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Geometry of ℓ_p-direct sums of normed linear spaces

Abstract. We consider ℓ_p-direct sums ($1 \leq p < \infty$) and c_0-direct sums of countably many normed spaces and find the dual of these spaces. We characterize the support functionals of arbitrary elements in these spaces to characterize smoothness and approximate smoothness, both locally and globally. These results let us answer the Chmieliński, Khurana, and Sain question raised in [4] on the existence of a non-approximately smooth normed space whose every element is smooth. We also characterize Birkhoff-James orthogonality and its pointwise symmetry in these spaces.

Introduction

The aim of the present article is to study the geometry of the normed linear spaces constructed by taking countably infinite ℓ_p direct sums (for $1 \leq p < \infty$) of normed linear spaces. We also consider c_0 analogues of the direct sums and find the duals of these spaces. We further characterize the support functionals of a non-zero element in these spaces. Consequently, we characterize smoothness and approximate smoothness in them and answer a question about the approximate smoothness of a space raised by Chmieliński, Khurana, and Sain in [4]. We finish by characterizing Birkhoff-James orthogonality and its pointwise symmetry in these spaces. A similar analysis was done for ℓ_p direct sums of a pair of normed linear spaces in [4] by Chmieliński, Khurana, and Sain, where the support functionals and approximate smoothness in these finite direct sums were studied. In this article, we consider ℓ_p-direct sums of countably many normed linear spaces and study

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support functionals, approximate smoothness, Birkhoff-James orthogonality, and its pointwise symmetry in countably infinite ℓ_p-direct sums.

Let us establish the relevant notations and terminologies to be used throughout the article. Throughout, $\mathbb{K}$ will denote the field of scalars (also called the *ground field*), which is either $\mathbb{R}$ or $\mathbb{C}$. Recall the sign function on $\mathbb{K}$ given by $\operatorname{sgn}\colon \mathbb{K} \to \mathbb{K}$,

$$\operatorname{sgn}(x) := \begin{cases} \frac{x}{|x|}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Given a normed linear space $\mathbb{X}$ over $\mathbb{K}$, let $B_{\mathbb{X}}$ denote the closed unit ball of the space. For any $C \subseteq \mathbb{X}$ convex, we denote the collection of all extreme points of C by $\operatorname{Ext}(C)$. Let $\mathbb{X}^*$ stand for the continuous dual of the space $\mathbb{X}$ and define the support functional of a non-zero element $x \in \mathbb{X}$ to be any $f \in \mathbb{X}^*$ such that

$$\|f\| = 1, \quad f(x) = \|x\|.$$

Let $J(x)$ denote the collection of support functionals of a non-zero x. Clearly, $J(x)$ is convex and weak* compact. The diameter of $J(x)$ for a non-zero x is denoted by $D(x)$. It is trivial to note that as $\|f\| = 1$ for every $f \in D(x)$, $0 \leq D(x) \leq 2$. We also define $\mathcal{D}(\mathbb{X})$ to be the supremum of $D(x)$ over all non-zero x, i.e.

$$D(x) := \operatorname{diam}(J(x)), \quad \mathcal{D}(\mathbb{X}) := \sup\{D(x) : \ x \in \mathbb{X} \setminus \{0\}\}.$$

A non-zero element $x \in \mathbb{X}$ is said to be *smooth* if it has a unique support functional. Hence, a non-zero x is smooth if and only if $D(x) = 0$. A normed space $\mathbb{X}$ is *smooth* if every non-zero element of the space is smooth, i.e. $\mathcal{D}(\mathbb{X}) = 0$.

A non-zero element $x \in \mathbb{X}$ is called *approximately ϵ-smooth* for $0 \leq \epsilon < 2$ if

$$D(x) \leq \epsilon.$$

A space $\mathbb{X}$ is called approximately ϵ-smooth for $0 \leq \epsilon < 2$ if

$$\mathcal{D}(\mathbb{X}) \leq \epsilon.$$

A non-zero point in a space or a space itself is called *approximately smooth* if the point or the space is approximately ϵ-smooth for some $0 \leq \epsilon < 2$ respectively. In [4], Chmieliński, Khurana, and Sain proved the following result pertaining to the approximate smoothness in finite-dimensional Banach spaces, which serves as a motivation for this work:

Theorem 1
If $\mathbb{X}$ is a finite-dimensional normed linear space, then $\mathbb{X}$ is smooth if and only if every non-zero element of $\mathbb{X}$ is approximately smooth.

