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Some application of Grunsky coefficients in the theory of univalent functions

Abstract. Let function f be normalized, analytic and univalent in the unit disk $\mathbb{D} = \{z : |z| < 1\}$ and $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. Using a method based on Grusky coefficients we study several problems over that class of univalent functions: upper bounds of the special case of the generalized Zalcman conjecture $|a_2a_3 - a_4|$, of the third logarithmic coefficient, and of the second Hankel determinant for the logarithmic coefficients.

1. Introduction and definitions

Let $\mathcal{A}$ be the class of functions f which are analytic in the open unit disc $\mathbb{D} = \{z : |z| < 1\}$ of the form

$$f(z) = z + a_2z^2 + a_3z^3 + \cdots , \tag{1.1}$$

and let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of functions that are univalent in $\mathbb{D}$.

Although the famous Bieberbach conjecture $|a_n| \le n$ for $n \ge 2$, was proved by de Branges in 1985 [2], a great many other problems concerning the coefficients a_n remain open.

One of them is the generalized Zalcman conjecture

$$|a_na_m - a_{n+m-1}| \le (n-1)(m-1),$$

$n \ge 2$, $m \ge 2$, closed by Ma for the class of starlike functions and for the class of univalent functions with real coefficients and by Ravichandran and Verma in

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[12] for the classes of starlike and convex functions of given order and for the class of functions with bounded turning. In [11] the authors studied the generalized Zalcman conjecture for the class

$$\mathcal{U} = \left\{ f \in \mathcal{A} : \ \left| \left(\frac{z}{f(z)} \right)^2 f'(z) - 1 \right| < 1, \, z \in \mathbb{D} \right\}$$

and proved it for the cases $m = 2$, $n = 3$; and $m = 2$, $n = 4$. In this paper we prove the estimate $2.10064\ldots$ for the general class when $m = 2$ and $n = 3$ which is close to the conjectured value 2.

Another, still open problem, is finding sharp estimates of logarithmic coefficient, γ_n, of a univalent function $f(z) = z + a_2z^2 + a_3z^3 + \cdots$, defined by

$$F_f(z) := \log \frac{f(z)}{z} = 2\sum_{n=1}^{\infty} \gamma_n z^n. \tag{1.2}$$

Relatively little exact information is known about the coefficients. The natural conjecture $|\gamma_n| \le 1/n$, inspired by the Koebe function (whose logarithmic coefficients are $1/n$) is false even in order of magnitude (see Duren [3, Section 8.1]). For the class $\mathcal{S}$ the sharp estimates of single logarithmic coefficients S are known only for γ_1 and γ_2, namely,

$$|\gamma_1| \le 1 \quad \text{and} \quad |\gamma_2| \le \frac{1}{2} + \frac{1}{e} = 0.635\ldots,$$

and are unknown for $n \ge 3$. In this paper we give the estimate $|\gamma_3| \le 0.5566178\ldots$ for the general class of univalent functions. This is an improvement of $|\gamma_3| \le 0.7688\ldots$ obtained in [10]. For the subclasses of univalent functions the situation is not a great deal better. Only the estimates of the initial logarithmic coefficients are available. For details see [1].

The upper bound of the Hankel determinant is a problem rediscovered and extensively studied in recent years. Over the class $\mathcal{A}$ of functions $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ analytic on the unit disk, this determinant is defined by

$$H_{q,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix},$$

where $q \ge 1$ and $n \ge 1$. The second order Hankel determinants is

$$H_{2,2}(f) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2a_4 - a_3^2,$$

and for the logarithmic coefficients:

$$H_{2,1}(F_f/2) = \gamma_1\gamma_3 - \gamma_2^2 = \frac{1}{4}\left(a_2a_4 - a_3^2 + \frac{1}{12}a_2^4 \right). \tag{1.3}$$

For the general class $\mathcal{S}$ of univalent functions in the class $\mathcal{A}$ there are very few results concerning the Hankel determinant. The best known for the second order case is due to Hayman ([4]), saying that $|H_2(n)| \le An^{1/2}$, where A is an absolute constant, and that this rate of growth is the best possible. There are much more results for the subclasses of $\mathcal{S}$ and some references are [5, 6, 13]. Much less is known about the bounds of the modulus of the Hankel determinant for the logarithmic coefficients. In [7, 8] the authors considered the cases of starlike, convex, strongly starlike and strongly convex functions and found the best possible results. In this paper we give estimate for the second order Hankel determinant for the general class of univalent functions.

For the study of the probles defined above we will use method based on Grunsky coefficients. In the proofs we will use mainly the notations and results given in the book of N. A. Lebedev ([9]).

Here are basic definitions and results.

