}{n} - 1\right)p_i + 2, \theta\left(\frac{\beta}{n} p_i + t\right)\right) dt \right]^{1/p_i},\end{aligned} \tag{2.2}$$

provided that all the integrals involved converge.

The proof of this theorem is given in detail in Section 3. The main novelty is the introduction of two adjustable parameters, β and θ, and the inclusion of the incomplete lower gamma function in the integrals involved. This gives the theorem a high degree of flexibility, in contrast to most multivariate Hardy-Hilbert-type integral inequalities in the literature. More essentially, as explained in the introduction, it has the property of unifying [12, Theorem 3] and [13, Theorem 2]. Indeed, if we take $\beta = 0$ and "$\theta = +\infty$", we have

$$\gamma\left(\alpha, \theta\left(\beta + \sum_{i=1}^{n} t_i\right)\right) = \Gamma(\alpha), \quad \beta + t = t, \quad \beta + \sum_{i=1}^{n} t_i = \sum_{i=1}^{n} t_i,$$

and, for any $i = 1, \ldots, n$,

$$\gamma(\alpha - p_i + 1, \theta(\beta + t)) = \Gamma(\alpha - p_i + 1),$$

$$\gamma\left(\left(\frac{\alpha-1}{n} - 1\right)p_i + 2, \theta\left(\frac{\beta}{n} p_i + t\right)\right) = \Gamma\left(\left(\frac{\alpha-1}{n} - 1\right)p_i + 2\right)$$

and

$$\frac{\beta}{n} p_i + t = t.$$

It follows from Theorem 2.1 that

– if α such that $\alpha > \max_{i=1,\ldots,n}(p_i - 1)$,

$$\begin{aligned}
&\int_{(0,+\infty)^n} \frac{\prod_{i=1}^n f_i(t_i)}{(\sum_{i=1}^n t_i)^\alpha} \prod_{i=1}^n dt_i \\
&\quad\le \frac{1}{\Gamma(\alpha)} \sum_{i=1}^n \frac{1}{p_i} \int_{(0,+\infty)} t^{p_i-\alpha-1} f_i^{p_i}(t)\Gamma(\alpha - p_i + 1)dt \\
&\quad= \frac{1}{\Gamma(\alpha)} \sum_{i=1}^n \frac{\Gamma(\alpha - p_i + 1)}{p_i} \int_{(0,+\infty)} t^{p_i-\alpha-1} f_i^{p_i}(t)dt,
\end{aligned}$$

provided that all the integrals involved converge, which corresponds to the inequality in [12, Theorem 3],

– if $\alpha > \max[0, n(1 - 2/\max_{i=1,\ldots,n} p_i) + 1]$,

$$\begin{aligned}
&\int_{(0,+\infty)^n} \frac{\prod_{i=1}^n f_i(t_i)}{(\sum_{i=1}^n t_i)^\alpha} \prod_{i=1}^n dt_i \\
&\quad\le \frac{1}{\Gamma(\alpha)} \prod_{i=1}^n \left[\int_{(0,+\infty)} t^{[1-(\alpha-1)/n]p_i-2} f_i^{p_i}(t)\Gamma\Big(\Big(\frac{\alpha-1}{n} - 1\Big)p_i + 2\Big)dt\right]^{1/p_i} \\
&\quad= \frac{1}{\Gamma(\alpha)} \prod_{i=1}^n \left[\Gamma\Big(\Big(\frac{\alpha-1}{n} - 1\Big)p_i + 2\Big)\right]^{1/p_i} \\
&\qquad\times \left[\int_{(0,+\infty)} t^{[1-(\alpha-1)/n]p_i-2} f_i^{p_i}(t)dt\right]^{1/p_i},
\end{aligned}$$

provided that all the integrals involved converge, which corresponds to the inequality in [13, Theorem 2].

2.2. A functional generalization

With little effort, Theorem 2.1 can be extended to a broader framework. To support this claim, a functional generalization is proposed in the theorem below.

Theorem 2.2

Let $a \in \mathbb{R}$ or "$a = -\infty$", $b \in \mathbb{R}$ or "$b = +\infty$" with $b > a$, $n \in \mathbb{N}\backslash\{0\}$, $p_1, \ldots, p_n$ be such that $\min_{i=1,\ldots,n} p_i > 1$ and $\sum_{i=1}^n (1/p_i) = 1$, $\beta \ge 0$, $\theta > 0$ or "$\theta = +\infty$", $f_1, \ldots, f_n \colon (a, b) \to (0, +\infty)$ be n functions and $g_1, \ldots, g_n \colon (a, b) \to (0, +\infty)$ be n differentiable non-decreasing functions such that, for any $i = 1, \ldots, n$, $\lim_{t\to a} g_i(t) = 0$ and $\lim_{t\to b} g_i(t) = +\infty$, under some integrability assumptions (clarification of this aspect is described below).

(a) *Let* α *such that* $\alpha > \max_{i=1,\ldots,n}(p_i - 1)$. *Then the following applies:*

$$\int_{(a,b)^n} \frac{\prod_{i=1}^n f_i(t_i)}{[\beta + \sum_{i=1}^n g_i(t_i)]^\alpha} \gamma\left(\alpha, \theta\left[\beta + \sum_{i=1}^n g_i(t_i)\right]\right) \prod_{i=1}^n dt_i \\ \leq \sum_{i=1}^n \frac{1}{p_i} \int_{(a,b)} [\beta + g_i(t)]^{p_i - \alpha - 1} \frac{f_i^{p_i}(t)}{[g_i'(t)]^{p_i - 1}} \\ \times \gamma(\alpha - p_i + 1, \theta[\beta + g_i(t)])dt, \tag{2.3}$$

where $\gamma(\epsilon, \zeta) = \int_0^\zeta t^{\epsilon - 1} e^{-t} dt$ *with* $\epsilon > 0$ *and* $\zeta > 0$ *denotes the standard lower incomplete gamma function, provided that all the integrals involved converge.*

