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Notes on Jordan type of an Artinian algebra

Abstract. These notes were made for the Preparatory School on Lefschetz Properties, held from 6th to 10th May 2024, in Kraków, ahead of the conference "Lefschetz properties in algebra, geometry, topology, and combinatorics". They are a short introduction to Jordan type of an Artinian algebra, and collect basic results known so far, with examples to illustrate them along the way.

1. Introduction

The study of Jordan types as invariants has a long history, but their relations with the weak and the strong Lefschetz properties form a new subject of study that is drawing attention in the commutative algebra community. The Jordan type of an Artinian algebra tells us whether the algebra has the weak or strong Lefschetz properties, so it is a finer invariant.

The two main sources used to write these notes were the introductory paper "Artinian algebras and Jordan type" written by Anthony Iarrobino, Chris McDaniel, and the author [14], and "Jordan type of an Artinian algebra, a survey" written with Nasrin Altafi and Anthony Iarrobino [1].

2. Preliminaries

Let us start by defining Jordan type. The definition below is the one we are going to adopt in these notes, but clearly it could be extended to any nilpotent

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endomorphism of a finite-dimensional vector space (see Definition 1.1 in [1]). Let us fix notation and conventions.

Setting 2.1
We consider a field k, of any characteristic, and either the local regular ring $\mathcal{R} = \mathsf{k}\{x_1, \dots, x_r\}$ or the polynomial ring $R = \mathsf{k}[x_1, \dots, x_r]$. Let A be an Artinian algebra, either quotient of $\mathcal{R}$ by an Artinian ideal I, or quotient of R by a homogeneous Artinian ideal I. In case A is local, $A = \mathcal{R}/I$, we denote its maximal ideal by m; if A is graded, $A = R/I = \bigoplus_{i\geq 0} A_i$, we take $\mathsf{m} = \bigoplus_{i\geq 1} A_i$. If A has a non-standard grading, we assume $A_0 = \mathsf{k}$.

The *socle* of A is the ideal $(0 : \mathsf{m})$. The *socle degree* of A is the unique integer j such that $\mathsf{m}^j \neq 0 = \mathsf{m}^{j+1}$. If A is standard graded, the socle degree is the highest degree of a non-zero element in the socle, i.e. if we write $A = \bigoplus_{i=0}^{j} A_i$, with $A_j \neq 0$, then $A_j \subseteq (0 : \mathsf{m})$. Note that some authors only use the name "socle degree" in the case of Artinian level algebras, i.e. graded Artinian algebras $A = \bigoplus_{i=0}^{j} A_i$ satisfying $A_j = (0 : \mathsf{m})$.

Definition 2.2 (Jordan type)
Let M be a finitely generated module over an Artinian algebra A as above, and let $\ell \in \mathsf{m}$. The Jordan type of ℓ in M is the partition of $\dim_{\mathsf{k}} M$, denoted by

$$P_{\ell,M} = (p_1, \dots, p_s),$$

where $p_1 \geq \cdots \geq p_s$, whose parts p_i are the block sizes in the Jordan canonical form matrix of the multiplication map

$$m_\ell : M \to M, \quad x \mapsto \ell x.$$

We may write P_ℓ instead of $P_{\ell,M}$ when M is understood.

Example 2.3
Let $A = \mathsf{k}[x, y]/(x^2, xy^2, y^5)$. Then A is a graded Artinian algebra admitting a monomial basis

$$\begin{matrix} 1 & x & xy & y^3 & y^4 \\ & y & y^2 & & \end{matrix}$$

as a vector space over k. Being monomial, this is in particular a homogeneous basis, in the sense that all its elements are homogeneous, and therefore it can be partitioned into bases for each homogeneous summand of A:

$$A_0 = \langle 1 \rangle; \qquad A_1 = \langle x, y \rangle; \qquad A_2 = \langle xy, y^2 \rangle; \qquad A_3 = \langle y^3 \rangle; \qquad A_4 = \langle y^5 \rangle.$$

Let us compute the Jordan types of a few elements of the maximal ideal of A, regarding A as a module over itself. When we consider $\ell_1 = x + y$ for instance, the multiplication map $m_{\ell_1} : A \to A$ sends 1 to $x + y$, then it sends $x + y$ to

$$(x + y)^2 = x^2 + 2xy + y^2 = 2xy + y^2.$$

and so on, giving the following string:

$$1 \longmapsto x + y \longmapsto 2xy + y^2 \longmapsto y^3 \longmapsto y^4 \longmapsto 0.$$

This string has five elements composing a linearly independent subset of A. Since A has dimension 7 as a vector space over k, we may still consider an element that is not in the span of those elements, as $x \in A_1$, for example. We see that the multiplication map m_{ℓ_1} sends x to xy, and sends xy to 0. So we get two strings:

$$\begin{array}{c} 1 \longmapsto x+y \longmapsto 2xy+y^2 \longmapsto y^3 \longmapsto y^4 \longmapsto 0 \\ x \longmapsto xy \longmapsto 0. \end{array} \tag{2.1}$$

Using these strings, we may consider the basis

$$\mathcal{B} = \{1,\, x+y,\, 2xy+y^2,\, y^3,\, y^4,\, x,\, xy\},$$

and reorder its elements in the sequence

$$(y^4,\, y^3,\, 2xy+y^2,\, x+y,\, 1,\, xy,\; x),$$

to obtain the following matrix representing m_{ℓ_1} with respect to $\mathcal{B}$:

	y^4	y^3	$2xy+y^2$	$x+y$	1	xy	x
y^4	0	1	0	0	0	0	0
y^3	0	0	1	0	0	0	0
$2xy+y^2$	0	0	0	1	0	0	0
$x+y$	0	0	0	0	1	0	0
1	0	0	0	0	0	0	0
xy	0	0	0	0	0	0	1
x	0	0	0	0	0	0	0

This matrix is in the Jordan canonical form, having two Jordan blocks, one of size 5, the other of size 2, so we get that the Jordan type of $x+y$ in A is $P_{x+y,A} = (5,2)$. We call the basis $\mathcal{B}$ a Jordan basis for m_{ℓ_1} (see Definition 2.5 below). Note that even in the case when the field k has characteristic 2, the set $\mathcal{B}$ remains a basis of A.

The way we got this basis is not the standard way of computing a Jordan basis, but in the case of Artinian algebras this approach works. For a good explanation of how to find a Jordan basis of a vector space endomorphism (not necessarily nilpotent), you may see sections 7.7 and 7.8 in [17], or the proof of Proposition 4.7.1 in [2, Section 4.7], where the convention for the Jordan canonical form is lower triangular.

Now if we look back at the strings (2.1), we see that the first string has 5 non-zero elements, and the second has 2, so we can conclude directly from the strings that the Jordan type is $(5, 2)$.

We may now consider another element of the maximal ideal of A, say $\ell_2 = x$, and we get the strings

$$\begin{array}{ccccccccccc}
1 & \longmapsto & x & \longmapsto & 0 & & & & & & \\
 & & y & \longmapsto & xy & \longmapsto & 0 & & & & \\
 & & & & y^2 & \longmapsto & 0 & & & & \\
 & & & & & & y^3 & \longmapsto & 0 & & \\
 & & & & & & & & y^4 & \longmapsto & 0.
\end{array} \tag{2.2}$$

Therefore the Jordan type of x is $P_{x,A} = (2, 2, 1, 1, 1)$.

If we consider $\ell_3 = x + y^2$, a non-homogeneous element, still using this naïve approach to find the Jordan type, we get the strings

$$\begin{array}{ccccccccc}
1 & \longmapsto & x + y^2 & \longmapsto & & \longrightarrow & y^4 & \longmapsto & 0 \\
 & & y & \longmapsto & xy + y^3 & \longmapsto & 0 & & \\
 & & & & y^2 & \longmapsto & y^4 & & \\
 & & & & & & y^3 & \longmapsto & 0.
\end{array} \tag{2.3}$$

Note that this time we did not obtain a Jordan basis, because y^2 is not sent to zero, it is sent to an element in a previous string. But we can modify that string, if we observe that $\ell_3(x + y^2) = \ell_3 y^2$, and therefore

$$\ell_3 x = \ell_3(x + y^2) - \ell_3 y^2 = 0.$$

So we may replace the third string and obtain

$$\begin{array}{ccccccccc}
1 & \longmapsto & x + y^2 & \longmapsto & & \longrightarrow & y^4 & \longmapsto & 0 \\
 & & y & \longmapsto & xy + y^3 & \longmapsto & 0 & & \\
 & & x & \longmapsto & 0 & & & & \\
 & & & & & & y^3 & \longmapsto & 0.
\end{array} \tag{2.4}$$

Therefore the Jordan type of $x + y^2$ is $P_{x+y^2,A} = (3, 2, 1, 1)$.

In this example, both Jordan bases coming from strings in (2.1) and (2.2) are homogeneous (and the strings are presented with each column corresponding to a degree). In a graded module M over a graded algebra A, if $\ell \in \mathsf{m}$ is a homogenous element, it is always possible to find a homogeneous Jordan basis for m_ℓ (see [14, Lemma 2.2] for the case of a linear element). However, if ℓ is not homogeneous, or M is not graded, any Jordan basis will be also non-homogeneous, but we have arranged the strings by order. Here is a definition.

Definition 2.4
Let A be an Artinian algebra as in Setting 2.1. The *order* of a non-zero element a in A is the unique integer i such that $a \in \mathsf{m}^i \setminus \mathsf{m}^{i+1}$.

The basis $\{1,\, x+y^2,\, y^4,\, y,\, xy+y^3,\, y^2,\, y^3\}$ coming from strings in (2.3) is what we will call a pre-Jordan basis. Here is Definition 1.2 in [1]:

Definition 2.5 (Jordan basis, pre-Jordan basis)
With the notation of Definition 2.2, a *pre-Jordan basis* for ℓ is a basis of M as a vector space over k of the form

$$\mathcal{B} = \{\ell^i z_k :\ 1 \le k \le s,\, 0 \le i \le p_k - 1\}, \tag{2.5}$$

where $P_{\ell,M} = (p_1, \ldots, p_s)$ is the Jordan type of ℓ. We call the sequences

$$S_k = (z_k, \ell z_k, \ldots, \ell^{p_k-1} z_k)$$

strings, or *ℓ-strings*, of the basis $\mathcal{B}$, and each element $\ell^i z_k$ a *bead* of the string. The Jordan blocks of the multiplication m_ℓ are determined by the strings S_k, and M is the direct sum

$$M = \langle S_1 \rangle \oplus \cdots \oplus \langle S_s \rangle. \tag{2.6}$$

If the elements $z_1, \ldots, z_s \in M$ satisfy $\ell^{p_k} z_k = 0$ for each k, we call $\mathcal{B}$ a *Jordan basis* for ℓ, recovering the usual definition in linear algebra, since a matrix representing the multiplication by ℓ with respect to $\mathcal{B}$, ordering elements as

$$(\ell^{p_1-1} z_1, \ldots, z_1, \ell^{p_2-1} z_2, \ldots, z_2, \ldots, \ell^{p_s-1} z_s, \ldots, z_s),$$

is a canonical Jordan form. In that case, each $\langle S_k \rangle$ is a cyclic $\mathsf{k}[\ell]$-submodule of M.