Let $f \in \mathcal{S}$ and let

$$\log \frac{f(t) - f(z)}{t - z} = \sum_{p,q=0}^{\infty} \omega_{p,q} t^p z^q,$$

where $\omega_{p,q}$ are called Grunsky's coefficients with property $\omega_{p,q} = \omega_{q,p}$. For those coefficients we have the next Grunsky's inequality ([3, 9]),

$$\sum_{q=1}^{\infty} q \left| \sum_{p=1}^{\infty} \omega_{p,q} x_p \right|^2 \le \sum_{p=1}^{\infty} \frac{|x_p|^2}{p}, \tag{1.4}$$

where x_p are arbitrary complex numbers such that last series converges.

Further, it is well-known that if f given by (1.1) belongs to $\mathcal{S}$, then also

$$f_2(z) = \sqrt{f(z^2)} = z + c_3 z^3 + c_5 z^5 + \cdots \tag{1.5}$$

belongs to the class $\mathcal{S}$. Then for the function f_2 we have the appropriate Grunsky's coefficients of the form $\omega_{2p-1,2q-1}$ and the inequality (1.4) has the form

$$\sum_{q=1}^{\infty} (2q-1) \left| \sum_{p=1}^{\infty} \omega_{2p-1,2q-1} x_{2p-1} \right|^2 \le \sum_{p=1}^{\infty} \frac{|x_{2p-1}|^2}{2p-1}. \tag{1.6}$$

Here, and futher in the paper we omit the upper index (2) in $\omega^{(2)}_{2p-1,2q-1}$ if compared with Lebedev's notation.

From inequality (1.6), when $x_{2p-1} = 0$ and $p = 2, 3, \ldots$, we have

$$|\omega_{11} x_1 + \omega_{31} x_3|^2 + 3|\omega_{13} x_1 + \omega_{33} x_3|^2 + 5|\omega_{15} x_1 + \omega_{35} x_3|^2 \le |x_1|^2 + \frac{|x_3|^2}{3}. \tag{1.7}$$

As it has been shown in [9, p.57], if f is given by (1.1) then the coefficients a_2, a_3, a_4 and a_5 are expressed by Grunsky's coefficients $\omega_{2p-1,2q-1}$ of the function

f_2 given by (1.5) in the following way:

$$\begin{aligned} a_2 &= 2\omega_{11}, \\ a_3 &= 2\omega_{13} + 3\omega_{11}^2, \\ a_4 &= 2\omega_{33} + 8\omega_{11}\omega_{13} + \frac{10}{3}\omega_{11}^3, \\ 0 &= 3\omega_{15} - 3\omega_{11}\omega_{13} + \omega_{11}^3 - 3\omega_{33}. \end{aligned} \tag{1.8}$$

2. Generalized Zalcman conjecture

In this section we consider the generalized Zalcman conjecture in the case $n = 2$ and $m = 3$.

Theorem 1
If $f \in \mathcal{S}$ is given by (1.1), *then*

$$|a_2a_3 - a_4| \le 2.10064\dots.$$

Proof. Using (1.8) we have

$$|a_2a_3 - a_4| = \left|2\omega_{33} + 4\omega_{11}\omega_{13} - \frac{8}{3}\omega_{11}^3\right|,$$

and further by (3.1),

$$\begin{aligned} |a_2a_3 - a_4| &= \left|2\omega_{15} + 2\omega_{11}\omega_{13} - 2\omega_{11}^3\right| \\ &= \left|2\omega_{15} + (2\omega_{13} - \omega_{11}^2)\omega_{11} - \omega_{11}^3\right| \\ &\le 2|\omega_{15}| + |2\omega_{13} - \omega_{11}^2||\omega_{11}| + |\omega_{11}|^3. \end{aligned} \tag{2.1}$$

Since $|2\omega_{13} - \omega_{11}^2| = |a_3 - a_2^2| \le 1$ (see [13, p.5]) for the class $\mathcal{S}$ and using (3.3), from (2.1) we obtain

$$|a_2a_3 - a_4| \le x + x^3 + \frac{2}{\sqrt{5}}\sqrt{1 - x^2 - 3y^3} \equiv f_3(x, y),$$

where we put $|\omega_{11}| = x$, $|\omega_{13}| = y$ and $(x, y) \in \{(x, y) :\ 0 \le x \le 1, 0 \le y \le \frac{1}{\sqrt{3}}\sqrt{1 - x^2}\} \equiv E_3$. Since $\frac{\partial f_3}{\partial x} = \frac{-6y}{\sqrt{5}\sqrt{1-x^2-3y^2}} = 0$ if, and only if, $y = 0$, we realize that f_3 has no singular points inside E_3. On the boundary we have

- $f_3(x, 0) = x + x^3 + \frac{2}{\sqrt{5}}\sqrt{1 - x^2} \le 2.10064\dots$ for $0 \le x \le 1$, obtained for $x = 0.9740\dots$;
- $f_3(0, y) = \frac{2}{\sqrt{5}}\sqrt{1 - 3y^2} \le \frac{2}{\sqrt{5}} < 1$ for $0 \le y \le 1/\sqrt{3}$;
- $f_3(1, y) = f_3(1, 0) = 2$;
- $f_3\big(x, \frac{1}{\sqrt{3}}\sqrt{1 - x^2}\big) = x + x^3 \le 2$ for $0 \le x \le 1$.