(b) *Let* α *such that* $\alpha > \max\left[0, n(1 - 2/\max_{i=1,\ldots,n} p_i) + 1\right]$. *Then the following applies:*

$$\int_{(a,b)^n} \frac{\prod_{i=1}^n f_i(t_i)}{[\beta + \sum_{i=1}^n g_i(t_i)]^\alpha} \gamma\left(\alpha, \theta\left[\beta + \sum_{i=1}^n g_i(t_i)\right]\right) \prod_{i=1}^n dt_i \\ \leq \prod_{i=1}^n \left[\int_{(a,b)} \left[\frac{\beta}{n} p_i + g_i(t)\right]^{[1 - (\alpha - 1)/n]p_i - 2} \frac{f_i^{p_i}(t)}{[g_i'(t)]^{p_i - 1}} \right. \\ \left. \times \gamma\left(\left(\frac{\alpha - 1}{n} - 1\right)p_i + 2, \theta\left[\frac{\beta}{n} p_i + g_i(t)\right]\right) dt \right]^{1/p_i}, \tag{2.4}$$

provided that all the integrals involved converge.

The proof of this theorem is given in detail in Section 3. It is mainly based on Theorem 2.1 and appropriate changes of variables. Obviously, taking $g_i(t) = t$ for any $i = 1, \ldots n$, Theorem 2.2 reduces to Theorem 2.1.

Three examples of applications of Theorem 2.2 are given below.

(1) If we take $a = 1$, "$b = +\infty$", $g_1, \ldots, g_n \colon (1, +\infty) \to (0, +\infty)$ defined by $g_i(x) = \log(x)$ for any $i = 1, \ldots, n$ and α such that $\alpha > \max_{i=1,\ldots,n}(p_i - 1)$, then we have

$$\int_{(1,+\infty)^n} \frac{\prod_{i=1}^n f_i(t_i)}{[\beta + \sum_{i=1}^n \log(t_i)]^\alpha} \gamma\left(\alpha, \theta\left[\beta + \sum_{i=1}^n \log(t_i)\right]\right) \prod_{i=1}^n dt_i \\ \leq \sum_{i=1}^n \frac{1}{p_i} \int_{(1,+\infty)} [\beta + \log(t)]^{p_i - \alpha - 1} t^{p_i - 1} f_i^{p_i}(t) \\ \times \gamma\left(\alpha - p_i + 1, \theta[\beta + \log(t)]\right) dt,$$

provided that all the integrals involved converge.

In addition, for α such that $\alpha > \max[0, n(1 - 2/\max_{i=1,\ldots,n} p_i) + 1]$, we have

$$\int_{(1,+\infty)^n} \frac{\prod_{i=1}^n f_i(t_i)}{[\beta + \sum_{i=1}^n \log(t_i)]^\alpha} \gamma\Big(\alpha, \theta\Big[\beta + \sum_{i=1}^n \log(t_i)\Big]\Big) \prod_{i=1}^n dt_i \\ \leq \prod_{i=1}^n \Big[\int_{(1,+\infty)} \Big[\frac{\beta}{n} p_i + \log(t)\Big]^{[1-(\alpha-1)/n]p_i - 2} t^{p_i - 1} f_i^{p_i}(t) \\ \times \gamma\Big(\Big(\frac{\alpha - 1}{n} - 1\Big)p_i + 2, \theta\Big[\frac{\beta}{n} p_i + \log(t)\Big]\Big) dt\Big]^{1/p_i},$$

provided that all the integrals involved converge.

(2) If we take "$a = -\infty$", "$b = +\infty$", $g_1, \ldots, g_n \colon \mathbb{R} \to (0, +\infty)$ defined by $g_i(x) = e^x$ for any $i = 1, \ldots, n$ and α such that $\alpha > \max_{i=1,\ldots,n}(p_i - 1)$, then we have

$$\int_{\mathbb{R}^n} \frac{\prod_{i=1}^n f_i(t_i)}{(\beta + \sum_{i=1}^n e^{t_i})^\alpha} \gamma\Big(\alpha, \theta\Big(\beta + \sum_{i=1}^n e^{t_i}\Big)\Big) \prod_{i=1}^n dt_i \\ \leq \sum_{i=1}^n \frac{1}{p_i} \int_{\mathbb{R}} (\beta + e^t)^{p_i - \alpha - 1} e^{t(1-p_i)} f_i^{p_i}(t) \gamma(\alpha - p_i + 1, \theta(\beta + e^t)) dt,$$

provided that all the integrals involved converge.

In addition, for α such that $\alpha > \max[0, n(1 - 2/\max_{i=1,\ldots,n} p_i) + 1]$, we have

$$\int_{\mathbb{R}^n} \frac{\prod_{i=1}^n f_i(t_i)}{(\beta + \sum_{i=1}^n e^{t_i}))^\alpha} \gamma\Big(\alpha, \theta\Big(\beta + \sum_{i=1}^n e^{t_i}\Big)\Big) \prod_{i=1}^n dt_i \\ \leq \prod_{i=1}^n \Big[\int_{\mathbb{R}} \Big(\frac{\beta}{n} p_i + e^t\Big)^{[1-(\alpha-1)/n]p_i - 2} e^{t(1-p_i)} f_i^{p_i}(t) \\ \times \gamma\Big(\Big(\frac{\alpha - 1}{n} - 1\Big)p_i + 2, \theta\Big(\frac{\beta}{n} p_i + e^t\Big)\Big) dt\Big]^{1/p_i},$$

provided that all the integrals involved converge.