In the same article, the authors raised the following two questions about possible generalizations of the above result, which we have answered negatively in our present work.

(1) If all non-zero elements in a normed linear space are approximately smooth, is the space approximately smooth as well?

(2) If all finite-dimensional subspaces of a normed linear space are approximately smooth, is the space approximately smooth as well?

Given two elements $x, y \in \mathbb{X}$, x is defined to be *Birkhoff-James orthogonal* to y [1], denoted by $x \perp_B y$ if

$$\|x + \lambda y\| \geq \|x\| \quad \text{for every scalar } \lambda.$$

James proved in [8] that $x \perp_B y$ if and only if $f(y) = 0$ for some support functional f of x. In the same article, he proved that a non-zero point $x \in \mathbb{X}$ is smooth if and only if Birkhoff-James orthogonality is right additive at x, i.e. for any $y, z \in \mathbb{X}$,

$$x \perp_B y,\ x \perp_B z \ \Rightarrow\ x \perp_B (y + z).$$

James proved in [7] that in a normed linear space of dimension 3 or more, Birkhoff-James orthogonality is symmetric if and only if the space is an inner product space. However, the importance of studying the point-wise symmetry of Birkhoff-James orthogonality in describing the geometry of normed linear spaces has been illustrated in [3, Theorem 2.11], [19, Corollary 2.3.4]. Let us recall the following definition in this context from [18], which will play an important part in our present study.

Definition 2
An element x of a normed linear space $\mathbb{X}$ is said to be *left-symmetric* (*resp. right-symmetric*) if

$$x \perp_B y \ \Rightarrow\ y \perp_B x \quad (\text{resp. } y \perp_B x \ \Rightarrow\ x \perp_B y),$$

for every $y \in \mathbb{X}$.

Note that we refer to the left-symmetric and right-symmetric points of a given normed linear space by the term *point-wise symmetry of Birkhoff-James orthogonality.* Birkhoff-James orthogonality and its pointwise symmetry have been the focus of tremendous research aimed at understanding the geometry of a normed space. We refer the readers to [2], [3],[5], [6], [10], [11], [12], [13], [15], [16], [17], [20] [19],[18], [21] for some of the prominent works in this direction.

A *semi-inner product* [9, Definition 1] on a $\mathbb{K}$ vector space $\mathbb{V}$ is defined to be a map $[\cdot,\cdot]\colon \mathbb{V} \times \mathbb{V} \to \mathbb{K}$ such that for $x, y, z \in \mathbb{V}$ and $\lambda \in \mathbb{K}$,

(1) $[x, x] \geq 0$ with equality if and only if $x = 0$.
(2) $[x, z] + \lambda[y, z] = [x + \lambda y, z]$.
(3) $|[x, y]|^2 \leq [x, x][y, y]$.

Clearly, if $\mathbb{V}$ is a semi-inner product space, then $\|x\| := [x, x]^{\frac{1}{2}}$ is a norm on $\mathbb{V}$. Further, a *semi-inner product on a Banach space* $\mathbb{X}$ is a map $[\cdot,\cdot]\colon \mathbb{X} \times \mathbb{X} \to \mathbb{K}$ satisfying the above three properties along with $[x, x] = \|x\|^2$ for every $x \in \mathbb{X}$. Construction of a semi-inner product on $\mathbb{X}$ [9, Theorem 2] follows by choosing a support functional f_y for every y and defining

$$[x, y] := \|y\| f_y(x).$$

We now note down a proposition that relates the semi-inner products on a Banach space with notions of smoothness and Birkhoff-James orthogonality.