Finally,

$$|a_2a_3 - a_4| \le 2.10064\dots.$$

Remark 1
We believe that $|a_2a_3 - a_4| \le 2$ is true for the class $\mathcal{S}$.

3. The third logarithmic coefficient

We now give upper bound of the third logarithmic coefficient over the class $\mathcal{S}$.

THEOREM 2
Let $f \in \mathcal{S}$ *and be given by* (1.1). *Then*

$$|\gamma_3| \leq 0.5566178\ldots.$$

Proof. From (1.2), after differentiation and comparison of coefficients we receive

$$\gamma_3 = \frac{1}{2}\Big(a_4 - a_2a_3 + \frac{1}{3}a_2^3\Big).$$

The fifth relation in (1.8) gives

$$\omega_{33} = \omega_{15} - \omega_{11}\omega_{13} + \frac{1}{3}\omega_{11}^3, \tag{3.1}$$

which, together with the other expressions from (1.8) implies

$$\gamma_3 = \omega_{33} + 2\omega_{11}\omega_{13} = \omega_{15} + \omega_{11}\omega_{13} + \frac{1}{3}\omega_{11}^3.$$

Therefore,

$$|\gamma_3| \leq \frac{1}{3}|\omega_{11}|^3 + |\omega_{11}||\omega_{13}| + |\omega_{15}|. \tag{3.2}$$

Now, choosing $x_1 = 1$ and $x_3 = 0$ in (1.7) we have

$$|\omega_{11}|^2 + 3|\omega_{13}|^2 + 5|\omega_{15}|^2 \leq 1,$$

and also from here

$$|\omega_{11}|^2 + 3|\omega_{13}|^2 \leq 1.$$

The last two relations imply

$$|\omega_{13}| \leq \frac{1}{\sqrt{3}}\sqrt{1 - |\omega_{11}|^2} \quad \text{and} \quad |\omega_{15}| \leq \frac{1}{\sqrt{5}}\sqrt{1 - |\omega_{11}|^2 - 3|\omega_{13}|^2}. \tag{3.3}$$

Using (3.2) and (3.3) we have

$$|\gamma_3| \leq \frac{1}{3}|\omega_{11}|^3 + |\omega_{11}||\omega_{13}| + \frac{1}{\sqrt{5}}\sqrt{1 - |\omega_{11}|^2 - 3|\omega_{13}|^2} \equiv f_1(|\omega_{11}|, |\omega_{13}|),$$

where

$$f_1(x,y) = \frac{1}{3}x^3 + xy + \frac{1}{\sqrt{5}}\sqrt{1 - x^2 - 3y^2}$$

and $0 \leq x \leq 1$, $0 \leq y \leq \frac{1}{\sqrt{3}}\sqrt{1-x^2}$ ($|a_2| = |2\omega_{11}| \leq 2$ implies $0 \leq |\omega_{11}| \leq 1$).

So, we need to find maximum of the function f_1 over the region

$$E_1 = \Big\{(x,y):\ 0 \leq x \leq 1, 0 \leq y \leq \frac{1}{\sqrt{3}}\sqrt{1-x^2}\Big\}.$$

The system

$$\begin{cases} \frac{\partial f_1}{\partial x} = x^2 + y - \frac{x}{\sqrt{5}\sqrt{1-x^2-3y^2}} = 0 \\ \frac{\partial f_1}{\partial y} = x - \frac{3y}{\sqrt{5}\sqrt{1-x^2-3y^2}} = 0 \end{cases}$$

has only one solution in the interior of E_1, that is $(x_1, y_1) = (0.81267\ldots, 0.243532\ldots)$ such that $f_1(x_1, y_1) = 0.5566178\ldots$.

Now, let consider the function f_1 on the boundary of E_1,

- $f_1(x,0) = \frac{1}{3}x^3 + \frac{1}{\sqrt{5}}\sqrt{1-x^2} \le \frac{1}{\sqrt{5}} = 0.4472\ldots$ for $0 \le x \le 1$, with maximum obtained for $x = 0$;
- $f_1(0,y) = \frac{1}{\sqrt{5}}\sqrt{1-3y^2} \le \frac{1}{\sqrt{5}} = 0.4472\ldots$ for $0 \le y \le 1/\sqrt{3}$, with maximum obtained for $y = 0$;
- $f_1(1,y) = f_1(1,0) = \frac{1}{3}$;
- $f_1\big(x, \frac{1}{\sqrt{3}}\sqrt{1-x^2}\big) = \frac{1}{3}x^3 + \frac{1}{\sqrt{3}}x\sqrt{1-x^2} \le \frac{1}{\sqrt{5}} = 0.4472\ldots$ for $0 \le x \le 1$, with maximum obtained for $x = 0.898344\ldots$.