(3) If we take $a = 0$, $b = 1$, $g_1, \ldots, g_n \colon (0,1) \to (0, +\infty)$ defined by $g_i(x) = -1/\log(x)$ for any $i = 1, \ldots, n$ and α such that $\alpha > \max_{i=1,\ldots,n}(p_i - 1)$, then we have

$$\int_{(0,1)^n} \frac{\prod_{i=1}^n f_i(t_i)}{[\beta - \sum_{i=1}^n 1/\log(t_i)]^\alpha} \gamma\Big(\alpha, \theta\Big[\beta - \sum_{i=1}^n \frac{1}{\log(t_i)}\Big]\Big) \prod_{i=1}^n dt_i \\ \leq \sum_{i=1}^n \frac{1}{p_i} \int_{(0,1)} \Big[\beta - \frac{1}{\log(t)}\Big]^{p_i - \alpha - 1} t^{p_i - 1} \log^{2(p_i - 1)}(t) f_i^{p_i}(t) \\ \times \gamma\Big(\alpha - p_i + 1, \theta\Big[\beta - \frac{1}{\log(t)}\Big]\Big) dt,$$

provided that all the integrals involved converge.

In addition, for α such that $\alpha > \max\left[0, n(1 - 2/\max_{i=1,\ldots,n} p_i) + 1\right]$, we have

$$\int_{(0,1)^n} \frac{\prod_{i=1}^n f_i(t_i)}{[\beta - \sum_{i=1}^n 1/\log(t_i)]^\alpha} \gamma\left(\alpha, \theta\left[\beta - \sum_{i=1}^n \frac{1}{\log(t_i)}\right]\right) \prod_{i=1}^n dt_i$$
$$\leq \prod_{i=1}^n \left[\int_{(0,1)} \left[\frac{\beta}{n} p_i - \frac{1}{\log(t)}\right]^{[1-(\alpha-1)/n]p_i - 2} t^{p_i - 1} \log^{2(p_i-1)}(t) f_i^{p_i}(t)\right.$$
$$\left. \times\, \gamma\left(\left(\frac{\alpha - 1}{n} - 1\right)p_i + 2, \theta\left[\frac{\beta}{n} p_i - \frac{1}{\log(t)}\right]\right) dt\right]^{1/p_i},$$

provided that all the integrals involved converge.

More complex examples can be presented similarly, varying the natures of the functions $g_1, \ldots, g_n$.

The rest of the article is devoted to the proof of the two theorems.

3. Proofs of Theorems 2.1 and 2.2

Before developing the proofs, two technical lemmas must be presented. These are the subject of the next subsection.

3.1. Two key lemmas

The lemma below can be seen as a generalization of the standard Young inequality.

Lemma 3.1 ([12, Lemma 2])
Let $n \in \mathbb{N}\backslash\{0\}$, $a_1, \ldots, a_n$ be such that $\min_{i=1,\ldots,n} a_i \geq 0$ and $p_1, \ldots, p_n$ be such that $\min_{i=1,\ldots,n} p_i > 1$ and $\sum_{i=1}^n (1/p_i) = 1$. Then the following applies:

$$\prod_{i=1}^n a_i \leq \sum_{i=1}^n \frac{1}{p_i} a_i^{p_i}.$$

The proof proposed in [12] is by induction.

The lemma below is a possible statement of the generalized Hölder integral inequality.

Lemma 3.2
Let $n \in \mathbb{N}\backslash\{0\}$, $p_1, \ldots, p_n$ be such that $\min_{i=1,\ldots,n} p_i > 1$ and $\sum_{i=1}^n (1/p_i) = 1$, $\theta > 0$ or "$\theta = +\infty$", and $h_1, \ldots, h_n \colon (0, \theta) \to (0, +\infty)$ be n functions. Then the following applies:

$$\int_{(0,\theta)} \left[\prod_{i=1}^n h_i(s)\right] ds \leq \prod_{i=1}^n \left[\int_{(0,\theta)} h_i^{p_i}(s) ds\right]^{1/p_i},$$

provided that all the integrals involved converge.

The historical facts and technical details of the generalized Hölder integral inequality can be found in [3, 16].

3.2. Proof of Theorem 2.1

The proofs of the two points have the same mathematical basis. They use ingredients similar to those in the proofs of [12, Theorem 3] and [13, Theorem 2], but with more theoretical developments to consider a bounded integration interval depending on θ, i.e. $(0, \theta)$, and the parameter β.

First, let us introduce the notion of Laplace transform. For a given function $h\colon (0, +\infty) \to (0, +\infty)$, we define the Laplace transform of h by

$$\mathcal{L}(h)(s) = \int_{(0,+\infty)} h(t)e^{-st}dt$$

with $s > 0$. See [18].

Based on this, the key term of the proof of Theorem 2.1 is defined by

$$\Upsilon = \int_{(0,\theta)} s^{\alpha-1}e^{-\beta s}\left[\prod_{i=1}^{n} \mathcal{L}(f_i)(s)\right]ds. \tag{3.1}$$

We will show that Υ is in fact equal to the common left term in equations (2.1) and (2.2), and we will majorize it in two different ways, distinguishing the points (a) and (b), respectively.

Using standard integral developments, we can express Υ as

$$\begin{aligned}
\Upsilon &= \int_{(0,\theta)} s^{\alpha-1}e^{-\beta s}\left[\prod_{i=1}^{n} \int_{(0,+\infty)} f_i(t)e^{-st}dt\right]ds \\
&= \int_{(0,\theta)} s^{\alpha-1}e^{-\beta s}\left[\prod_{i=1}^{n} \int_{(0,+\infty)} f_i(t_i)e^{-st_i}dt_i\right]ds \qquad (3.2)\\
&= \int_{(0,\theta)} \int_{(0,+\infty)^n} s^{\alpha-1}e^{-s(\beta+\sum_{i=1}^{n} t_i)}\left[\prod_{i=1}^{n} f_i(t_i)\right]\left(\prod_{i=1}^{n} dt_i\right)ds.
\end{aligned}$$