Proposition 3
A non-zero point $x \in \mathbb{X}$ is smooth if and only if

$$[y,x]_1 = [y,x]_2 \quad \textit{for every } y \in \mathbb{X}, \tag{0.1}$$

for every pair $[\cdot,\cdot]_1$, $[\cdot,\cdot]_2$ of semi-inner products on $\mathbb{X}$. Also, for $x, y \in \mathbb{X}$, $x \perp_B y$ if and only if $[y,x] = 0$ for some semi-inner product $[\cdot,\cdot]$ on $\mathbb{X}$.

Proof. Observe that $y \mapsto \frac{1}{\|x\|}[y,x]$ is a support functional of x and from the construction of semi-inner products in [9, Theorem 2], for any $f \in J(x)$, there exists a semi-inner product $[\cdot,\cdot]$ on $\mathbb{X}$ such that $[y,x] = \|x\| f(y)$.

Now, for every pair of semi-inner products $[\cdot,\cdot]_1$, $[\cdot,\cdot]_2$ and $x \neq 0$, $[y,x]_1 = [y,x]_2$ if and only if $\frac{1}{\|x\|}[y,x]_1 = \frac{1}{\|x\|}[y,x]_2$ if and only if x has a unique support functional, i.e. x is smooth.

Again, $[y,x] = 0$ for some semi-inner product $[\cdot,\cdot]$ on $\mathbb{X}$ if and only if $x = 0$ or $\|x\| f(y) = 0$ for some $f \in J(x)$. Hence, by the James' characterization of Birkhoff-James orthogonality, $[y,x] = 0$ for some semi-inner product $[\cdot,\cdot]$ on $\mathbb{X}$ if and only if $x \perp_B y$.

We now define a notion of pointwise symmetry of semi-inner products.

Definition 4
Let $1 < p < \infty$. A point $x \in \mathbb{X}$ is said to be *p-left (resp.p-right) s.i.p. commuting* with $y \in \mathbb{X} \setminus \{0\}$ if given any semi-inner product $[\cdot,\cdot]$, there exists a semi-inner product $[\cdot,\cdot]'$ such that

$$[x,y]' = \left| \frac{[y,x]}{\|x\|\|y\|} \right|^{p-2} \overline{[y,x]} \quad (\textit{resp.}\ [x,y] = \left| \frac{[y,x]'}{\|x\|\|y\|} \right|^{p-2} \overline{[y,x]'}).$$

A point $x \in \mathbb{X}$ is said to be *p-left (resp. p-right) s.i.p. symmetric* if x is p-left (resp. p-right) s.i.p. commuting with every $y \in \mathbb{X}$. Also, if $x \in \mathbb{X}$ is said to be *left (resp. right) s.i.p. symmetric* if x is 2-left (resp. 2-right) s.i.p. symmetric.

We can immediately note the following proposition relating s.i.p. symmetry with pointwise symmetry of Birkhoff-James orthogonality.

Proposition 5
If $x \in \mathbb{X}$ is p-left (resp. right) s.i.p. symmetric, for some $1 < p < \infty$, then x is a left-symmetric (resp. right-symmetric) point.

Proof. We use the fact from Proposition 3 that $x \perp_B y$ if and only if $[y,x] = 0$ for some semi-inner product $[\cdot,\cdot]$ on $\mathbb{X}$. First assume that $x \in \mathbb{X}$ is p-left s.i.p.-symmetric for some $1 < p < \infty$. Let $x \perp_B y$ for some $y \in \mathbb{X}$. If $y = 0$, then clearly $y \perp_B x$. Otherwise, $[y,x] = 0$ for some semi-inner product $[\cdot,\cdot]$ on $\mathbb{X}$. However, by p-left s.i.p. commuting property of x with respect to y, there exists a semi-inner product $[\cdot,\cdot]'$ on $\mathbb{X}$ such that

$$[x,y]' = \left| \frac{[y,x]}{\|x\|\|y\|} \right|^{p-2} \overline{[y,x]} = 0.$$

Hence, $y \perp_B x$. The proof for the p-right symmetric case follows similarly.

In the first section, we define the ℓ_p-direct sums $(1 \leq p < \infty)$ and c_0-direct sums of normed spaces and characterize the duals of these spaces. In the second section, we characterize the support functionals of these elements and obtain values of diameters of any point in the space characterizing approximate smoothness completely. We also use these results to answer a question raised by Chmieliński, Khurana, and Sain in [4]. In the final section, we characterize Birkhoff-James orthogonality and its pointwise symmetry in these spaces.