Remark 2.6
Note that in the previous examples, strings were displayed in a diagram

$$z_k \longmapsto \ell z_k \longmapsto \cdots \longmapsto \ell^{p_k-1} z_k,$$

and not in the sequence notation $(z_k, \ell z_k, \ldots, \ell^{p_k-1} z_k)$ given above, because it is often easier to understand the action of multiplication by an element ℓ if we use a diagram. Note also that the notation used in Definition 1.2 in [1], and also in Definition 2.1 in [14], is that of a set $\{z_k, \ell z_k, \ldots, \ell^{p_k-1} z_k\}$. We are adopting the sequence notation here to emphasise the order of the beads, but even in the set notation, there is a natural order in a string, given by the power of ℓ in each bead $\ell^i z_k$.

The maximal ideal m is a vector subspace of A, so looking at A as an affine algebraic set, if the field k is infinite, m is an irreducible algebraic subset. If A is graded, the same holds for A_1. Therefore, it makes sense to consider a general element in m or in A_1. This motivates the following definition (see Definition 1.2 in [1]).

Definition 2.7 (Generic Jordan type of an Artinian algebra)
Suppose k is infinite. The *generic Jordan type* of A, denoted by P_A, is the Jordan type $P_{\ell,A}$ for a general element ℓ of A_1 (when A is graded), or of m (when A is local).

In the next example, we will see that the Jordan type may depend on the characteristic of the field k.

Example 2.8
Let $A = \mathsf{k}[x,y]/(x^2,y^2)$. Then $\{1,x,y,xy\}$ is a monomial basis for A as a vector space over k and any element ℓ in its maximal ideal can be written as

$$\ell = ax + by + cxy,$$

with $a,b,c \in \mathsf{k}$.

Suppose $\operatorname{char} \mathsf{k} \neq 2$. It is easy to check that if $ab \neq 0$ then $P_{\ell,A} = (3,1)$. In fact, if the pair (a',b') satisfies $ab' - a'b \neq 0$, we see that the strings

$$\begin{array}{c} 1 \longmapsto \ell \longmapsto 2abxy \longmapsto 0 \\ a'x + b'y \longmapsto (ab' + a'b)xy \end{array}$$

give the pre-Jordan basis $\{1,\ \ell,\ 2abxy,\ a'x+b'y\}$. We may omit the zeros and the elements that are in the span of previous strings and get the simpler diagram

$$\begin{array}{c} 1 \longmapsto \ell \longmapsto 2abxy \\ a'x + b'y \end{array}$$

If k is an infinite field, since the set

$$\{ax + by + cxy \in \mathsf{m} :\ ab \neq 0\}$$

is open and dense in m, we get that the generic Jordan type of A is $P_A = (3,1)$. Outside this set, we can consider $b = 0$, and assume $a = 1$, making $\ell = x + cxy$, therefore we get the strings

$$\begin{array}{c} 1 \longmapsto x + cxy \\ y \longmapsto xy \end{array}$$

so $P_{x+cxy,A} = (2,2)$. By symmetry on the variables, also $P_{y+cxy,A} = (2,2)$. Finally, we can easily check that A admits two further Jordan types, namely $P_{xy,A} = (2,1,1)$ and the Jordan type of the zero map $P_{0,A} = (1,1,1,1)$.

Now suppose char $\mathsf{k} = 2$. Then $\ell^2 = 0$, so if $(a, b) \neq (0, 0)$, we can check that $P_{\ell,A} = (2, 2)$. Taking a pair (a', b') satisfying $ab' - a'b \neq 0 \neq ab' + a'b$, we get the strings

$$\begin{aligned} 1 &\longmapsto \ell \\ a'x + b'y &\longmapsto (ab' + a'b)xy. \end{aligned}$$

Again, if the field k is infinite, this is the generic Jordan type, as the set

$$\{ax + by + cxy \in \mathsf{m} : \ (a, b) \neq (0, 0)\}$$

is open and dense in m. The two other possible Jordan types for A are again $P_{xy,A} = (2, 1, 1)$ and $P_{0,A} = (1, 1, 1, 1)$.

Example 2.9
Let $A = \mathsf{k}[x, y, z]/(yz - x^3,\ z^2,\ y^3)$, a non-graded complete intersection (see the proof of Theorem 3.2 in [13] for a further discussion involving this example). A monomial basis for A is

$$\begin{array}{ccccccc} 1 & x & x^2 & x^3 & x^4 & x^5 & x^5y \\ & y & xy & x^2y & x^3y & x^4y & \\ & z & xz & x^2z & x^2y^2 & & \\ & & y^2 & xy^2 & & & \end{array}$$

Note that the element yz has order 3, because in A, $yz = x^3 \in \mathsf{m}^3$. The basis above is organised by order, and we have chosen representatives of each element that make their order apparent (that is why, instead of having yz in the second column, we have x^3 in the third).

Consider $\ell_1 = x + y + z$. Staring with 1 and then taking lower-order elements for the next strings, as in the previous examples, we get:

$$\begin{array}{l} 1 \longmapsto \ell_1 \longmapsto \ell_1^2 \longmapsto \ell_1^3 \longmapsto \ell_1^4 \longmapsto \ell_1^5 \longmapsto \ell_1^6 \\ \quad y \longmapsto \ell_1 y \longmapsto \ell_1^2 y \longmapsto \ell_1^3 y \longmapsto \ell_1^4 y \\ \quad z \longmapsto \ell_1 z \longmapsto \ell_1^2 z \\ \qquad y^2 \longmapsto \ell_1 y^2 \longmapsto \ell_1^2 y^2 \end{array} \tag{2.7}$$

where

$$\begin{aligned} \ell_1^2 &= x^2 + 2xy + 2xz + y^2 + 2x^3 \\ \ell_1^3 &= x^3 + 3x^2y + 3x^2z + 3xy^2 + 6x^4 + 3x^3y \\ \ell_1^4 &= x^4 + 4x^3y + 6x^2y^2 + 12x^5 + 12x^4y \\ \ell_1^5 &= x^5 + 5x^4y + 30x^5y \\ \ell_1^6 &= 6x^5y \end{aligned}$$

$$
\begin{aligned}
\ell_1 y &= xy + y^2 + x^3 & \ell_1 z &= xz + x^3\\
\ell_1^2 y &= x^2y + 2xy^2 + 2x^4 + 2x^3y & \ell_1^2 z &= x^2z + 2x^4 + x^3y\\
\ell_1^3 y &= x^3y + 3x^2y^2 + 3x^5 + 6x^4y & \ell_1 y^2 &= xy^2 + x^3y\\
\ell_1^4 y &= x^4y + 12x^5y & \ell_1^2 z &= x^2y^2 + 2x^4y
\end{aligned}
$$

Let us assume that $\operatorname{char} \mathsf{k} \notin \{2,3,5\}$, so that all coefficients of these elements are non-zero. In particular $\ell_1^6 \neq 0$. Do the strings in (2.7) give us a pre-Jordan basis? Can we conclude from here that the Jordan type of ℓ_1 is $(7,5,3,3)$? Let us look closer at how we built the strings.

After getting the first string, which we know cannot be longer, since $\ell_1^7 = 0$, we chose two elements $y, z \in \mathsf{m} \setminus \mathsf{m}^2$ as starting beads of the next two strings. These elements were chosen because the classes of ℓ_1, y, z in the quotient m/m^2 form a linearly independent set. Let us look at the second string, whose first bead is y. Could we have chosen another starting bead and get a longer string? If we could, this would be true for a general element in A, in case k is infinite. Let us add this assumption and suppose we choose a bead b outside the maximal ideal (we can adapt the following argument for a finite k). We can assume $b = 1 + c$, with $c \in \mathsf{m}$. Then the first two strings would be

$$
\begin{array}{l}
1 \longmapsto \ell_1 \longmapsto \ell_1^2 \longmapsto \ell_1^3 \longmapsto \ell_1^4 \longmapsto \ell_1^5 \longmapsto \ell_1^6\\
1 + c \longmapsto \ell_1 + \ell_1 c \longmapsto \ell_1^2 + \ell_1^2 c \longmapsto \ell_1^3 + \ell_1^3 c \longmapsto \ell_1^4 + \ell_1^4 c.
\end{array}
$$

Note that $\ell_1(\ell_1^4 + \ell_1^4 c) = \ell_1^5 + \ell_1^5 c$, and since $\ell_1^5 c \in \mathsf{m}^6 = \langle x^5 y\rangle = \langle \ell_1^6\rangle$, we get that $\ell_1(\ell_1^4 + \ell_1^4 c)$ is a linear combination of elements in the first string. So the second string cannot have length greater than 5. With some patience and looking carefully at the details, we can show that the third string cannot have a larger length that 3, and conclude that $P_{\ell_1,A} = (7,5,3,3)$. Using arguments similar to these, we can show that the generic Jordan type of A is $P_A = (7,5,3,3)$.

We will see in Lemma 2.13 below that there is an algorithmic way of computing the Jordan type of an element that does not require taking strings.

Let us now consider $\ell_2 = y + z$. Using the same approach as before, we get the strings

$$
\begin{array}{l}
1 \longmapsto y + z \longmapsto y^2 + 2x^3 \longmapsto 3x^3y\\
x \longmapsto xy + xz \longmapsto xy^2 + 2x^4 \longmapsto 3x^4y\\
z \longmapsto x^3\\
x^2 \longmapsto x^2y + x^2z \longmapsto x^2y^2 + 2x^5 \longmapsto 3x^5y\\
xz \longmapsto x^4\\
x^2z \longmapsto x^5
\end{array}
\tag{2.8}
$$

corresponding to the Jordan type $P_{\ell_2,A} = (4,4,4,2,2,2)$. Now if we add x^2 to ℓ_2

and consider $\ell_3 = y + z + x^2$, we obtain

$$\begin{array}{l}
1 \longmapsto \ell_3 \longmapsto \ell_3^2 \longmapsto \ell_3^3 \longmapsto \ell_3^4 \\
x \longmapsto \ell_3 x \longmapsto \ell_3^2 x \longmapsto \ell_3^3 x \\
z \longmapsto \ell_3 z \longmapsto \ell_3^2 z \\
x^2 \longmapsto \ell_3 x^2 \longmapsto \ell_3^2 x^2 \\
xz \longmapsto \ell_3 xz \\
x^2 z
\end{array} \tag{2.9}$$

where

$$\begin{aligned}
\ell_3^2 &= y^2 + 2x^3 + 2x^2y + 2x^2z + x^4 & \ell_3 x &= xy + xz + x^3 \\
\ell_3^3 &= 3x^3y + 3x^2y^2 + 6x^5 + 3x^4y & \ell_3^2 x &= xy^2 + 2x^4 + 2x^3y + x^5 \\
\ell_3^4 &= 12x^5y & \ell_3^3 x &= 3x^4y + 3x^5y \\
\ell_3 z &= x^3 + x^2z & \ell_3 x^2 &= x^2y + x^2z + x^4 \\
\ell_3^2 z &= x^3y + 2x^5 & \ell_3^2 x^2 &= x^2y^2 + 2x^5 + 2x^4y \\
\ell_3^2 xz &= x^4
\end{aligned}$$

and we get the Jordan type $P_{\ell_3,A} = (5,4,3,3,2,1)$.

The following result is well known and is an easy consequence of the existence and construction of the Jordan canonical form of a nilpotent endomorphism (see [2, Section 4.7], or [19]).