Changing the order of integration by the Fubini-Tonell integral theorem (the integrand is non-negative), applying the change of variables $u = s\left(\beta + \sum_{i=1}^{n} t_i\right)$ and identifying the lower incomplete gamma function, we find that

$$\begin{aligned}
&\int_{(0,\theta)} \int_{(0,+\infty)^n} s^{\alpha-1}e^{-s(\beta+\sum_{i=1}^{n} t_i)}\left[\prod_{i=1}^{n} f_i(t_i)\right]\left(\prod_{i=1}^{n} dt_i\right)ds \\
&\quad= \int_{(0,+\infty)^n} \left[\prod_{i=1}^{n} f_i(t_i)\right]\left[\int_{(0,\theta)} s^{\alpha-1}e^{-s(\beta+\sum_{i=1}^{n} t_i)}ds\right]\prod_{i=1}^{n} dt_i \\
&\quad= \int_{(0,+\infty)^n} \left[\prod_{i=1}^{n} f_i(t_i)\right]\left[\int_{(0,\theta(\beta+\sum_{i=1}^{n} t_i))} \left(\frac{u}{\beta + \sum_{i=1}^{n} t_i}\right)^{\alpha-1}\right. \\
&\qquad \left.\times\, e^{-u}\frac{1}{\beta + \sum_{i=1}^{n} t_i}du\right]\prod_{i=1}^{n} dt_i \qquad (3.3)\\
&\quad= \int_{(0,+\infty)^n} \frac{\prod_{i=1}^{n} f_i(t_i)}{(\beta + \sum_{i=1}^{n} t_i)^{\alpha}}\left[\int_{(0,\theta(\beta+\sum_{i=1}^{n} t_i))} u^{\alpha-1}e^{-u}du\right]\prod_{i=1}^{n} dt_i
\end{aligned}$$

$$= \int_{(0,+\infty)^n} \frac{\prod_{i=1}^n f_i(t_i)}{(\beta + \sum_{i=1}^n t_i)^\alpha} \gamma\left(\alpha, \theta\left(\beta + \sum_{i=1}^n t_i\right)\right) \prod_{i=1}^n dt_i.$$

Putting (3.2) and (3.3) together, we get

$$\Upsilon = \int_{(0,+\infty)^n} \frac{\prod_{i=1}^n f_i(t_i)}{(\beta + \sum_{i=1}^n t_i)^\alpha} \gamma\left(\alpha, \theta\left(\beta + \sum_{i=1}^n t_i\right)\right) \prod_{i=1}^n dt_i. \tag{3.4}$$

This gives us the common left term in (2.1) and (2.2). Let us now distinguish between (a) and (b) by using different mathematical techniques to majorize Υ.

(a) The key is the expression of Υ given in (3.1). Using the inequality in Lemma 3.1 with, for any $i = 1, \ldots, n$, $a_i = \mathcal{L}(f_i)(s)$ and $s \in (0, \theta)$, we obtain

$$\prod_{i=1}^n \mathcal{L}(f_i)(s) \leq \sum_{i=1}^n \frac{1}{p_i} \mathcal{L}^{p_i}(f_i)(s),$$

so that

$$\begin{aligned} \Upsilon &= \int_{(0,\theta)} s^{\alpha-1} e^{-\beta s} \left[\prod_{i=1}^n \mathcal{L}(f_i)(s)\right] ds \\ &\leq \int_{(0,\theta)} s^{\alpha-1} e^{-\beta s} \left[\sum_{i=1}^n \frac{1}{p_i} \mathcal{L}^{p_i}(f_i)(s)\right] ds \\ &= \sum_{i=1}^n \frac{1}{p_i} \int_{(0,\theta)} s^{\alpha-1} e^{-\beta s} \mathcal{L}^{p_i}(f_i)(s) ds. \end{aligned} \tag{3.5}$$

For any $i = 1, \ldots, n$, the classical Hölder integral inequality applied with the parameter p_i gives

$$\begin{aligned} \mathcal{L}^{p_i}(f_i)(s) &= \left[\int_{(0,+\infty)} f_i(t) e^{-st} dt\right]^{p_i} \\ &= \left[\int_{(0,+\infty)} f_i(t) e^{-st/p_i} e^{-st(1-1/p_i)} dt\right]^{p_i} \\ &\leq \left[\int_{(0,+\infty)} f_i^{p_i}(t) e^{-st} dt\right] \left[\int_{(0,+\infty)} e^{-st} dt\right]^{p_i - 1} \\ &= s^{1-p_i} \int_{(0,+\infty)} f_i^{p_i}(t) e^{-st} dt, \end{aligned} \tag{3.6}$$

so that

$$\begin{aligned} &\int_{(0,\theta)} s^{\alpha-1} e^{-\beta s} \mathcal{L}^{p_i}(f_i)(s) ds \\ &\qquad \leq \int_{(0,\theta)} s^{\alpha-1} e^{-\beta s} s^{1-p_i} \left[\int_{(0,+\infty)} f_i^{p_i}(t) e^{-st} dt\right] ds \end{aligned} \tag{3.7}$$

$$= \int_{(0,\theta)} \int_{(0,+\infty)} s^{\alpha-p_i} f_i^{p_i}(t) e^{-s(\beta+t)} dt ds.$$

Changing the order of integration by the Fubini-Tonell integral theorem (the integrand is non-negative), applying the change of variables $v = s(\beta+t)$ and identifying the lower incomplete gamma function, we establish that

$$\begin{aligned}
&\int_{(0,\theta)} \int_{(0,+\infty)} s^{\alpha-p_i} f_i^{p_i}(t) e^{-s(\beta+t)} dt ds \\
&\quad = \int_{(0,+\infty)} \int_{(0,\theta)} s^{\alpha-p_i} f_i^{p_i}(t) e^{-s(\beta+t)} ds dt \\
&\quad = \int_{(0,+\infty)} f_i^{p_i}(t) \left[\int_{(0,\theta)} s^{\alpha-p_i} e^{-s(\beta+t)} ds \right] dt \\
&\quad = \int_{(0,+\infty)} f_i^{p_i}(t) \left[\int_{(0,\theta(\beta+t))} \left(\frac{v}{\beta+t} \right)^{\alpha-p_i} e^{-v} \frac{1}{\beta+t} dv \right] dt \\
&\quad = \int_{(0,+\infty)} (\beta+t)^{p_i-\alpha-1} f_i^{p_i}(t) \left[\int_{(0,\theta(\beta+t))} v^{\alpha-p_i} e^{-v} dv \right] dt \\
&\quad = \int_{(0,+\infty)} (\beta+t)^{p_i-\alpha-1} f_i^{p_i}(t) \gamma(\alpha - p_i + 1, \theta(\beta+t)) dt.
\end{aligned} \tag{3.8}$$