1. ℓ_p-direct sums

We now recall the definition of the ℓ_p direct sums of a sequence of normed linear spaces. [22, Section 2.3.]

Throughout, $\mathbb{X}_n$ would denote a sequence of non-trivial (of dimension at least one) normed linear spaces. We define the following direct sums:

Definition 6
Let $1 \leq p < \infty$. Then the ℓ_p-*direct sum* of $\mathbb{X}_n$ is defined as:

$$\bigoplus_p \mathbb{X}_n := \left\{ \{x_n\}_{n\in\mathbb{N}} : \ x_n \in \mathbb{X}_n, \ \sum_{n=1}^{\infty} \|x_n\|^p < \infty \right\}.$$

Also define:

$$\bigoplus_\infty \mathbb{X}_n := \left\{ \{x_n\}_{n\in\mathbb{N}} : \ x_n \in \mathbb{X}_n, \ \sup_{n\geq 1} \|x_n\| < \infty \right\},$$
$$\bigoplus_0 \mathbb{X}_n := \left\{ \{x_n\}_{n\in\mathbb{N}} : \ x_n \in \mathbb{X}_n, \ \lim_{n\to\infty} \|x_n\| = 0 \right\}.$$

We define the norms for these spaces as follows:

Definition 7
Let $1 \leq p < \infty$. Then for $\{x_n\}_{n\in\mathbb{N}} \in \bigoplus_p \mathbb{X}_n$, define:

$$\|\{x_n\}_{n\in\mathbb{N}}\| := \left(\sum_{n=1}^{\infty} \|x_n\|^p \right)^{\frac{1}{p}}.$$

Similarly, for $\{x_n\}_{n\in\mathbb{N}} \in \bigoplus_\infty \mathbb{X}_n$, define

$$\|\{x_n\}_{n\in\mathbb{N}}\|_\infty := \sup_{n\geq 1} \|x_n\|.$$

We also define for $\{x_n\}_{n\in\mathbb{N}} \in \bigoplus_0 \mathbb{X}_n$,

$$\|\{x_n\}_{n\in\mathbb{N}}\|_0 := \max_{n\geq 1} \|x_n\|.$$

Proposition 8
For $p \in [1,\infty] \cup \{0\}$, $\left(\bigoplus_p \mathbb{X}_n, \|\cdot\|_p\right)$ is a normed linear space. Also, it is a Banach space if and only if $\mathbb{X}_n$ is a Banach space for every $n \in \mathbb{N}$.

Proof. The first statement follows from the sub-additivity of the norms and Minkowski's inequality. Also, if any $\{x_k^n\}_{k\in\mathbb{N}} \subset \mathbb{X}_n$ is a Cauchy sequence which is not convergent, then defining $\{z^k\}_{k\in\mathbb{N}} \subset \bigoplus_p \mathbb{X}_n$ as:

$$z^k := \{z_j^k\}_{j\in\mathbb{N}}, \quad z_j^k := \begin{cases} x_k^n, & j = n, \\ 0, & \text{otherwise}, \end{cases}$$

gives a Cauchy sequence in $\bigoplus_p \mathbb{X}_n$ which is not convergent.

Now, assume all the $\mathbb{X}_n$ to be Banach spaces. Let

$$x^k = \{x_n^k\}_{n\in\mathbb{N}} \in \bigoplus_p \mathbb{X}_n$$

be such that $\{x^k\}_{k\in\mathbb{N}}$ is a Cauchy sequence. Then clearly as $|x^k|_p \geq |x_n^k|$ for every $k, n \in \mathbb{N}$, $\{x_n^k\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{X}_n$. Let $y_n := \lim_{k\to\infty} x_n^k$. Let $p \in [1,\infty)$. Now fix $N \in \mathbb{N}$. Then for $k, j > K$ for K sufficiently large,

$$\sum_{n=1}^{N} \|x_n^k - x_n^j\|^p \leq \sum_{n=1}^{\infty} \|x_n^k - x_n^j\|^p = \|x^k - x^j\|_p^p < \epsilon^p.$$