Lemma 2.10
If M has a pre-Jordan basis $\mathcal{B}$ as in (2.5)*, then for each k, we have*

$$\ell^{p_k} z_k \in \langle \ell^a z_i : a \geq p_k,\ i < k \rangle.$$

There is a Jordan basis of M derived from the pre-Jordan basis, and having the same partition invariant $P_{\ell,M}$ giving the lengths of strings.

Proof. See Remark A.3.

Definition 2.11 (Conjugate partition)
Let $P = (p_1, \dots, p_s)$, with $p_1 \geq \dots \geq p_s$, be a partition of an integer $n > 0$. The *conjugate partition* $P^\vee$ of P is the partition $P^\vee = (p_1^\vee, \dots, p_t^\vee)$ defined by

$$p_i^\vee = \#\{j :\ p_j \geq i\}.$$

This corresponds to swapping rows and columns in the Ferrers diagram.

Notation 2.12
Let $d = (d_0, d_1, \dots, d_t)$ be a sequence of integers. We denote the sequence of its first differences (or first-order differences) by $\Delta(d)$, i.e.

$$\Delta(d) = (\delta_1, \delta_2, \dots, \delta_t),$$

with $\delta_i = d_i - d_{i-1}$ for each $i \in \{1, \dots, t\}$.

The following well-known result allows us to compute the Jordan type, see Lemma 3.60 in [6].

Lemma 2.13
Let A be an Artinian graded or local algebra with maximal ideal m and socle degree j, and let $\ell \in \mathsf{m}$. Let M be a finite length A-module, and consider the increasing dimension sequence

$$d(\ell) = (d_0, d_1, \ldots, d_j, d_{j+1}), \text{ where } d_i = \dim_{\mathsf{k}} M/\ell^i M. \tag{2.10}$$

Its sequence of first differences $\Delta\bigl(d(\ell)\bigr)$ satisfies

$$P_{\ell,M} = \Delta\bigl(d(\ell)\bigr)^{\vee}. \tag{2.11}$$

Remark 2.14
Note that the increasing sequence $\hat{r}(\ell) = (r_{j+1}, r_j, r_{j-1}, \ldots, r_1, r_0)$, where $r_i = \dim_{\mathsf{k}} \ell^i M$ satifies $\Delta\bigl(\hat{r}(\ell)\bigr) = \Delta\bigl(d(\ell)\bigr) = P^{\vee}_{\ell,M}$ (see Lemma 2.3 in [14], where r_ℓ is a decreasing sequence).

3. Lefschetz properties

Lefschetz properties are defined for graded algebras. Here we will see that Jordan type gives us a way of extending these definitions to the non-graded case. We start by giving a summary of the graded case.

Definition 3.1 (Hilbert function)
Let A be a graded k-algebra, with $A_0 = \mathsf{k}$, and let M be a graded module over A. The *Hilbert function* of M is the function

$$H(M) : \mathbb{N} \to \mathbb{N} \quad \text{given by} \quad i \mapsto \dim_{\mathsf{k}} M_i.$$

We often denote $H(M)_i = \dim_{\mathsf{k}} M_i$, and write $H(M)$ as a sequence $(h_0, h_1, h_2, \ldots)$, with $h_i = H(M)_i$. If there is an integer k such that for all $i > k$, $h_i = 0$, we may write $H(M) = (h_0, h_1, \ldots, h_k)$. We say that $H(M)$ is *unimodal* if there is an integer k such that for all $i < k$,

$$H(M)_i \leq H(M)_{i+1},$$

and for all $i \geq k$,

$$H(M)_i \geq H(M)_{i+1}.$$

If A is a local algebra, with maximal ideal m, we can consider its associated graded algebra

$$A^* = \bigoplus_{i \geq 0} \frac{\mathsf{m}^i}{\mathsf{m}^{i+1}}.$$

We define the Hilbert function of A as that of A^*, i.e. $H(A) = H(A^*)$, so

$$H(A)_i = \dim_{\mathsf{k}} \frac{\mathsf{m}^i}{\mathsf{m}^{i+1}}.$$

More generally, if M is a module over a local algebra $(A, \mathsf{m}, \mathsf{k})$, we consider the *Hilbert function with respect to the* m*-adic filtration* as

$$H_{\mathsf{m}}(M) = \dim_{\mathsf{k}} \frac{\mathsf{m}^i M}{\mathsf{m}^{i+1} M}.$$

Definition 3.2 (Sperner number)
Let A be a graded Artinian k-algebra and let M be a finite-length graded module over A. The Sperner number of M is the maximum value of the Hilbert function of M:

$$\operatorname{Sperner} M = \max\{H(M)_i :\ i \geq 0\}.$$

The following definitions are standard. Here we are adopting Definitions 3.1 and 3.8 in [6] (see also, for instance, Definition 2.4 in [18]).

Definition 3.3 (Weak Lefschetz property)
Let A be a graded Artinian algebra and consider a linear form $\ell \in A_1$. We say that ℓ is a *weak Lefschetz* (WL) element if for each integer $i \geq 0$, the map

$$\times\ell : A_i \to A_{i+1}, \quad x \mapsto \ell x,$$

has maximal rank. We say that A satisfies the *weak Lefschetz property* (WLP) if it has a weak Lefschetz element.

Definition 3.4 (Strong Lefschetz property)
Let A be a graded Artinian algebra and consider a linear form $\ell \in A_1$. We say that ℓ is a *strong Lefschetz* (SL) element if for each pair of integers $i, d \geq 0$, the map

$$\times\ell^d : A_i \to A_{i+d}, \quad x \mapsto \ell^d x,$$

has maximal rank. We say that A satisfies the *strong Lefschetz property* (SLP) if it has a strong Lefschetz element.

Note 3.5
If you read or hear *Lefschetz element*, in these notes or elsewhere, you should understand it as "strong Lefschetz element".

Example 3.6
Consider the graded Artinian algebra $A = \mathsf{k}[x, y]/(x^2, xy^2, y^5)$, from Example 2.3. It is easy to check that $x + y$ is a strong Lefschetz element, and, furthermore that any linear form $\ell = ax + by$, with $b \neq 0$ is also SL.

Example 3.7
Consider the graded Artinian algebra $A = \mathsf{k}[x, y]/(x^2, y^2)$, from Example 2.8. It is easy to check that any non-zero linear form $\ell = ax + by$ is a weak Lefschetz element. Furthermore, if $ab \neq 0$ and $\operatorname{char} \mathsf{k} \neq 2$ then ℓ is strong Lefschetz, because $\ell^2 \neq 0$, so the map $\times\ell^2 : A_0 \to A_2$ is an isomorphism.

The following result is well known (see [8, Remark 3] or [6, Proposition 3.2]). We include the proof from [6] because its main idea is the core of the proof of Proposition 3.9 below.

Proposition 3.8
Let A be a standard-graded Artinian algebra and suppose that A satisfies the weak Lefschetz property. Then the Hibert function of A is unimodal.

Proof. (See proof of Proposition 3.2 in [6].) Since A satisfies the WLP, there is a WL element $\ell \in A_1$. Let $\mathsf{m} = \bigoplus_{i \geq 1} A_i$, and let

$$k = \min\{i \geq 0 : \ H(A)_i > H(A)_{i+1}\}.$$

Since ℓ is WL and $\dim_{\mathsf{k}} A_k > \dim_{\mathsf{k}} A_{k+1}$, the map $\times \ell : A_k \to A_{k+1}$ is surjective. Since A has the standard grading, for any degree i, the ideal m^i is generated by the homogeneous summand A_i. In particular, since $\ell A_k = A_{k+1}$, we get $\mathsf{m}^{k+1} = \ell \mathsf{m}^k$. By induction on i, we can check that for all $i \geq k$, $\mathsf{m}^{i+1} = \ell \mathsf{m}^i$. Therefore, for any $i \geq k$, the map $\times \ell : A_k \to A_{k+1}$ is surjective, which implies $H(A)_i \geq H(A)_{i+1}$. Hence $H(A)$ is unimodal.

The results that follow will show why Jordan type is a finer invariant than the Lefschetz properties, and therefore could be a useful tool to study them. We start with the weak Lefschetz property, and a result from Tadahito Harima and Junzo Watanabe [8].

Proposition 3.9 (See [8, Remark 3 and Proposition 14])
Let A be a standard-graded Artinian algebra. Then a linear form $\ell \in A_1$ is a weak Lefschetz element if and only if the number of parts in the Jordan type $P_{\ell,A}$ equals the Sperner number of A.

Proof. (See also the proof of Proposition 3.5 in [6].) Let $\ell \in A_1$ be any linear form. Note that the number of parts in $P_{\ell,A}$ is the number of blocks in the canonical Jordan form of $\times \ell : A \to A$, and equals the dimension of the kernel of this map, since any Jordan basis of a nilpotent endomorphism has one kernel element per Jordan block. If ℓ is WL then $H(A)$ is unimodal, by Proposition 3.8. Let

$$k = \min\{i \geq 0 : \ H(A)_i > H(A)_{i+1}\}.$$

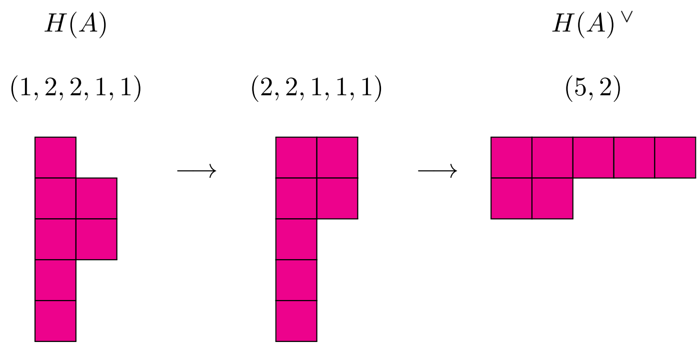
Then, since ℓ is homogeneous, the dimension of the kernel of $\times \ell : A \to A$ is the sum

$$\begin{aligned} &\dim\ker(\times \ell : A_k \to A_{k+1}) + \dim\ker(\times \ell : A_{k+1} \to A_{k+2}) + \cdots \\ &\qquad = (\dim A_k - \dim A_{k+1}) + (\dim A_{k+1} - \dim A_{k+2}) + \cdots \\ &\qquad\qquad = \dim A_k = \operatorname{Sperner} A. \end{aligned}$$

The converse is easy to check, since this last computation shows that Sperner A is a lower bound for the dimension of the kernel of $\times \ell : A \to A$. So if the number of parts in $P_{\ell,A}$ equals Sperner A then $H(A)$ must be unimodal and ℓ must be a WL element.

Definition 3.10 (Conjugate partition of a Hilbert function)
If H is the Hilbert function of an Artinian algebra, or a finite-length module over an Artinian algebra, we consider the partition whose parts are the values of H, after reordering them to become non-increasing; we call the conjugate of this partition the *conjugate partition* of H.