Combining (3.5), (3.7), and (3.8), we get

$$\Upsilon \le \sum_{i=1}^{n} \frac{1}{p_i} \int_{(0,+\infty)} (\beta+t)^{p_i-\alpha-1} f_i^{p_i}(t) \gamma(\alpha - p_i + 1, \theta(\beta+t)) dt. \tag{3.9}$$

It follows from (3.4) and (3.9) that

$$\begin{aligned}
&\int_{(0,+\infty)^n} \frac{\prod_{i=1}^{n} f_i(t_i)}{(\beta + \sum_{i=1}^{n} t_i)^\alpha} \gamma\left(\alpha, \theta\left(\beta + \sum_{i=1}^{n} t_i\right)\right) \prod_{i=1}^{n} dt_i \\
&\quad \le \sum_{i=1}^{n} \frac{1}{p_i} \int_{(0,+\infty)} (\beta+t)^{p_i-\alpha-1} f_i^{p_i}(t) \gamma(\alpha - p_i + 1, \theta(\beta+t)) dt,
\end{aligned}$$

which is the desired inequality, i.e. the one in (2.1). The proof of (a) is achieved.

(b) A different strategy is used for (b). Based on the expression of Υ in (3.1), a basic product manipulation gives

$$\begin{aligned}
\Upsilon &= \int_{(0,\theta)} s^{\alpha-1} e^{-\beta s} \left[\prod_{i=1}^{n} \mathcal{L}(f_i)(s) \right] ds \\
&= \int_{(0,\theta)} \left[\prod_{i=1}^{n} s^{(\alpha-1)/n} e^{-\beta s/n} \mathcal{L}(f_i)(s) \right] ds.
\end{aligned} \tag{3.10}$$

Using the inequality in Lemma 3.2 with, for any $i = 1, \ldots, n$, $h_i(s) = s^{(\alpha-1)/n} e^{-\beta s/n} \mathcal{L}(f_i)(s)$ and $s \in (0, \theta)$, we obtain

$$\begin{aligned}\int_{(0,\theta)} \Big[\prod_{i=1}^{n} s^{(\alpha-1)/n} e^{-\beta s/n} \mathcal{L}(f_i)(s) \Big] ds \\ \leq \prod_{i=1}^{n} \Big[\int_{(0,\theta)} s^{(\alpha-1)p_i/n} e^{-\beta s p_i/n} \mathcal{L}^{p_i}(f_i)(s) ds \Big]^{1/p_i}.\end{aligned} \tag{3.11}$$

If we re-use the exact developments in (3.6), for any $i = 1, \ldots, n$ we have

$$\mathcal{L}^{p_i}(f_i)(s) \leq s^{1-p_i} \int_{(0,+\infty)} f_i^{p_i}(t) e^{-st} dt,$$

so that

$$\begin{aligned}&\int_{(0,\theta)} s^{(\alpha-1)p_i/n} e^{-\beta s p_i/n} \mathcal{L}^{p_i}(f_i)(s) ds \\ &\quad \leq \int_{(0,\theta)} s^{(\alpha-1)p_i/n} e^{-\beta s p_i/n} s^{1-p_i} \Big[\int_{(0,+\infty)} f_i^{p_i}(t) e^{-st} dt \Big] ds \\ &\quad = \int_{(0,\theta)} \int_{(0,+\infty)} s^{[(\alpha-1)/n-1]p_i+1} f_i^{p_i}(t) e^{-s(\beta p_i/n+t)} dt ds.\end{aligned} \tag{3.12}$$

Changing the order of integration by the Fubini-Tonell integral theorem (the integrand is non-negative), applying the change of variables $w = s(\beta p_i/n + t)$ and identifying the lower incomplete gamma function, we find that

$$\begin{aligned}&\int_{(0,\theta)} \int_{(0,+\infty)} s^{[(\alpha-1)/n-1]p_i+1} f_i^{p_i}(t) e^{-s(\beta p_i/n+t)} dt ds \\ &\quad = \int_{(0,+\infty)} f_i^{p_i}(t) \Big[\int_{(0,\theta)} s^{[(\alpha-1)/n-1]p_i+1} e^{-s(\beta p_i/n+t)} ds \Big] dt \\ &\quad = \int_{(0,+\infty)} f_i^{p_i}(t) \Big[\int_{(0,\theta(\beta p_i/n+t))} \Big(\frac{w}{\beta p_i/n + t} \Big)^{[(\alpha-1)/n-1]p_i+1} \\ &\qquad \times e^{-w} \frac{1}{\beta p_i/n + t} dw \Big] dt \\ &\quad = \int_{(0,+\infty)} \Big(\frac{\beta}{n} p_i + t \Big)^{[1-(\alpha-1)/n]p_i-2} f_i^{p_i}(t) \\ &\qquad \times \Big[\int_{(0,\theta(\beta p_i/n+t))} w^{[(\alpha-1)/n-1]p_i+1} e^{-w} dw \Big] dt \\ &\quad = \int_{(0,+\infty)} \Big(\frac{\beta}{n} p_i + t \Big)^{[1-(\alpha-1)/n]p_i-2} f_i^{p_i}(t) \\ &\qquad \times \gamma\Big(\Big(\frac{\alpha-1}{n} - 1 \Big) p_i + 2, \theta\Big(\frac{\beta}{n} p_i + t \Big) \Big) dt.\end{aligned} \tag{3.13}$$