Taking $j \to \infty$, we get

$$\sum_{n=1}^{N} \|x_n^k - y_n\|^p < \epsilon^p \tag{1.1}$$

Hence we have

$$\left(\sum_{n=1}^{N} \|y_n\|^p\right)^{\frac{1}{p}} \leq \left(\sum_{n=1}^{N} \|x_n^k - y_n\|^p\right)^{\frac{1}{p}} + \left(\sum_{n=1}^{N} \|x_n^k\|^p\right)^{\frac{1}{p}} < \epsilon + \|x^k\|_p.$$

Taking $N \to \infty$ now yields $y = \{y_n\}_{n\in\mathbb{N}} \in \bigoplus_p \mathbb{X}_n$. Now taking $N \to \infty$ in (1.1), we get that for $k > K$,

$$\|x^k - y\|_p < \epsilon.$$

For $p = \infty, 0$, the proof proceeds similarly.

We now characterize the dual of $\bigoplus_p \mathbb{X}_n$ for $p = [1,\infty) \cup \{0\}$.

Theorem 9
Let $p = [1,\infty)$. Let $q \in (1,\infty]$ such that $\frac{1}{p} + \frac{1}{q} = 1$. For $p = 0$, set $q = 1$ by definition. Then the dual of $\bigoplus_p \mathbb{X}_n$ is isometrically isomorphic to $\bigoplus_q \mathbb{X}_n^*$ with $f = \{f_n\}_{n\in\mathbb{N}} \in \bigoplus_q \mathbb{X}_n^*$ acting on $\bigoplus_p \mathbb{X}_n$ as

$$f(x) := \sum_{n=1}^{\infty} f_n(x_n), \quad x = \{x_n\}_{n\in\mathbb{N}} \in \bigoplus_p \mathbb{X}_n.$$

Proof. First let $1 < p < \infty$. Clearly for $f = \{f_n\}_{n\in\mathbb{N}} \in \bigoplus_q \mathbb{X}_n^*$, we have by Holder's inequality,

$$\left|\sum_{n=1}^{\infty} f_n(x_n)\right| \leq \sum_{n=1}^{\infty} \|f_n\|\|x_n\| \leq \|f\|_q\|x\|_p,$$

for any $x = \{x_n\}_{n\in\mathbb{N}} \in \bigoplus_p \mathbb{X}_n$. Further for $\epsilon > 0$, find $x_n \in S_{\mathbb{X}_n}$ such that

$$f_n(x_n) > \|f_n\| - \frac{\epsilon\|f\|_q^{q-1}}{2^n\|f_n\|^{q-1}},$$

whenever $f_n \neq 0$ (such an x_n is guaranteed to exist as we can find $y_n \in S_{X_n}$ such that $|f_n(y_n)| > \|f_n\| - \frac{\epsilon\|f\|_q^{q-1}}{2^n\|f_n\|^{q-1}}$ and now taking $x_n = \overline{\operatorname{sgn}(f_n(y_n))}y_n$ gives the desired inequality) and take $x_n = 0$ otherwise. Then for $y = \{y_n\}_{n\in\mathbb{N}}$ given by $y_n = \frac{\|f_n\|^{q-1}}{\|f\|_q^{q-1}}x_n$ is an element of the unit sphere of $\bigoplus_p \mathbb{X}_n$. Further,

$$f(y) = \sum_{n=1}^{\infty} \frac{\|f_n\|^{q-1}}{\|f\|_q^{q-1}} f_n(x_n) > \sum_{n=1}^{\infty} \frac{\|f_n\|^{q-1}}{\|f\|_q^{q-1}}\left(\|f_n\| - \frac{\epsilon\|f\|_q^{q-1}}{2^n\|f_n\|^{q-1}}\right) = \|f\|_q - \epsilon.$$

Hence f is a continuous functional on $\bigoplus_p \mathbb{X}_n$.