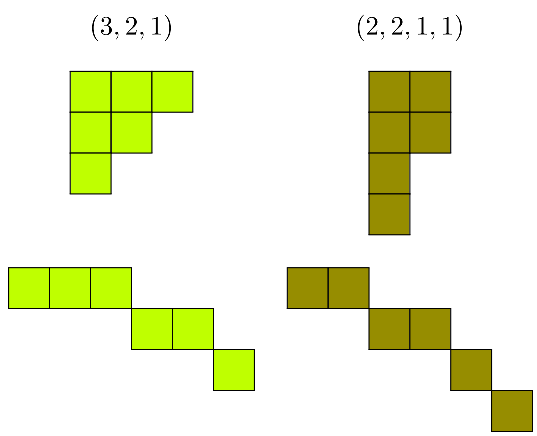
Example 3.11
Consider the Artinian algebra $A = \mathsf{k}[x,y]/(x^2, xy^2, y^5)$, from Example 2.3. The Hilbert function of A is $H(A) = (1,2,2,1,1)$, and its conjugate is $H(A)^\vee = (5,2)$.



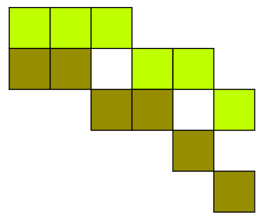
Definition 3.12 (Dominance)
Let $P = (p_1, \dots, p_s)$ and $Q = (q_1, \dots, q_r)$, where $p_1 \geq \cdots \geq p_s$, and $q_1 \geq \cdots \geq q_r$, are two partitions of a positive integer n. We say that P *dominates* Q (written $P \geq Q$), if for each $k \in \{1, \dots, \min\{s, r\}\}$, we have

$$\sum_{i=1}^{k} p_i \geq \sum_{i=1}^{k} q_i.$$

One way of viewing dominance partial order graphically is to take a partition and redraw it, putting the beginning of each new row at the point where the previous one ends. Here is the case of two partitions of 6:

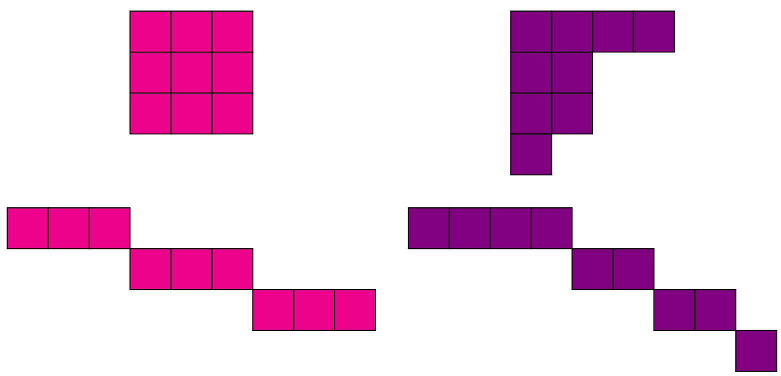


Then, if it is possible to draw one of the partitions right below the other, without a space between first rows, and without overlapping, as in the next picture, we see that the first partition dominates the second.

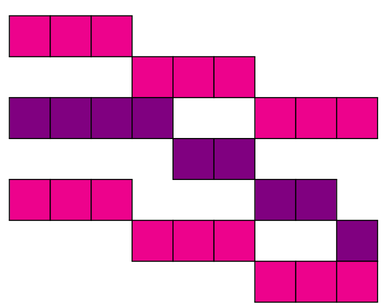


In this case, $(2,2,1,1) < (3,2,1)$. Let us look at the case of the following two partitions of 9:

$(3,3,3)$ $(4,2,2,1)$



We see that we cannot fit one below the other, without space between first rows, or without overlapping:



So $(3,3,3)$ and $(4,2,2,1)$ are incomparable.

Using the definitions of conjugate partition of the Hilbert function and dominance of partitions, Harima et al. [6] obtained the following result, giving an upper bound for the Jordan type of a homogeneous element, in the case of a standard graded Artinian algebra whose Hilbert function is unimodal, and a characterisation of strong Lefschetz elements in terms of their Jordan type.

PROPOSITION 3.13 ([6, Proposition 3.64])
Let A be a standard graded Artinian algebra, and let $\ell \in \mathsf{m}$ be a homogeneous element.

1. *If $H(A)$ is unimodal, then*
$$P_{\ell,A} \le H(A)^\vee.$$
2. *If $\ell \in A_1$ is a linear form then ℓ is a strong Lefschetz element if and only if*
$$P_{\ell,A} = H(A)^\vee.$$

Tony Iarrobino, Chris McDaniel, and the author [14] showed that the upper bound in the first statement of this result can be generalised to local algebras, or graded modules whose grading is non-standard, and that the hypothesis of the Hilbert function being unimodal can be dropped.

THEOREM 3.14 ([14, Theorem 2.5])
Let $(A, \mathsf{m}, \mathsf{k})$ be a local Artinian algebra over k and let $M \subset A^k$ be an A-module, with Hilbert function $H(M)$. For any $\ell \in \mathsf{m}$, its Jordan type satisfies
$$P_{\ell,M} \le H(M)^\vee. \tag{3.1}$$
If A has weight function w, for which $A_0 = \mathsf{k}$, and if M is a graded module over A with w-Hilbert function $H_{\mathsf{w}}(M)$, then for any w-homogeneous element $\ell \in \mathsf{m}$ its Jordan type also satisfies
$$P_{\ell,M} \le H_{\mathsf{w}}(M)^\vee. \tag{3.2}$$

Furthermore, in the same paper, the second statement of Proposition 3.13 was generalised to non-standard graded Artinian algebras.

PROPOSITION 3.15 ([14, Proposition 2.10])
Let A be a (possibly non-standard) graded Artinian algebra and $\ell \in A_1$. Then the following statements are equivalent:

1. *For each integer b, the multiplication maps $\times \ell^b : A_i \to A_{i+b}$ have maximal rank in each degree i. (That is, ℓ is a strong Lefschetz element.)*
2. *The Jordan type of ℓ is equal to the conjugate partition of the Hilbert function, i.e.*
$$P_{\ell,A} = H(A)^\vee.$$
3. *There is a set of strings $S_1, \dots, S_s$ as in Equation (2.6), composed of homogeneous elements, for the multiplication map $\times \ell^b : A \to A$ such that for each degree u and each integer i we have the equivalence*
$$\dim_{\mathsf{k}} A_u \ge i \quad \text{if and only if} \quad A_u \cap S_a \ne \emptyset \ \text{ for all } a \le i. \tag{3.3}$$

EXAMPLE 3.16
Consider the Artinian algebra $A = \mathsf{k}[x,y]/(x^2,y^2)$, from Example 2.8, and suppose $\operatorname{char} \mathsf{k} \ne 2$. Its Hilbert function is $H(A) = (1,2,1)$, so we can see that the number of parts in both Jordan types $(3,1)$ and $(2,2)$ is the Sperner number of A.

Therefore, since $P_{ax+by,A} = (3,1)$, if $ab \neq 0$, and $P_{x,A} = P_{y,A} = (2,2)$, we have a different way of checking that $ax+by$, x, and y are all weak Lefschetz. Furthermore, since $H(A)^\vee = (3,1)$, we have that $ax+by$ is a strong Lefschetz element, if $ab \neq 0$.

The previous results motivated the extension of the definitions of Lefschetz properties to non-graded algebras (see [14, Definition 2.12]).

Definition 3.17
Let $(A, \mathsf{m}, \mathsf{k})$ be a local Artinian algebra over k with Hilbert function $H(A)$. We say that an element $\ell \in \mathsf{m}$ is

1. a *strong Lefschetz element* if $P_{\ell,A} = H(A)^\vee$.
2. a *weak Lefschetz element* if the number of parts in $P_{\ell,A}$ equals the Sperner number of A.

Additionally if A is graded via a weight function w with w-Hilbert function $H_\mathsf{w}(A)$, then we say that a w-homogeneous element $\ell \in \mathsf{m}$ has

3. w-*strong Lefschetz Jordan type* (w-SLJT) if $P_\ell = H_\mathsf{w}(A)^\vee$.

We say that A is strong Lefschetz, respectively weak Lefschetz, respectively w-SLJT if it has an element $\ell \in \mathsf{m}$ of that type.

In light of this extended definition, we can recover a well-known result on Lefschetz properties (see [14, Lemma 2.15]), shown by Joël Briançon in [4] in characteristic zero, and extended by Roberta Basili and Tony Iarrobino in [3] to the case of large enough characteristic. In the case of graded Artinian algebras, this result has also been proved in [7, Proposition 4.4], for characteristic zero, and in [5, Theorem 4.11], for monomial ideals in any characteristic.

Lemma 3.18 (Height two Artinian algebras are strong Lefschetz [4], [3, Theorem 2.16])
Let $A = \mathsf{k}[x,y]/I$ be a standard graded Artinian algebra, or a local Artinian algebra of socle degree j, and suppose $\operatorname{char}\mathsf{k} = 0$ *or* $\operatorname{char}\mathsf{k} > j$. *Let ℓ be a general element of m. Then ℓ is a strong Lefschetz element and A has the strong Lefschetz property.*

4. Finer invariants, Jordan type, and their behaviour under deformations

One nice feature of Jordan type is its behaviour along flat families:

Lemma 4.1 (Generic Jordan type of a module [14, Lemma 2.54])
Given an A-module M, there is an open dense subset $U_M \subset \mathsf{m}$ for which $\ell \in U_M$ implies that the partition $P_{\ell,M}$ satisfies $P_{\ell,M} \geq P_{\ell',M}$ for any other element $\ell' \in \mathsf{m}$.

Likewise, if A admits a weight function w, then for each weight i, there is a dense open set $U_{i,M} \subset A_i(\mathsf{w})$ for which $\ell \in U_{i,M}$ implies that $P_{\ell,M} \geq P_{\ell',M}$ for any other $\ell' \in A_i(\mathsf{w})$.

Example 4.2
Consider the Artinian algebra $A = \mathsf{k}[x,y]/(x^2,y^2)$, from Example 2.8, and suppose that k is infinite, and $\operatorname{char}\mathsf{k} \neq 2$. We have seen that the generic Jordan type of A is $P_A = (3,1)$, and we can easily check that all other Jordan types that occur for A are dominated by $(3,1)$. In fact, in this case, we have a chain

$$P_A = (3,1) > P_{x,A} = (2,2) > P_{xy,A} = (2,1,1) > P_{0,A} = (1,1,1,1).$$

Proposition 4.3 (Semicontinuity of Jordan type [14, Corollary 2.44])

1. *Let $(M_t)_{t\in T}$ be a family of constant-length modules over a parameter topological space T. Then for a neighbourhood $U \subseteq T$ of t_0, we have that the generic Jordan types satisfy*

$$t \in U \Rightarrow P_{M_t} \geq P_{M_{t_0}}.$$

2. *Let $(A_t)_{t\in T}$ be a constant-length family of local or graded Artinian algebras. Then for a neighbourhood $U \subseteq T$ of t_0, we have*

$$t \in U \Rightarrow P_{A_t} \geq P_{A_{t_0}}.$$

3. *Let $(\ell_t)_{t\in T}$, $\ell_t \in \mathcal{M}_n(\mathsf{k})$, be a family of $n \times n$ nilpotent matrices, and let P_t be their Jordan type. Then there is a neighbourhood $U \subseteq T$ of t_0 such that*

$$P_t \geq P_{t_0} \text{ for all } t \in U.$$

We have seen in the previous chapter that Jordan type is a finer invariant than the Lefschetz properties. We will see now some refinements of Jordan type, starting with Jordan degree type, for the graded case.