Combining (3.10), (3.11), (3.12), and (3.13), we get

$$\begin{aligned}\Upsilon \leq \prod_{i=1}^{n} \Bigg[\int_{(0,+\infty)} \Big(\frac{\beta}{n} p_i + t\Big)^{[1-(\alpha-1)/n]p_i - 2} f_i^{p_i}(t) \\ \times \gamma\Big(\Big(\frac{\alpha-1}{n} - 1\Big)p_i + 2, \theta\Big(\frac{\beta}{n}p_i + t\Big)\Big) dt \Bigg]^{1/p_i}.\end{aligned} \tag{3.14}$$

It follows from (3.4) and (3.14) that

$$\begin{aligned}&\int_{(0,+\infty)^n} \frac{\prod_{i=1}^{n} f_i(t_i)}{(\beta + \sum_{i=1}^{n} t_i)^{\alpha}} \gamma\Big(\alpha, \theta\Big(\beta + \sum_{i=1}^{n} t_i\Big)\Big) \prod_{i=1}^{n} dt_i \\ &\quad\leq \prod_{i=1}^{n} \Bigg[\int_{(0,+\infty)} \Big(\frac{\beta}{n} p_i + t\Big)^{[1-(\alpha-1)/n]p_i - 2} f_i^{p_i}(t) \\ &\quad\quad \times \gamma\Big(\Big(\frac{\alpha-1}{n} - 1\Big)p_i + 2, \theta\Big(\frac{\beta}{n}p_i + t\Big)\Big) dt \Bigg]^{1/p_i},\end{aligned}$$

which is the desired inequality, i.e. the one in (2.2). The proof of (b) is concluded.

The two points having been established, the proof of Theorem 2.1 ends.

3.3. Proof of Theorem 2.2

The proofs of the points (a) and (b) have the same mathematical basis. Making the changes of variables $u_i = g_i(t_i)$ for any $i = 1, \ldots, n$, so that $t_i = g_i^{-1}(u_i)$ and $dt_i = 1/g_i'(g_i^{-1}(u_i))$ for any $i = 1, \ldots, n$, we get

$$\begin{aligned}&\int_{(a,b)^n} \frac{\prod_{i=1}^{n} f_i(t_i)}{[\beta + \sum_{i=1}^{n} g_i(t_i)]^{\alpha}} \gamma\Big(\alpha, \theta\Big[\beta + \sum_{i=1}^{n} g_i(t_i)\Big]\Big) \prod_{i=1}^{n} dt_i \\ &= \int_{(0,+\infty)^n} \frac{\prod_{i=1}^{n} f_i(g_i^{-1}(u_i))}{(\beta + \sum_{i=1}^{n} u_i)^{\alpha}} \gamma\Big(\alpha, \theta\Big(\beta + \sum_{i=1}^{n} u_i\Big)\Big) \prod_{i=1}^{n} \frac{1}{g_i'(g_i^{-1}(u_i))} du_i \\ &= \int_{(0,+\infty)^n} \frac{\prod_{i=1}^{n} k_i(u_i)}{(\beta + \sum_{i=1}^{n} u_i)^{\alpha}} \gamma\Big(\alpha, \theta\Big(\beta + \sum_{i=1}^{n} u_i\Big)\Big) \prod_{i=1}^{n} du_i,\end{aligned} \tag{3.15}$$

where, for any $i = 1, \ldots, n$,

$$k_i(t) = \frac{f_i(g_i^{-1}(t))}{g_i'(g_i^{-1}(t))}.$$

Let us now distinguish between (a) and (b).

(a) It follows from the point (a) of Theorem 2.1 applied to $k_1, \ldots k_n$ instead of $f_1, \ldots, f_n$, respectively, that

$$\begin{aligned}
&\int_{(0,+\infty)^n} \frac{\prod_{i=1}^n k_i(u_i)}{(\beta+\sum_{i=1}^n u_i)^\alpha}\gamma\left(\alpha,\theta\left(\beta+\sum_{i=1}^n u_i\right)\right)\prod_{i=1}^n du_i \\
&\quad\le \sum_{i=1}^n \frac{1}{p_i}\int_{(0,+\infty)} (\beta+u)^{p_i-\alpha-1} k_i^{p_i}(u) \\
&\qquad \times \gamma(\alpha-p_i+1,\theta(\beta+u))du \qquad (3.16)\\
&\quad= \sum_{i=1}^n \frac{1}{p_i}\int_{(0,+\infty)} (\beta+u)^{p_i-\alpha-1} \frac{f_i^{p_i}(g_i^{-1}(u))}{[g_i'(g_i^{-1}(u))]^{p_i}} \\
&\qquad \times \gamma(\alpha-p_i+1,\theta(\beta+u))du.
\end{aligned}$$

Making the change of variables $u = g_i(t)$ with $i = 1,\dots,n$, we obtain

$$\begin{aligned}
&\sum_{i=1}^n \frac{1}{p_i}\int_{(0,+\infty)} (\beta+u)^{p_i-\alpha-1} \frac{f_i^{p_i}(g_i^{-1}(u))}{[g_i'(g_i^{-1}(u))]^{p_i}} \\
&\qquad \times \gamma(\alpha-p_i+1,\theta(\beta+u))du \\
&\quad= \sum_{i=1}^n \frac{1}{p_i}\int_{(a,b)} [\beta+g_i(t)]^{p_i-\alpha-1} \frac{f_i^{p_i}(t)}{[g_i'(t)]^{p_i}} \qquad (3.17)\\
&\qquad \times \gamma(\alpha-p_i+1,\theta[\beta+g_i(t)])g_i'(t)dt \\
&\quad= \sum_{i=1}^n \frac{1}{p_i}\int_{(a,b)} [\beta+g_i(t)]^{p_i-\alpha-1} \frac{f_i^{p_i}(t)}{[g_i'(t)]^{p_i-1}} \\
&\qquad \times \gamma(\alpha-p_i+1,\theta[\beta+g_i(t)])dt.
\end{aligned}$$