Now let ψ be a continuous functional on $\bigoplus_p \mathbb{X}_n$. Then for any $n \in \mathbb{N}$,

$$x \mapsto \psi(xe_n), \quad x \in \mathbb{X}_n,$$

is a bounded linear functional on $\mathbb{X}_n$. Let us denote this functional by $\psi_n \in \mathbb{X}_n^*$. Again consider $x_n \in S_{\mathbb{X}_n}$ such that

$$\psi_n(x_n) > \|\psi_n\| - \frac{\epsilon}{2^{\frac{n}{p}}}.$$

Then for any $\{c_n\}_{n\in\mathbb{N}} \in \ell_p$, we have

$$\sum_{n=1}^{\infty} \|\psi_n\||c_n| < \sum_{n=1}^{\infty} \psi\left(\sum_{n=1}^{\infty} c_n x_n e_n\right) + \epsilon\|\{c_n\}_{n\in\mathbb{N}}\|_p < \infty.$$

Hence $\{\|\psi_n\|\}_{n\in\mathbb{N}} \in \ell_q$. The proofs for $p = 1, 0$ cases follow similarly.

2. Geometry of ℓ_p direct sums

Again, let $\mathbb{X}_n$ be a given sequence of normed spaces. Since our aim is to characterize the smoothness and approximate smoothness, We begin by characterizing the support functionals of a non-zero element of $\bigoplus_p \mathbb{X}_n$.

Theorem 10
Let $p \in (1,\infty)$ and let q be its conjugate. Then we have for any non-zero $x = \{x_n\}_{n\in\mathbb{N}} \in \bigoplus_p \mathbb{X}_n$, we have for any $f = \{f_n\}_{n\in\mathbb{N}} \in \bigoplus_q {\mathbb{X}_n}^*$, $f \in J(x)$ if and only if

$$\frac{\|x\|_p^{p-1}}{\|x_n\|^{p-1}} f_n \in J\left(x_n\right), \tag{2.1}$$

if $x_n \neq 0$ and $f_n = 0$ otherwise.

Proof. The sufficiency follows from elementary computations. For the necessity, observe that by Hölder's inequality

$$\begin{aligned} \|x\|_p = f(x) = \sum_{n=1}^{\infty} f_n x_n &\leq \sum_{n=1}^{\infty} \|f_n\|\|x_n\| \\ &\leq \left(\sum_{n=1}^{\infty} \|f_n\|^q\right)^{\frac{1}{q}} \left(\sum_{n=1}^{\infty} x_n\|^p\right)^{\frac{1}{p}} = \|f\|_q\|x\|_p. \end{aligned} \tag{2.2}$$

Hence, equality must hold in the two above inequalities. From the first inequality, we get, since $f_n(x_n) \leq \|f_n\|\|x_n\|$ for every $n \in \mathbb{N}$,

$$f_n(x_n) = \|f_n\|\|x_n\| \quad \text{for every } n \in \mathbb{N}.$$

Hence $\frac{f_n}{\|f_n\|} \in J(x_n)$ of $x_n \neq 0$. Also, from the second inequality, by the condition of equality in Hölder's inequality, we get:

$$\frac{\|f_n\|^q}{\|x_n\|^p} = \frac{\|f\|_q^q}{\|x\|_p^p} = \frac{1}{\|x\|_p^p} \;\Rightarrow\; \|f_n\| = \frac{\|x_n\|^{p-1}}{\|x\|^{p-1}},$$

for every $n \in \mathbb{N}$. Combining the two results yields (2.1).

Recall that for any non-zero x in a normed space, $D(x) := \operatorname{diam}(J(x))$. We now find $D(x)$ for any $x \in \bigoplus_p \mathbb{X}_n$.

Theorem 11
Let $p \in (1,\infty)$. For $x = \{x_n\}_{n\in\mathbb{N}} \in \bigoplus_p \mathbb{X}_n$,

$$D(x) = \left(\sum_{n=1}^{\infty} \frac{\|x_n\|^p}{\|x\|_p^p} (D(x_n))^q\right)^{\frac{1}{q}}. \tag{2.3}$$