Example 4.4
Assume that k is infinite and $\operatorname{char}\mathsf{k} \notin \{2,3,5\}$. Consider the family $(A_t)_{t\in\mathsf{k}}$ of a non-graded complete intersections given by

$$A_t = \mathsf{k}[x,y,z]/(yz - ty^2 - x^3,\ y^3,\ z^2 - tx^3).$$

For any $t \in \mathsf{k}$, the algebra A_t has a constant Hilbert function

$$H(A_t) = (1,3,4,4,3,2,1)$$

. We can see that A_0 is the algebra from Example 2.9, whose generic Jordan type is $P_{A_0} = (7,5,3,3)$. However, we can check that for $t \neq 0$, the generic Jordan type of A_t is $P_{A_t} = (7,5,4,2)$, which strictly dominates P_{A_0}. In fact, if we choose $\ell = x + y + z$, we can see that the following strings give us a pre-Jordan basis for $\times\ell : A_t \to A_t$, with $t \neq 0$:

$$\begin{array}{l} 1 \longmapsto \ell \longmapsto \ell^2 \longmapsto \ell^3 \longmapsto \ell^4 \longmapsto \ell^5 \longmapsto \ell^6 \\ x \longmapsto \ell x \longmapsto \ell^2 x \longmapsto \ell^3 x \longmapsto \ell^4 x \\ y \longmapsto \ell y \longmapsto \ell^2 y \longmapsto \ell^3 y \\ y^2 \longmapsto \ell y^2. \end{array} \tag{4.1}$$

Note that $(7,5,4,2) = H(A_t)^\vee$, so it is an upper bound for the Jordan type of any element in the maximal ideal of A_t. Therefore, the fact that this Jordan type is attained shows that it is the generic Jordan type of A_t for $t \neq 0$.

Definition 4.5 (Jordan degree type)
Let A be a graded Artinian algebra, let M be a finite graded A-module, and let $\ell \in A_1$ be any linear element. Let $\mathcal{B}$ be a homogeneous Jordan basis for ℓ, as in Definition 2.5, and consider the decomposition of M as a direct sum

$$M = \langle S_1 \rangle \oplus \cdots \oplus \langle S_s \rangle$$

of cyclic $\mathsf{k}[\ell]$-modules generated by ℓ-strings, with homogeneous beads, of the form

$$S_k = \{z_k,\ \ell z_k, \ldots, \ell^{p_k - 1} z_k\}.$$

The *Jordan degree type of* ℓ *in* M is the sequence of pairs of integers

$$\mathcal{S}_{\ell,M} = \big((p_1,\nu_1), \ldots, (p_s,\nu_s)\big), \tag{4.2}$$

where $P_{\ell,M} = (p_1, \ldots, p_s)$ is the Jordan type of ℓ in M and for each k, ν_k is the degree of the initial bead z_k. For any $k, k' \in \{1, \ldots, s\}$ if $k < k'$ and $p_k = p_{k'}$, we assume $\nu_k \leq \nu_{k'}$.

Remark 4.6
By Lemma 2.2 in [14], a homogeneous Jordan basis always exists, and the sequence of pairs (4.2) is an invariant of (M, ℓ), so the Jordan degree type is always well defined.

In the non-graded local case, the closest we have to degree is the order of an element, (see Definition 2.4). The following invariants are introduced and studied in the unpublished work with Tony Iarrobino and Johanna Steinmeyer [15]:

Definition 4.7 (Sequential Jordan type, Loewy sequential Jordan type, and double sequential Jordan type)
Let A be an Artinian local algebra of socle degree j, let m be its maximal ideal, and let $\ell \in \mathsf{m}$.

The *sequential Jordan type* (SJT) of ℓ in A is given by the sequence

$$(P_{\ell,A/\mathsf{m}^i}),\ i \in \{1, \ldots, j+1\}$$

of Jordan types of successive quotients of A by powers of the maximal ideal.

The *Loewy sequential Jordan type* (LJT) of ℓ in A is given by the sequence

$$\big(P_{\ell,A/(0:\mathsf{m}^{j-k})}\big),\ k \in \{0, \ldots, j\}$$

of Jordan types of successive quotients of A by the Loewy ideals.

The *double sequential Jordan type* (DSJT) is given by the table whose (a,i) entry is the partition

$$P_{\ell,B_{a,i}},\ \text{where } B_{a,i} = A/\big(\mathsf{m}^i \cap (0:\mathsf{m}^{j+1-a-i})\big),\ 0 \leq a \leq j,\ 0 \leq i \leq j+1-a$$

giving the Jordan type of the quotient of A by intersections of a Loewy ideal with a power of the maximal ideal.

As a consequence of Corollary 2.44 in [14], these invariants also enjoy semicontinuity, as we can see in the following result.

PROPOSITION 4.8 ([15])
Let $\mathcal{R} = \mathsf{k}\{x_1, \ldots, x_r\}$ be the local regular ring, and consider an element ℓ in the maximal ideal $(x_1, \ldots, x_r)$ of $\mathcal{R}$. Let $(A_w)_{w\in W}$ be a family of Artinian algebras, quotients of R, and denote m_w the maximal ideal of each A_w. Let $w_0 \in W$.

1. *If the Hilbert function $H(A_w)$ is constant along the family, then there is a neighbourhood U of w_0 such that the sequential Jordan type satisfies*
$$P_{\ell, A_w/\mathsf{m}_w^i} \geq P_{\ell, A_{w_0}/\mathsf{m}_{w_0}^i} \text{ for all } i.$$
2. *If the dimensions of the Loewy ideals $(0 : \mathsf{m}_w^i)$ are constant along the family, then there is a neighbourhood U of w_0 such that the sequential Loewy Jordan type satisfies*
$$P_{\ell, A_w/(0:\mathsf{m}_w^i)} \geq P_{\ell, A_{w_0}/(0:\mathsf{m}_{w_0}^i)} \text{ for all } i.$$
3. *If the dimensions of the ideals $\mathsf{m}_w^i \cap (0 : \mathsf{m}_w^k)$ are constant along the family, then there is a neighbourhood U of w_0 such that the double sequential Jordan type satisfies*
$$P_{\ell, A_w/(\mathsf{m}_w^i \cap (0:\mathsf{m}_w^k))} \geq P_{\ell, A_{w_0}/(\mathsf{m}_{w_0}^i \cap (0:\mathsf{m}_{w_0}^k))} \text{ for all } i \text{ and } k.$$

5. Artinian Gorenstein algebras

We will now focus on an important class of Artinian algebras, namely those that are Gorenstein. An Artinian algebra A as in Setting 2.1 is Gorenstein if and only if its socle $(0 : \mathsf{m})$ is a one-dimensional vector space over k. In this case, we will have $(0 : \mathsf{m}) = \mathsf{m}^j$, where j is the socle degree of A.

Iarrobino showed in [9, 10, 11] that the associated graded algebra
$$A^* = \bigoplus_{i\geq 0} \frac{\mathsf{m}^i}{\mathsf{m}^{i+1}}$$
of an Artinian Gorenstein (AG) algebra A has a canonical stratification by ideals $C(a)$ whose successive quotients $Q(a) = C(a)/C(a+1)$ yield an exact pairing:
$$Q(a)_i \times Q(a)_{j-a-i} \to \mathsf{k}. \tag{5.1}$$
Each graded piece of the module $Q(a)$ admits a presentation
$$Q(a)_i \cong \frac{\mathsf{m}^i \cap (0 : \mathsf{m}^{j+1-a-i})}{\mathsf{m}^i \cap (0 : \mathsf{m}^{j-a-i}) + \mathsf{m}^{i+1} \cap (0 : \mathsf{m}^{j+1-a-i})} \tag{5.2}$$
(see [10, page 350] or [9, Section 3]). To better understand this quotient, we may observe that A admits the m-adic filtration
$$A \supset \mathsf{m} \supset \mathsf{m}^2 \supset \cdots \supset \mathsf{m}^j \supset \mathsf{m}^{j+1} = 0$$

and the Loewy filtration

$$A \supset (0 : \mathsf{m}^j) \supset (0 : \mathsf{m}^{j-1}) \supset \cdots \supset (0 : \mathsf{m}) \supset 0.$$

Note that an element of A of order i, i.e. an element in $\mathsf{m}^i \setminus \mathsf{m}^{i+1}$, clearly belongs to the Loewy ideal $(0 : \mathsf{m}^{j+1-i})$, since $\mathsf{m}^{j+1} = 0$. However, a general element of order i will not annihilate $\mathsf{m}^{j+1-a-i}$ for $a > 0$. The graded piece $Q(a)_i$ helps us understand which elements of order i annihilate a lower power of m than we would expect, and the integer a is telling us how much lower that is. Note that it is the quotient of an intersection of a power of m and a Loewy ideal by the sum of the next two such intersections contained in it.

A very useful tool when studying AG algebras is their dual generator. Consider the local regular ring $\mathcal{R} = \mathsf{k}\{x_1, \ldots, x_r\}$, as in Setting 2.1, and the divided-power ring $\mathfrak{D} = \mathsf{k}_{DP}[X_1, \ldots, X_r]$ (for details see [12, Appendix A]). The ring $\mathcal{R}$ acts on $\mathfrak{D}$ by contraction:

$$x_i^k \circ X_i^K = \begin{cases} X_i^{K-k}, & \text{if } K \geq k, \\ 0, & \text{if } K < k. \end{cases} \tag{5.3}$$

We have ([16], [11, Lemma 1.1]).

Lemma 5.1 (AG algebras and k-linear maps of $\mathcal{R}$)
There is a one-to-one isomorphism of sets

$$\left\{ \begin{matrix} \textit{AG quotients } A \textit{ of } \mathcal{R} \\ \textit{having socle degree } j \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \mathsf{k}\textit{-linear homomorphisms } \phi : \mathcal{R} \to \mathsf{k}, \\ \textit{with } \phi|_{\mathsf{m}^{j+1}} = 0 \textit{ but } \phi|_{\mathsf{m}^j} \neq 0 \end{matrix} \right\}.$$

Here $A = \mathcal{R}/I$ *with* $I = \{h : \ \phi(\mathcal{R} \cdot h) = 0\}$.

Definition 5.2
Let A be an AG algebra, quotient of $\mathcal{R}$. We call an element $F \in \mathfrak{D}$ such that $A = \mathcal{R}/\operatorname{Ann} F$ a *dual generator* for A.

Example 5.3
Let $A = \mathsf{k}[x, y, z]/(yz - x^3, y^3, z^2)$, from Example 2.9. Being a complete intersection, we know that A is an AG algebra, and we can check that its dual generator is $F = X^5Y + X^2Y^2Z$. Note that the degree of F is the socle degree of A.