Combining (3.15), (3.16) and (3.17), we get

$$\begin{aligned}
&\int_{(a,b)^n} \frac{\prod_{i=1}^n f_i(t_i)}{[\beta+\sum_{i=1}^n g_i(t_i)]^\alpha}\gamma\left(\alpha,\theta\left[\beta+\sum_{i=1}^n g_i(t_i)\right]\right)\prod_{i=1}^n dt_i \\
&\quad\le \sum_{i=1}^n \frac{1}{p_i}\int_{(a,b)} [\beta+g_i(t)]^{p_i-\alpha-1} \frac{f_i^{p_i}(t)}{[g_i'(t)]^{p_i-1}}\gamma(\alpha-p_i+1,\theta[\beta+g_i(t)])dt,
\end{aligned}$$

which is the desired inequality, i.e. the one in (2.3). The proof of (a) is completed.

(b) It follows from (b) of Theorem 2.1 applied to $k_1,\dots k_n$ instead of $f_1,\dots,f_n$, respectively, that

$$\begin{aligned}
&\int_{(0,+\infty)^n} \frac{\prod_{i=1}^n k_i(u_i)}{(\beta+\sum_{i=1}^n u_i)^\alpha}\gamma\left(\alpha,\theta\left(\beta+\sum_{i=1}^n u_i\right)\right)\prod_{i=1}^n du_i \\
&\quad\le \prod_{i=1}^n \left[\int_{(0,+\infty)} \left(\frac{\beta}{n}p_i+u\right)^{[1-(\alpha-1)/n]p_i-2} k_i^{p_i}(u)\right.
\end{aligned}$$

$$\times \gamma\Big(\Big(\frac{\alpha-1}{n}-1\Big)p_i+2,\theta\Big(\frac{\beta}{n}p_i+u\Big)\Big)du\Bigg]^{1/p_i} \tag{3.18}$$
$$= \prod_{i=1}^{n}\Bigg[\int_{(0,+\infty)}\Big(\frac{\beta}{n}p_i+u\Big)^{[1-(\alpha-1)/n]p_i-2}\frac{f_i^{p_i}(g_i^{-1}(u))}{[g_i'(g_i^{-1}(u))]^{p_i}}$$
$$\times \gamma\Big(\Big(\frac{\alpha-1}{n}-1\Big)p_i+2,\theta\Big(\frac{\beta}{n}p_i+u\Big)\Big)du\Bigg]^{1/p_i}.$$

Making the change of variables $u = g_i(t)$ with $i = 1, \ldots, n$, we obtain

$$\prod_{i=1}^{n}\Bigg[\int_{(0,+\infty)}\Big(\frac{\beta}{n}p_i+u\Big)^{[1-(\alpha-1)/n]p_i-2}\frac{f_i^{p_i}(g_i^{-1}(u))}{[g_i'(g_i^{-1}(u))]^{p_i}}$$
$$\times \gamma\Big(\Big(\frac{\alpha-1}{n}-1\Big)p_i+2,\theta\Big(\frac{\beta}{n}p_i+u\Big)\Big)du\Bigg]^{1/p_i}$$
$$= \prod_{i=1}^{n}\Bigg[\int_{(a,b)}\Big[\frac{\beta}{n}p_i+g_i(t)\Big]^{[1-(\alpha-1)/n]p_i-2}\frac{f_i^{p_i}(t)}{[g_i'(t)]^{p_i}} \tag{3.19}$$
$$\times \gamma\Big(\Big(\frac{\alpha-1}{n}-1\Big)p_i+2,\theta\Big[\frac{\beta}{n}p_i+g_i(t)\Big]\Big)g_i'(t)dt\Bigg]^{1/p_i}$$
$$= \prod_{i=1}^{n}\Bigg[\int_{(a,b)}\Big[\frac{\beta}{n}p_i+g_i(t)\Big]^{[1-(\alpha-1)/n]p_i-2}\frac{f_i^{p_i}(t)}{[g_i'(t)]^{p_i-1}}$$
$$\times \gamma\Big(\Big(\frac{\alpha-1}{n}-1\Big)p_i+2,\theta\Big[\frac{\beta}{n}p_i+g_i(t)\Big]\Big)dt\Bigg]^{1/p_i}.$$

Combining (3.15), (3.18) and (3.19), we get

$$\int_{(a,b)^n}\frac{\prod_{i=1}^{n} f_i(t_i)}{[\beta+\sum_{i=1}^{n} g_i(t_i)]^{\alpha}}\gamma\Big(\alpha,\theta\Big[\beta+\sum_{i=1}^{n} g_i(t_i)\Big]\Big)\prod_{i=1}^{n} dt_i$$
$$\le \prod_{i=1}^{n}\Bigg[\int_{(a,b)}\Big[\frac{\beta}{n}p_i+g_i(t)\Big]^{[1-(\alpha-1)/n]p_i-2}\frac{f_i^{p_i}(t)}{[g_i'(t)]^{p_i-1}}$$
$$\times \gamma\Big(\Big(\frac{\alpha-1}{n}-1\Big)p_i+2,\theta\Big[\frac{\beta}{n}p_i+g_i(t)\Big]\Big)dt\Bigg]^{1/p_i},$$

which is the desired inequality, i.e. the one in (2.4). The proof of (b) ends.

The two points having been established, the proof of Theorem 2.2 is concluded.

4. Conclusion

In conclusion, this article has presented a unified framework that significantly extends the applicability of [12, Theorem 3] and [13, Theorem 2] by introducing two additional adjustable parameters. An originality of this extension is the inclusion and modulation of the incomplete lower gamma function in the integrals involved. A functional version of the main theorem is also established. providing detailed proofs and comprehensive techniques, such as the generalized Young inequality and the generalized Hölder integral inequality, this work may inspire further development of multivariate inequalities in mathematical analysis.