Proof. Let $f^1 = \{f_n^1\}_{n\in\mathbb{N}}$, $f^2 = \{f_n^2\}_{n\in\mathbb{N}} \in J(x)$. Then $\frac{\|x\|_p^{p-1}}{\|x_n\|^{p-1}} f_n^i \in J(x_n)$ whenever $x_n \neq 0$ for $i = 1, 2$. Hence

$$\|f_n^1 - f_n^2\| \leq \frac{\|x_n\|^{p-1}}{\|x\|_p^{p-1}} D(x_n).$$

Hence

$$\|f^1 - f^2\|_q^q = \sum_{n=1}^{\infty} \|f_n^1 - f_n^2\|^q \leq \sum_{n=1}^{\infty} \frac{\|x_n\|^p}{\|x\|_p^p}(D(x_n))^q.$$

Taking supremum over $f^1, f^2 \in J(x)$ now yields

$$D(x) \leq \left(\sum_{n=1}^{\infty} \frac{\|x_n\|^p}{\|x\|_p^p}(D(x_n))^q\right)^{\frac{1}{q}}.$$

Again, fix $\epsilon > 0$. Find $f_n^1, f_n^2 \in J(x_n)$ such that

$$\|f_n^1 - f_n^2\|^q > (D(x_n))^q - \frac{\|x\|_p^p}{\|x_n\|^p}\frac{\epsilon}{2^n}.$$

Set $f^i = \left\{\frac{\|x_n\|^{p-1}}{\|x\|_p^{p-1}} f_n^i\right\}$ for $i = 1, 2$. Hence we have $f^1, f^2 \in J(x)$. However,

$$\|f^1 - f^2\|^q = \sum_{n=1}^{\infty} \frac{\|x_n\|^p}{\|x\|_p^p}\|f_n^1 - f_n^2\|^q \geq \left(\sum_{n=1}^{\infty} \frac{\|x_n\|^p}{\|x\|_p^p}(D(x_n))^q\right) - \epsilon.$$

Since $\epsilon > 0$ is arbitrary, (2.3) follows.

Corollary 12
An non-zero element $x = \{x_n\}_{n\in\mathbb{N}} \in \bigoplus_p \mathbb{X}_n$ is smooth if and only if $x_n \in \mathbb{X}_n$ is smooth whenever $x_n \neq 0$.

Recall that $\mathcal{D}(\mathbb{X})$ is defined to be $\sup\{D(x) : \ x \in \mathbb{X} \setminus \{0\}\}$. We can, therefore, conclude:

Corollary 13
Given normed linear spaces $\mathbb{X}_n$,

$$\mathcal{D}\Big(\bigoplus_p \mathbb{X}_n\Big) = \sup_{n\geq 1} \mathcal{D}(\mathbb{X}_n).$$

Proof. Clearly, if $x = \{x_n\} \in \bigoplus_p \mathbb{X}_n$, has $x_n = 0$ for every $n \neq n_0$, then $D(x) = D(x_{n_0})$. Hence $\mathcal{D}(\bigoplus_p \mathbb{X}_n) \geq \sup_{n\geq 1} \mathcal{D}(x_n)$. The reverse inequality follows directly from equation (2.3) of Theorem 11.

Note that this answers the question raised by Chmieliński, Khurana, and Sain in [4]: *Is there a Banach space $\mathbb{X}$ such that $D(x) < 2$ for every non-zero element x but $\mathcal{D}(\mathbb{X}) = 2$?* It was shown that no finite-dimensional Banach space has this property in [4]. Here, we show the existence of an infinite dimensional Banach space having this property.

Proposition 14
Let $\mathbb{X}_n$ be a sequence of Banach spaces such that $\mathcal{D}(\mathbb{X}_n) \in (2-\frac{1}{n}, 2)$. (For example, we can choose $\mathbb{X}_n$ to be suitable two-dimensional polygonal spaces as was shown in [4].) Then for any $p \in (1, \infty)$, for every $x \in \bigoplus_p \mathbb{X}_n$, $D(x) < 2$ but $\mathcal{D}\Big(\bigoplus_p \mathbb{X}_n\Big) = 2$.