We can compute the modules $Q(a)$ by acting on F. For instance, take $xy \in A$. Acting on F we get $xy \circ F = X^4 + XYZ$. Since this is a partial of F of degree 4, we know that for any element $g \in \mathsf{m}^5$, we have $g \circ (X^4 + XYZ) = 0$, i.e. $gxy \circ F = 0$, so in A we have $gxy = 0$, meaning that $xy \in (0 : \mathsf{m}^5)$. This means that it makes sense to look at the class of xy in

$$Q(0)_2 = \frac{\mathsf{m}^2 \cap (0 : \mathsf{m}^5)}{\mathsf{m}^2 \cap (0 : \mathsf{m}^4) + \mathsf{m}^3 \cap (0 : \mathsf{m}^5)}.$$

Note that if $h_1 \in \mathsf{m}^2 \cap (0 : \mathsf{m}^4)$ and $h_2 \in \mathsf{m}^3 \cap (0 : \mathsf{m}^5)$ we have that $(h_1 + h_2) \circ F$ is a partial of degree at most 3, since $h_1 \in (0 : \mathsf{m}^4)$ and $h_2 \in \mathsf{m}^3$. So we see that $xy \notin \mathsf{m}^2 \cap (0 : \mathsf{m}^4) + \mathsf{m}^3 \cap (0 : \mathsf{m}^5)$, so it represents a non-zero element in $Q(0)_2$.

Now take $y^2 \in A$. Acting on F we get $y^2 \circ F = X^2Z$. In this case, although y^2 is also an element of order 2, as xy is, its action on F gives us a partial of degree 3, and not 4. So $y^2 \in (0 : \mathsf{m}^4)$, and we can look at its class in

$$Q(1)_2 = \frac{\mathsf{m}^2 \cap (0 : \mathsf{m}^4)}{\mathsf{m}^2 \cap (0 : \mathsf{m}^3) + \mathsf{m}^3 \cap (0 : \mathsf{m}^4)}.$$

This time, if we take $h_1 \in \mathsf{m}^2 \cap (0 : \mathsf{m}^3)$ and $h_2 \in \mathsf{m}^3 \cap (0 : \mathsf{m}^4)$ we have that $h_1 \circ F$ is a partial of degree at most 2, since $h_1 \in (0 : \mathsf{m}^3)$, but $h_2 \circ F$ may have degree 3. It is an argument a bit more subtle than before, but it is not hard to check that $h_2 \circ F$ cannot have the term X^2Z, because this term cannot be obtained from F if we act with an element in m^3. Therefore $y^2 \notin \mathsf{m}^2 \cap (0 : \mathsf{m}^3) + \mathsf{m}^3 \cap (0 : \mathsf{m}^4)$, so it represents a non-zero element in $Q(1)_2$

Following these ideas, we get a complete description of the $Q(a)$ modules if we compute the space of partials of F:

	$Q(0)$	$Q(1)$
$Q(a)_0:$	$1 \circ F = F$	
$Q(a)_1:$	$x \circ F = X^4Y + XY^2Z$	
	$y \circ F = X^5 + X^2YZ$	
$Q(a)_2:$	$x^2 \circ F = X^3Y + Y^2Z$	$z \circ F = X^2Y^2$
	$xy \circ F = X^4 + XYZ$	
$Q(a)_3:$	$x^3 \circ F = X^2Y$	$xz \circ F = XY^2$
	$x^2y \circ F = X^3 + YZ$	$y^2 \circ F = X^2Z$
$Q(a)_4:$	$x^4 \circ F = XY$	$x^2z \circ F = Y^2$
	$x^3y \circ F = X^2$	$xy^2 \circ F = XZ$
$Q(a)_5:$	$x^5 \circ F = Y$	$x^2y^2 \circ F = Z$
	$x^4y \circ F = X$	
$Q(a)_6:$	$x^5y \circ F = 1$	

So the symmetric decomposition of the Hilbert function is

$H(A)$	1 3 4 4 3 2 1
$H\big(Q(0)\big)$	1 2 2 2 2 2 1
$H\big(Q(1)\big)$	0 1 2 2 1 0

We can also see the strings giving the Jordan type of an element of the maximal ideal by looking at its action on the dual generator. For example, if we want to compute the Jordan type of x in A, we can compute the following strings, starting at 1, y, z, and y^2:

$1 \circ F = F$ $\qquad\qquad$ $y \circ F = X^5 + X^2YZ$

$$\begin{aligned} x \circ F &= X^4Y + XY^2Z & xy \circ F &= X^4 + XYZ \\ x^2 \circ F &= X^3Y + Y^2Z & x^2y \circ F &= X^3 + YZ \\ x^3 \circ F &= X^2Y & x^3y \circ F &= X^2 \\ x^4 \circ F &= XY & x^4y \circ F &= X \\ x^5 \circ F &= Y & x^5y \circ F &= 1 \end{aligned}$$

$$\begin{aligned} z \circ F &= X^2Y^2 & y^2 \circ F &= X^2Z \\ xz \circ F &= XY^2 & xy^2 \circ F &= XZ \\ x^2z \circ F &= Y^2 & x^2y^2 \circ F &= Z \end{aligned}$$

We thus get $P_{x,A} = (6, 6, 3, 3)$.

A. Jordan basis *à la carte*

In this appendix, we will look at some technical features of Jordan, and pre--Jordan, bases. Since most results are valid for finite-dimensional vector spaces over a field k, we will present them in the more general setting of linear algebra, and then draw conclusions to the case of the Jordan type of Artinian algebras. The following material benefited from discussions with Tony Iarrobino and Johanna Steinmeyer while working on the unpublished paper [15].

To our knowledge, the results in this section are not written elsewhere, but are not surprising, and some versions of them have been used before, even if not explicitly. Example A.9, however, came as a surprise while writing these notes. It was the first time we could not find a pre-Jordan basis compatible with the Hilbert function (in the sense of Definition A.7 below), and the example shows that such a basis does not always exist, contrary to what this author believed.

The following definition is the usual one for a Jordan basis, in the case of nilpotent endomorphisms.

Definition A.1
Let V be a finite-dimensional vector space over a field k, let $\phi : V \to V$ be a nilpotent endomorphism, and let $(p_1, \dots, p_k)$ be a partition of $\dim V$, with $p_1 \geq \cdots \geq p_k$. We say that a basis

$$\{v_{1,1}, \dots, v_{1,p_1}, v_{2,1}, \dots, v_{2,p_2}, \dots, v_{k,1}, \dots, v_{k,p_k}\}$$

of V is a *Jordan basis* for ϕ if for every $s \in \{1, \dots, k\}$,

$$\phi(v_{s,m}) = \begin{cases} v_{s,m+1}, & \text{if } m < p_s, \\ 0, & \text{otherwise.} \end{cases} \tag{A.1}$$

If ϕ admits a Jordan basis as above we call the partition $(p_1, \dots, p_k)$ the *Jordan type* of ϕ.

We defined pre-Jordan bases for the multiplication map by an element of the maximal ideal of an Artinian algebra, but we can make the definition more general, in terms of linear algebra:

Definition A.2
Let V be a finite-dimensional vector space over a field k, let $\phi : V \to V$ be a nilpotent endomorphism, and let $(p_1, \dots, p_k)$, with $p_1 \geq \dots \geq p_k$, be its Jordan type. A *pre-Jordan basis* for ϕ is a basis

$$\{v_{1,1}, \dots, v_{1,p_1}, v_{2,1}, \dots, v_{2,p_2}, \dots, v_{k,1}, \dots, v_{k,p_k}\}$$

of V such that for every $s \in \{1, \dots, k\}$,

$$\phi(v_{s,m}) = v_{s,m+1}, \quad \text{if } m < p_s. \tag{A.2}$$

Remark A.3
We may construct a Jordan basis $(v'_{s,m})_{1\leq s\leq k,\, 1\leq m\leq p_s}$ from a given pre-Jordan basis $(v_{s,m})_{1\leq s\leq k,\, 1\leq m\leq p_s}$. We will show this by induction on s. First note that since ϕ is nilpotent, its only eigenvalue is zero. Also, we must have $\phi^{p_1} = 0$, for p_1 is the maximum size of a Jordan block for ϕ. Therefore $\phi(v_{1,p_1}) = \phi^{p_1}(v_{1,1}) = 0$. Write $v'_{1,m} = v_{1,m}$ for $1 \leq m \leq p_1$.

For the induction step, suppose that for some s, we have a list of vectors

$$v'_{1,1}, \dots, v'_{1,p_1}, \dots, v'_{s,1}, \dots, v'_{s,p_s}$$

satisfying (A.1) and such that

$$v'_{1,1}, \dots, v'_{1,p_1}, \dots, v'_{s,1}, \dots, v'_{s,p_s}, v_{s+1,1}, \dots, v_{s+1,p_{s+1}}, \dots, v_{k,1}, \dots, v_{k,p_k}$$

are linearly independent. Now, we know that the rank of $\phi^{p_{s+1}}$ is

$$(p_1 - p_{s+1}) + \dots + (p_s - p_{s+1}) = p_1 + \dots + p_s - sp_{s+1},$$

so its image is spanned by $(v'_{l,m})_{1\leq l\leq s,\, p_{s+1}<m\leq p_l}$. Since for each $l \in \{1, \dots, s\}$ and each $m \in \{p_{s+1}+1, \dots, p_l\}$ we have $v'_{l,m} = \phi^{p_{s+1}}(v'_{l,m-p_{s+1}})$, we obtain

$$\operatorname{im} \phi^{p_{s+1}} = \langle \phi^{p_{s+1}}(v'_{l,m}) : 1 \leq l \leq s,\, 1 \leq m \leq p_l - p_{s+1} \rangle.$$

Therefore there is $u \in \langle v'_{l,m} : 1 \leq l \leq s,\, 1 \leq m \leq p_l - p_{s+1} \rangle$ such that

$$\phi(v_{s+1,p_{s+1}}) = \phi^{p_{s+1}}(v_{s+1,1}) = \phi^{p_{s+1}}(u).$$

Let us write $v'_{s+1,1} = v_{s+1,1} - u$, and $v'_{s+1,m+1} = \phi(v'_{s+1,m})$ for $1 \leq m < p_{s+1}$. It is now straightforward to check that $v'_{1,1}, \dots, v'_{1,p_1}, \dots, v'_{s+1,1}, \dots, v'_{s+1,p_{s+1}}$ satisfy (A.1) and

$$v'_{1,1}, \dots, v'_{1,p_1}, \dots, v'_{s+1,1}, \dots, v'_{s+1,p_{s+1}}, v_{s+2,1}, \dots, v_{s+2,p_{s+2}}, \dots, v_{k,1}, \dots, v_{k,p_k}$$

are linearly independent, which concludes the induction step.

Note that, while the definition of Jordan basis does not depend on the Jordan type, the definition of pre-Jordan basis does. The following Lemma shows that we can check if a given basis of a vector space V is a pre-Jordan basis of an endomorphism $V \to V$ before we know its Jordan type.