Summarizing the above analysis brings the conclusion that

$$|\gamma_3| \le f_1(x_1, y_1) = 0.5566178\ldots.$$

4. The second Hankel determinant for logarithmic coefficients

Theorem 3
Let $f \in \mathcal{S}$ is given by (1.1). *Then*

$$|H_{2,1}(F_f/2)| = \left|\gamma_1\gamma_3 - \gamma_2^2\right| \le \frac{1}{3}.$$

Proof. For a function f from $\mathcal{S}$, using (1.3) and (1.8), we receive

$$\gamma_1\gamma_3 - \gamma_2^2 = \omega_{11}\omega_{33} + \omega_{11}^2\omega_{13} - \omega_{13}^2 - \frac{1}{4}\omega_{11}^4, \tag{4.1}$$

and from the last relation in (1.8),

$$\omega_{33} = \omega_{15} - \omega_{11}\omega_{13} + \frac{1}{3}\omega_{11}^3.$$

So,

$$\gamma_1\gamma_3 - \gamma_2^2 = \omega_{11}\omega_{15} - \omega_{13}^2 + \frac{1}{12}\omega_{11}^4,$$

and further,

$$\begin{aligned} \left|\gamma_1\gamma_3 - \gamma_2^2\right| &\le |\omega_{11}||\omega_{15}| + |\omega_{13}|^2 + \frac{1}{12}|\omega_{11}|^4 \\ &\le \frac{1}{\sqrt{5}}|\omega_{11}|\sqrt{1-|\omega_{11}|^2 - 3|\omega_{13}|^2} + |\omega_{13}|^2 + \frac{1}{12}|\omega_{11}|^4 \\ &=: f_4(|\omega_{11}|, |\omega_{13}|), \end{aligned}$$

where

$$f_4(x,y) = \frac{1}{\sqrt{5}} x\sqrt{1-x^2-3y^2} + y^2 + \frac{1}{12}x^4$$

and $(x,y) \in E_4 \equiv \left\{(x,y):\ 0 \le x \le 1, 0 \le y \le \frac{1}{\sqrt{3}}\sqrt{1-x^2}\right\}$.

Now, for the first partial derivatives of f_4 we have $\frac{\partial f_4}{\partial x} = \frac{x^3}{3} - \frac{x^2}{\sqrt{5}\sqrt{-x^2-3y^2+1}} + \frac{\sqrt{1-x^2-3y^2}}{\sqrt{5}}$ and $\frac{\partial f_4}{\partial y} = 2y - \frac{3xy}{\sqrt{5}\sqrt{1-x^2-3y^2}}$. The second one being zero implies $\sqrt{1-x^2-3y^2} = \frac{3x}{2\sqrt{5}}$, which brought in the first gives $\frac{\partial f_4}{\partial x} = \frac{x^3}{3} - \frac{11x}{30}$. It is easy to check that $\frac{x^3}{3} - \frac{11x}{30}$ is negative for $0 \le x \le 1$. Thus, f_4 has no critical points in the interior of E_4.

On the boundary of E_4 we have

- $f_4(x,0) = \frac{x^4}{12} + \frac{x\sqrt{1-x^2}}{\sqrt{5}} \le f_4(0.78167\ldots,0) = 0.2491\ldots$ for $0 \le x \le 1$;
- $f_4(0,y) = y^2 \le \frac{1}{3}(1-x^2) \le \frac{1}{3}$;
- $f_4(1,y) = f_4(1,0) = \frac{1}{12}$;
- $f_4\left(x, \frac{1}{\sqrt{3}}\sqrt{1-x^2}\right) = \frac{1}{12}(2-x^2)^2 \le \frac{1}{3}$ for $0 \le x \le 1$.

Finally, we conclude that

$$\left|\gamma_1\gamma_3 - \gamma_2^2\right| \le \frac{1}{3}.$$

Remark 2

The estimate from the previous theorem is probably not sharp since the natural way for a function f from $\mathcal{S}$ to reach equality sign in the estimate is to satisfy $x = |\omega_{11}| = 0$ and $y = |\omega_{13}| = \frac{1}{\sqrt{3}}$, leading to $a_2 = 0$ and $a_3 = \frac{2}{\sqrt{3}} = 1.1547\ldots$. This is in contradiction with the well known result that for functions $f \in \mathcal{S}$, $|a_3 - a_2^2| \le 1$, reducing in our case to $|a_3| \le 1$.

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