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References

[1] Adiyasuren, V., T. Batbold, and M. Krnić. "Hilbert-type inequalities involving differential operators, the best constants and applications." *Math. Inequal. Appl.* 18 (2015): 111-124. Cited on 6.

[2] Batbold, T., and Y. Sawano. "Sharp bounds for m-linear Hilbert-type operators on the weighted Morrey spaces." *Math. Inequal. Appl.* 20 (2017): 263-283. Cited on 6.

[3] Benaissa, B., and M.Z. Sarikaya. "On the refinements of some important inequalities with a finite set of positive numbers." *Math. Meth. Appl. Sci.* 47 (2024): 9589-9599. Cited on 12.

[4] Bényi, A., and C.T. Oh. "Best constant for certain multilinear integral operator." *J. Inequal. Appl.* 2006 (2006): Article ID 28582. Cited on 6.

[5] Chesneau, C. "Some four-parameter trigonometric generalizations of the Hilbert integral inequality." *Asia Mathematika* 8, no. 2 (2024): 45-59. Cited on 6.

[6] Hardy, G.H., J.E. Littlewood, and G. Polya. *Inequalities.* Cambridge: Cambridge University Press, 1934. Cited on 6.

[7] Hong, Y. "On multiple Hardy-Hilbert integral inequalities with some parameters." *J. Inequal. Appl.* 2006 (2006): Article ID 94960. Cited on 6.

[8] Huang, Z., and B.C. Yang. "A multidimensional Hilbert-type integral inequality." *J. Inequal. Appl.* 2015 (2015): Article ID 151. Cited on 6.

[9] Jian, S.W., and F.Z. Yang. "All-sided generalization about Hardy-Hilbert integral inequalities." *Acta Mathematica Sinica (China)* 44, no. 4 (2001): 619-626. Cited on 6.

[10] Li, Y., Y. Qian, and B. He. "On further analogs of Hilbert's inequality." *Int. J. Math. Math. Sci.* 2007 (2007): 1-6. Cited on 6.

[11] Sun, B. "A multiple Hilbert-type integral inequality with the best constant factor." *J. Inequal. Appl.* 2007 (2007): Article ID 071049. Cited on 6.

[12] Sulaiman, W.T. "A different type of Hardy-Hilbert's integral inequality." *AIP Conf. Proc.* 1046 (2008): 135-141. Cited on 6, 8, 9, 12, 13 and 19.

[13] Sulaiman, W.T. "General new forms of Hardy-Hilbert's integral inequality via new ideas." *Int. J. Contemp. Math. Sci.* 3, no. 22 (2008): 1059-1067. Cited on 6, 7, 8, 9, 13 and 19.

[14] Sulaiman, W.T. "New types of Hardy-Hilbert's integral inequality." *Gen. Math. Notes* 2, no. 2 (2011): 111-118. Cited on 6.

[15] Sulaiman, W.T. "On Hardy-Hilbert's integral inequality." *J. Inequal. Pure Appl. Math.* 5, no. 2 (2004): 1-9. Cited on 6.

[16] Tian, J.F. "Properties of generalized Hölder's inequalities." *J. Math. Inequal.* 9 (2015): 473-480. Cited on 12.

[17] Ullrich, D.C. "A simple elementary proof of Hilbert's inequality." *Amer. Math. Monthly* 120 (2013): 161-164. Cited on 6.

[18] Widder, D.V. *The Laplace Transform.* Princeton Mathematical Series. Princeton: Princeton University Press, 1941. Cited on 13.

[19] Xie, Z.T., Z. Zeng, and Y.F. Sun. "A new Hilbert-type inequality with the homogeneous kernel of degree -2." *Adv. Appl. Math. Sci.* 12, no. 7 (2013): 391-401. Cited on 6.

[20] Xin, D.M. "A Hilbert-type integral inequality with the homogeneous kernel of zero degree." *Math. Theory Appl.* 30, no. 2 (2010): 70-74. Cited on 6.

[21] Xu, J.S. "Hardy-Hilbert's inequalities with two parameters." *Adv. Math.* 36, no. 2 (2007): 63-76. Cited on 6.

[22] Yang, B.C. "On Hilbert's integral inequality." *J. Math. Anal. Appl.* 220, no. 2 (1998): 778-785. Cited on 6.

[23] Yang, B.C. *The Norm of Operator and Hilbert-Type Inequalities.* Beijing: Science Press, 2009. Cited on 6.

[24] Yang, B.C. *Hilbert-Type Integral Inequalities.* The United Arab Emirates: Bentham Science Publishers, 2009. Cited on 6.

[25] Yang, B.C. "On the norm of an integral operator and applications." *J. Math. Anal. Appl.* 321, no. 1 (2006): 182-192. Cited on 6.

[26] Yang, B.C. "On the norm of a Hilbert's type linear operator and applications." *J. Math. Anal. Appl.* 325 (2007): 529-541. Cited on 6.

[27] Yang, B.C. "Hilbert-type integral inequality with non-homogeneous kernel." *J. Shanghai Univ.* 17 (2011): 603-605. Cited on 6.

[28] Yang, B.C., and M. Krnić. "On the norm of a multi-dimensional Hilbert-type operator." *Sarajevo J. Math.* 7(20) (2011): 223-243. Cited on 6.

[29] Yang, B.C. "A multiple Hardy-Hilbert integral inequality." *Chinese Annals of Mathematics. Series A* 24, no. 6 (2003): 743-750. Cited on 6.

[30] Zhong, W.Y., and B.C. Yang. "On a multiple Hilbert-type integral inequality with the symmetric kernel." *J. Inequal. Appl.* 2007 (2007): Article ID 27962. Cited on 6.

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