Lemma A.4 (Characterisation of pre-Jordan bases)
Let V be a finite-dimensional vector space over an infinite field k, let $\phi : V \to V$ be a nilpotent endomorphism, and let $(p_1, \dots, p_k)$, with $p_1 \geq \cdots \geq p_k$, be a partition of $\dim_{\mathsf{k}} V$. Let

$$B = \{v_{1,1}, \dots, v_{1,p_1}, v_{2,1}, \dots, v_{2,p_2}, \dots, v_{k,1}, \dots, v_{k,p_k}\}$$

be a basis of V such that for every $s \in \{1, \dots, k\}$,

$$\phi(v_{s,m}) = v_{s,m+1}, \quad \text{if } m < p_s. \tag{A.3}$$

Then the following are equivalent:

(i) *B is a pre-Jordan basis for ϕ.*
(ii) *The Jordan type of ϕ is $(p_1, \dots, p_k)$.*
(iii) *For every $s \in \{1, \dots, k\}$,*

$$\phi(v_{s,p_s}) \in \langle v_{t,m} : t < s, m > p_s \rangle. \tag{A.4}$$

Proof. Equivalence between (i) and (ii) comes directly from Definition A.2. To show that (ii) implies (iii), take $s \in \{1, \dots, k\}$. If $s = 1$, then the set $\{v_{t,m} : \ t < s, m > p_s\}$ is empty, so its span is the zero space, and since p_1 is the largest part of the Jordan type of ϕ, we have $\phi^{p_1+1} = 0$, and in particular $\phi(v_{1,p_1}) = 0$. If $s > 1$, we can use the same idea of the induction step of the proof of Remark A.3: the dimension of $\operatorname{im} \phi^{p_s+1}$ is

$$(p_1 - p_s) + \cdots + (p_{s-1} - p_s) = p_1 + \cdots + p_{s-1} - (s-1)p_s,$$

which is the cardinality of the set $\{v_{t,m} : \ t < s, m > p_s\}$. Since this linearly independent set is contained in $\operatorname{im} \phi^{p_s+1}$ we get the equality

$$\operatorname{im} \phi^{p_s+1} = \langle v_{t,m} : t < s, m > p_s \rangle,$$

and therefore $\phi(v_{s,p_s}) \in \langle v_{t,m} : t < s, m > p_s \rangle$.

Conversely, suppose condition (iii) holds. We can construct a Jordan basis from B in a similar way we did in the proof of Remark A.3: we know that the first string $(v_{1,1}, \dots, v_{1,p_1})$ satisfies (A.1), since $\phi(v_{1,p_1}) = 0$. If we suppose we have

For the induction step, suppose that for some $s < k$, we have a list of vectors

$$v'_{1,1}, \dots, v'_{1,p_1}, \dots, v'_{s,1}, \dots, v'_{s,p_s}$$

satisfying (A.1) and such that

$$v'_{1,1}, \dots, v'_{1,p_1}, \dots, v'_{s,1}, \dots, v'_{s,p_s}, v_{s+1,1}, \dots, v_{s+1,p_{s+1}}, \dots, v_{k,1}, \dots, v_{k,p_k}$$

are linearly independent. Since $\phi(v_{s+1,p_{s+1}}) \in \langle v_{t,m} : t < s+1, m > p_{s+1} \rangle$, we can write

$$\begin{aligned}\phi^{p_{s+1}}(v_{s+1,1}) &= \phi(v_{s+1,p_{s+1}}) \\ &= \alpha_{1,p_{s+1}+1} v_{1,p_{s+1}+1} + \cdots + \alpha_{1,p_1} v_{1,p_1} + \cdots\end{aligned}$$

$$\begin{aligned} &+ \alpha_{s,p_{s+1}+1} v_{s,p_{s+1}+1} + \cdots + \alpha_{s,p_s} v_{s,p_s} \\ =\ & \alpha_{1,p_{s+1}+1} \phi^{p_{s+1}}(v_{1,1}) + \cdots + \alpha_{1,p_1} \phi^{p_{s+1}}(v_{1,p_1-p_{s+1}}) + \cdots \\ &+ \alpha_{s,p_{s+1}+1} \phi^{p_{s+1}}(v_{s,1}) + \cdots + \alpha_{s,p_s} \phi^{p_{s+1}}(v_{s,p_s-p_{s+1}}). \end{aligned}$$

Therefore, making

$$\begin{aligned} v'_{s+1,1} = v_{s+1,1} - \alpha_{1,p_{s+1}+1} v_{1,1} - \cdots - \alpha_{1,p_1} v_{1,p_1-p_{s+1}} - \cdots \\ - \alpha_{s,p_{s+1}+1} v_{s,1} - \cdots - \alpha_{s,p_s} v_{s,p_s-p_{s+1}}, \end{aligned}$$

and $v'_{s+1,m+1} = \phi(v'_{s+1,m})$ for $1 \le m < p_{s+1}$, we see that the vectors

$$v'_{1,1}, \ldots, v'_{1,p_1}, \ldots, v'_{s+1,1}, \ldots, v'_{s+1,p_{s+1}}$$

satisfy (A.1) and

$$v'_{1,1}, \ldots, v'_{1,p_1}, \ldots, v'_{s+1,1}, \ldots, v'_{s+1,p_{s+1}}, v_{s+2,1}, \ldots, v_{s+2,p_{s+2}}, \ldots, v_{k,1}, \ldots, v_{k,p_k}$$

are linearly independent. The result follows by induction on s.

Lemma A.6 below shows us that given an element ℓ in the maximal ideal m of an Artinian algebra A, we can construct a pre-Jordan basis for the multiplication by ℓ starting by elements of low order, as we did in most examples in these notes. This means that the first string can start in an element outside the maximal ideal, then the next strings can be chosen in $\mathsf{m} \setminus \mathsf{m}^2$, and so on. We will see the general case of a suitable filtration of a vector space V.

Lemma A.5
Let V be a finite-dimensional vector space over an infinite field k, let $\phi : V \to V$ be a nilpotent endomorphism, and let $(p_1, \ldots, p_k)$, with $p_1 \ge \cdots \ge p_k$, be its Jordan type. Let $V = V_0 \supset \cdots \supset V_j$ be a strictly-decreasing filtration of vector subspaces of V such that $\phi(V_i) \subseteq V_{i+1}$ for $0 \le i < j$. Then there is a pre-Jordan basis

$$B = \{v_{1,1}, \ldots, v_{1,p_1}, v_{2,1}, \ldots, v_{2,p_2}, \ldots, v_{k,1}, \ldots, v_{k,p_k}\}$$

for ϕ such that for each $s \in \{1, \ldots, k\}$ if $v_{s,1} \in V_i \setminus V_{i+1}$, and we define

$$W_s = \langle v_{1,1}, \ldots, v_{s,p_s} \rangle,$$

then the class of $v_{s,1}$ in

$$\frac{V_i}{(V_i \cap W_s) + V_{i+1}},$$

is non-zero.

Proof. If $p_1 > 1$, then $\phi^{p_1-1} \neq 0$, and therefore there is $v_{1,1} \in V$ such that $\phi^{p_1-1}(v_{1,1}) \neq 0$. Since k is infinite, we can ask that $v_{1,1} \in V_0 \setminus V_1$. For each $m < p_1$, define $v_{1,m+1} = \phi(v_{1,m})$, and let $W_1 = \langle v_{1,1}, \ldots, v_{1,p_1} \rangle$.

Now, ϕ induces an endomorphism $\phi_1 : \frac{V}{W_1} \to \frac{V}{W_1}$, whose Jordan type is $(p_2, \ldots, p_k)$. Let e_1 be the lowest integer such that

$$\frac{V_{e_1}}{(V_{e_1} \cap W_1) + V_{e_1+1}} \neq 0.$$

We can then choose $v_{2,1} \in V_{e_1} \setminus V_{e_1+1}$ such that $\phi_1^{p_2-1}(v_{2,1} + W_1) \neq 0$, for this is an open condition on V_{e_1}. Define $v_{2,m+1} = \phi(v_{2,m})$ for each $m < p_2$. Continuing in this manner, suppose for some $s < k$ we have vectors

$$v_{1,1}, \ldots, v_{1,p_1}, \ldots, v_{s,1}, \ldots, v_{s,p_s}$$

satisfying condition (A.2). In addition, suppose that for each $l \in \{1, \ldots, s-1\}$, $\phi_l^{p_{l+1}-1}(v_{l+1,1} + W_l) \neq 0$, where $W_l = \langle v_{1,1}, \ldots, v_{1,p_1}, \ldots, v_{l,1}, \ldots, v_{l,p_l} \rangle$ and

$$\phi_l : \frac{V}{W_l} \to \frac{V}{W_l}$$

is the endomorphism induced by ϕ, and $v_{l+1,1} \in V_{e_l} \setminus V_{e_l+1}$, where e_l is the lowest integer such that

$$\frac{V_{e_l}}{(V_{e_l} \cap W_l) + V_{e_l+1}} \neq 0.$$

By construction, we see that we get a basis for V

$$B = \{v_{1,1}, \ldots, v_{1,p_1}, \ldots, v_{k,1}, \ldots, v_{k,p_k}\}$$

satisfying (A.2) and for each $s \in \{1, \ldots, k\}$ if $v_{s,1} \in V_i \setminus V_{i+1}$ then the class of $v_{s,1}$ in

$$\frac{V_i}{(V_i \cap W_s) + V_{i+1}},$$

is non-zero.

Lemma A.6
Let A be an Artinian local algebra over a field k, let m be its maximal ideal, and let $\ell \in \mathsf{m}$. Suppose that the Jordan type of ℓ in A is $P_{\ell,A} = (p_1, \ldots, p_s)$. Then the multiplication by ℓ admits a pre-Jordan basis

$$\mathcal{B} = \{\ell^i z_k : \ 1 \leq k \leq s,\, 0 \leq i \leq p_k - 1\},$$

as in Definition 2.5 such that $z_0 \in A \setminus \mathsf{m}$ and for each $k \in \{2, \ldots, s\}$, if the initial bead z_k of the string S_k has order i then it represents a non-zero class in the quotient

$$\frac{\mathsf{m}^i}{(\mathsf{m}^i \cap W_{k-1}) + \mathsf{m}^{i+1}},$$

where $W_{k-1} = \langle S_1, \ldots, S_{k-1} \rangle$.

Proof. Apply Lemma A.5 to the filtration $A \supset \mathsf{m} \supset \mathsf{m}^2 \supset \cdots \supset \mathsf{m}^j$, where j is the socle degree of A.

Finally, we will look at a question on the possibility of having a basis of an Artinian algebra A as a vector space over a field k that helps us compute both the Hilbert function of A and the Jordan type of an element ℓ in its maximal ideal. This was the case of Jordan bases coming from strings (2.1) and (2.2) in Example 2.3, where A is graded and ℓ is homogeneous, and also pre-Jordan basis occuring in (2.3), for the non-homogeneous element $x + y^2$, where the number of elements of order i is $H(A)_i$ (in this case we must disregard the second occurrence of y^4). This is no longer the case of the Jordan basis coming from (2.4). In Example 2.9, pre-Jordan bases coming from (2.7) and (2.8) also have as many elements of a given order as the corresponding value of the Hilbert function, but this does not happen in (2.9).

As we will see below, if A is graded and ℓ is homogeneous, we can always find a Jordan basis composed of homogeneous elements that helps us compute the Hilbert function of A (this is made more precise in Definition A.7 below). However, we will see that if A is non-graded, we cannot always find such a Jordan basis, in some cases, not even a pre-Jordan basis.

Definition A.7
Let A be an Artinian local algebra over a field k and let m be its maximal ideal. Let B be a basis of A as a k-vector space. We say that B is *compatible with the Hilbert function* of A if for every $i \geq 0$, the cardinality of $B \cap \mathsf{m}^i$ equals $\dim \mathsf{m}^i$.

A simple linear algebra argument shows that any Artinian local algebra A over a field k always admits a basis as a k-vector space that is compatible with its Hilbert function, but the following example by Chris McDaniel shows that it is not always possible to find such a basis that is also a Jordan basis for the multiplication of an element in the maximal ideal.

Example A.8 (AG algebra A having no Jordan basis for the multiplication $\times x$ consistent with $H(A)$ — C. McDaniel, private communication)
Let

$$F = XY^3 + X^2Y \in \mathsf{k}_{DP}[X, Y],$$

$A = \mathsf{k}\{x, y\}/I$, $I = \operatorname{Ann} F = (x^2 - xy^2,\, y^4)$, having Hilbert function

$$H(A) = (1, 2, 2, 2, 1).$$

Considering $\ell = x$, we easily find a pre-Jordan basis for the multiplication $\times x$ compatible with the Hilbert function:

$$\begin{array}{ccccc} 1 \longmapsto & x & \longmapsto & xy^2 & \\ & y & \longmapsto & xy & \longmapsto xy^3 \\ & & y^2 & & \\ & & & y^3. & \end{array} \tag{A.5}$$

Note that this is not a Jordan basis, as y^2 and y^3 are not kernel elements. We have here, that multiplication by x on A does not admit a Jordan basis that is compatible with the Hilbert function of A. See [13, Example 2.14].

The following example shows that it may even not be possible to have a pre--Jordan basis compatible with the Hilbert function.

Example A.9 (Non-existence of a pre-Jordan basis compatible with the Hilbert function)

Let $A = \mathsf{k}[x, y, z]/(yz - x^3,\ y^3,\ z^2)$, from Example 2.9, whose Hilbert function is $(1, 3, 4, 4, 3, 2, 1)$, and recall that the following is a monomial basis for A:

$$\begin{array}{ccccccc} 1 & x & x^2 & x^3 & x^4 & x^5 & x^5y \\ & y & xy & x^2y & x^3y & x^4y & \\ & z & xz & x^2z & x^2y^2 & & \\ & & y^2 & xy^2 & & & \end{array}$$

Consider the element $\ell = y + z + x^2$ (this is ℓ_3 in the example). We will show that there is no pre-Jordan basis for the multiplication by ℓ that is compatible with the Hilbert function. Suppose there is such a basis. Since the Jordan type of ℓ is $P_{\ell,A} = (5, 4, 3, 3, 2, 1)$, the basis would be

$$\{b_1, \ell b_1, \dots, \ell^4 b_1,\ b_2, \dots, \ell^3 b_2,\ b_3, \ell b_3, \ell^2 b_3,\ b_4, \ell b_4, \ell^2 b_4,\ b_5, \ell b_5,\ b_6\},$$

composed of six strings, with initial beads $b_1, \dots, b_6$.

Then we would have a first string starting in order zero, and this must be the longest string, since $\ell^4 = 12x^5y$, so $\ell^4\mathsf{m} = 0$. By scaling, we can assume $b_1 = 1 + c$, with $c \in \mathsf{m}$. Now since $H(A)_1 = 3$, by Lemma A.6, we may assume that for some indices k, m, with $2 \le k, m \le 6$, we have $b_k, b_m \in \mathsf{m} \setminus \mathsf{m}^2$, so we may write

$$b_k = \alpha_1 x + \alpha_2 y + \alpha_3 z + d \qquad \text{and} \qquad b_m = \beta_1 x + \beta_2 y + \beta_3 z + e,$$

with $d, e \in \mathsf{m}^2$. Furthermore, since the basis is compatible with the Hilbert function, the classes of ℓb_1, b_k, and b_m in $\frac{\mathsf{m}}{\mathsf{m}^2}$ must be linearly independent. Since $\ell b_1 = y + z + x^2 + \ell c$ and $x^2 + \ell c \in \mathsf{m}^2$, we have that the determinant

$$\begin{vmatrix} 0 & 1 & 1 \\ \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \end{vmatrix}$$

must be non-zero. Multiplying ℓb_1, b_k, and b_m by ℓ, we get

$$\begin{aligned} \ell^2 b_1 &= y^2 + 2x^3 + 2x^2y + 2x^2z + x^4 + \ell^2 c \\ \ell b_k &= \alpha_1(xy + xz + x^3) + \alpha_2(y^2 + x^3 + x^2y) + \alpha_3(x^3 + x^2z) + \ell d \\ \ell b_m &= \beta_1(xy + xz + x^3) + \beta_2(y^2 + x^3 + x^2y) + \beta_3(x^3 + x^2z) + \ell e. \end{aligned}$$

Note that if $(\alpha_1, \alpha_2) \ne (0, 0)$, we have $\ell b_k \in \mathsf{m}^2 \setminus \mathsf{m}^3$. Likewise, if $(\beta_1, \beta_2) \ne (0, 0)$, we have $\ell b_m \in \mathsf{m}^2 \setminus \mathsf{m}^3$. However, we easily check that if both ℓb_k and ℓb_m are in $\mathsf{m}^2 \setminus \mathsf{m}^3$, the classes of of $\ell^2 b_1$, ℓb_k, and ℓb_m in $\frac{\mathsf{m}^2}{\mathsf{m}^3}$ are linearly dependent, which contradicts the hypothesis that the basis is compatible with the Hilbert function. Without loss of generality we may assume that $\beta_1 = \beta_2 = 0$, in which case we must have $\alpha_1 \ne 0$. So by scaling b_k and b_m we may also assume $\alpha_1 = \beta_3 = 1$. Then, we may compute

$$\begin{aligned} \ell^3 b_1 &= 3x^3y + 3x^2y^2 + 6x^5 + 3x^4y + \ell^3 c \\ \ell^4 b_1 &= 12x^5y \\ \ell^2 b_k &= xy^2 + 2x^4 + 2x^3y + x^5 + \alpha_2(2x^3y + 2x^2y^2 + 2x^5 + x^4y) \end{aligned}$$

$$
\begin{aligned}
&+\alpha_3(x^3y+2x^5)+\ell^2 d\\
\ell^3 b_k &= 3(x^4y+x^5y)+6\alpha_2 x^5y+\ell^3 d\\
\ell^2 b_m &= x^3y+2x^5+\ell^2 e.
\end{aligned}
$$

Since $\ell^3 b_k \notin \langle \ell^3 b_1, \ell^4 b_1\rangle$, and $\ell^4 b_k = 0$, we see that $(b_k, \ldots, \ell^3 b_k)$ is a string of length 4, so $k = 2$. Also, $\ell^2 b_m \notin \langle \ell^2 b_1, \ell^3 b_1, \ell^4 b_1, \ell^2 b_2, \ell^3 b_2\rangle$, but $\ell^3 b_m \in \langle \ell^4\rangle$, so $(b_m, \ell b_m, \ell^2 b_m)$ is a string of length 3, and we may assume $m = 3$. Therefore, so far, we have the three stings, arranged by order:

$$
\begin{array}{ccccccccc}
b_1 & \longmapsto & \ell b_1 & \longmapsto & \ell^2 b_1 & \longmapsto & \ell^3 b_1 & \longmapsto & \ell^4 b_1\\
 & & b_2 & \longmapsto & \ell b_2 & \longmapsto & \ell^2 b_2 & \longmapsto & \ell^3 b_2\\
 & & b_3 & \longmapsto & \ell b_3 & \longmapsto & \ell^2 b_3. & &
\end{array}
$$

Since $H(A)_2 = 4$ and so far we only have two elements in the basis of order two, again by Lemma A.6, we may assume that for some indices n, p, with $4 \le n, p \le 6$, we have $b_n, b_p \in \mathsf{m}^2 \setminus \mathsf{m}^3$, so we may write

$$
b_n = \gamma_1 x^2 + \gamma_2 xy + \gamma_3 xz + \gamma_4 y^2 + f \quad \text{and} \quad b_p = \epsilon_1 x^2 + \epsilon_2 xy + \epsilon_3 xz + \epsilon_4 y^2 + g,
$$

with $f, g \in \mathsf{m}^3$. As before, since the basis is compatible with the Hilbert function, the classes of $\ell^2 b_1$, ℓb_2, b_n and b_p in $\frac{\mathsf{m}^2}{\mathsf{m}^3}$ must be linearly independent, so the determinant

$$
\begin{vmatrix}
0 & 0 & 0 & 1\\
0 & 1 & 1 & \alpha_2\\
\gamma_1 & \gamma_2 & \gamma_3 & \gamma_4\\
\epsilon_1 & \epsilon_2 & \epsilon_3 & \epsilon_4
\end{vmatrix}
$$

must be non-zero. Looking at

$$
\ell b_n = \gamma_1(x^2y + x^2z + x^4) + \gamma_2(xy^2 + x^4 + x^3y) + \gamma_3 x^4 + \gamma_4(x^3y + x^2y^2) + \ell f,
$$

and considering the corresponding expression of ℓb_p, we see that if $(\gamma_1, \gamma_2) \neq (0, 0)$ then $\ell b_n \in \mathsf{m}^3 \setminus \mathsf{m}^4$, and if $(\epsilon_1, \epsilon_2) \neq (0, 0)$ then $\ell b_p \in \mathsf{m}^3 \setminus \mathsf{m}^4$. Note that the classes of $\ell^2 b_2$, ℓb_3, ℓb_n, and ℓb_p in $\frac{\mathsf{m}^3}{\mathsf{m}^4}$ are

$$
\begin{aligned}
\overline{\ell^2 b_2} &= \overline{xy^2}\\
\overline{\ell b_3} &= \overline{x^3 + x^2 z}\\
\overline{\ell b_n} &= \overline{\gamma_1(x^2y + x^2z) + \gamma_2 xy^2}\\
\overline{\ell b_p} &= \overline{\epsilon_1(x^2y + x^2z) + \epsilon_2 xy^2}.
\end{aligned}
$$

It is clear that they cannot be linearly independent, so we may assume that $\epsilon_1 = \epsilon_2 = 0$, which implies $\gamma_1 \neq 0 \neq \epsilon_3$, so we may take $\gamma_1 = \epsilon_3 = 1$. If we now compute

$$
\ell^2 b_n = x^2y^2 + 2x^5 + 2x^4y + \gamma_2(2x^4y + x^5y) + \gamma_3 x^4 y + \ell^2 f,
$$

we can check that $\ell^2 b_n \notin \langle \ell^2 b_1, \ell^3 b_1, \ell^4 b_1, \ell^2 b_2, \ell^3 b_2, \ell^2 b_3 \rangle$, but $\ell^3 b_n = 3x^5 y \in \langle \ell^4 \rangle$, so $(b_n, \ell b_n, \ell^2 b_n)$ is a new string of length 3, and therefore $n = 4$. We can also check that

$$\ell b_p = x^4 + \epsilon_4(x^3 y + x^2 y^2) + \ell g \notin \langle \ell b_1, \ldots, \ell^4 b_1, \ell b_2, \ell^2 b_2, \ell^3 b_2, \ell b_3, \ell^2 b_3, \ell b_4, \ell^2 b_4 \rangle,$$

but $\ell^2 b_p = x^4 y + \ell^2 g \in \langle \ell^3 b_2, \ell^4 \rangle$, so $(b_p, \ell b_p)$ is a new string of length 2, and therefore $p = 5$. However, we now see that the four elements $\ell^3 b_1$, $\ell^2 b_3$, $\ell^2 b_4$, and ℓb_5 are all in $\mathsf{m}^4 \setminus \mathsf{m}^5$, while $H(A)_4 = 3$, which contradicts the hypothesis that the basis is compatible with the Hilbert function.

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