Proof. Since $\mathcal{D}(\mathbb{X}_n) \in (2 - \frac{1}{n}, 2)$, for every $x_n \in \mathbb{X}_n$, $D(x_n) < 2$. Hence, for any $x = \{x_n\} \in \bigoplus_p \mathbb{X}_n$, by Theorem 11,

$$D(x) = \left(\sum_{n=1}^{\infty} \frac{\|x_n\|^p}{\|x\|_p^p} (D(x_n))^q\right)^{\frac{1}{q}} < \left(\sum_{n=1}^{\infty} \frac{\|x_n\|^p}{\|x\|_p^p} 2^q\right)^{\frac{1}{q}} = 2.$$

Thus, every non-zero element of $\bigoplus_p \mathbb{X}_n$ is approximately smooth but clearly,

$$\mathcal{D}\Big(\bigoplus_p \mathbb{X}_n\Big) = \sup_{n\geq 1} \mathcal{D}(x_n) \geq \sup_{n\geq 1} 2 - \frac{1}{n} = 2$$

from Corollary 13.

The above proposition, along with Theorem 1, also answers the second question on approximate smoothness raised in [4].

Corollary 15
If $\mathbb{X}_n$ is a sequence of Banach spaces such that $\mathcal{D}(\mathbb{X}_n) \in (2 - \frac{1}{n}, 2)$, then $\bigoplus_p \mathbb{X}_n$ has the property that any finite-dimensional subspace of this space is approximately smooth but the space itself is not.

We finish this section by characterizing the support functionals of elements of $\bigoplus_1 \mathbb{X}_n$ and $\bigoplus_0 \mathbb{X}_n$.

Theorem 16
For $x = \{x_n\}_{n\in\mathbb{N}} \in \bigoplus_1 \mathbb{X}_n$, $f = \{f_n\}_{n\in\mathbb{N}} \in \bigoplus_\infty \mathbb{X}_n$ is a support functional of x if and only if $f_n \in J(x_n)$ if $x_n \neq 0$ and $\|f_n\| \leq 1$ if $x_n = 0$.

For $x = \{x_n\}_{n\in\mathbb{N}} \in \bigoplus_0 \mathbb{X}_n$, $f = \{f_n\}_{n\in\mathbb{N}} \in \bigoplus_1 \mathbb{X}_n$ is a support functional of x if and only if $f_n \in \lambda_n J(x_n)$ if $\|x_n\| = \|x\|_0$ and $f_n = 0$ otherwise such that $\lambda_n \geq 0$, $\sum_{\|x_n\|=\|x\|_0} \lambda_n = 1$.

Proof. We first consider the case of $\bigoplus_1 \mathbb{X}_n$. Note that we have for $f \in J(x)$, $f(x) \leq \sum_{n=1}^{\infty} \|f_n\|\|x_n\|$ following similar computations as in (2.2). Further, by a similar application of Hölder's inequality for $p = 1, q = \infty$, we get

$$\|x\|_1 = f(x) \leq \sum_{n=1}^{\infty} \|f_n\|\|x_n\| \leq \left(\sum_{n=1}^{\infty} \|x_n\|\right)\Big(\sup_{n\geq 1} \|f_n\|\Big) = \|f\|_\infty \|x\|_1.$$

Hence equality must hold in all the inequalities. So, following the same argument as in Theorem 10, we get $f_n(x_n) = \|f_n\|\|x_n\|$ for every $n \in \mathbb{N}$ and $\|f_n\| = \|f\|_\infty$

whenever $x_n \neq 0$. Hence, for $x_n \neq 0$, $f_n(x_n) = \|f_n\|\|x_n\| = \|f\|_\infty\|x_n\| = \|x_n\|$, giving $f_n \in J(x_n)$. Further, if $x_n = 0$, then $\|f_n\| \leq \|f\|_\infty = 1$. The converse of course, follows by verifying $f(x) = \|x\|$ by directly computation and noting the fact that $\|f\|_\infty = \sup_{n\geq 1}\|f\| = 1$, since $x_n \neq 0$ for at least one $n \in \mathbb{N}$.

For the case of $\bigoplus\limits_0 \mathbb{X}_n$, we have for f support functional of x, by a similar application of Hölder's inequality: $\|x\|_0 = f(x) \leq \sum_{n=1}^\infty \|f_n\|\|x_n\| \leq \|x\|_0\|f\|_1 = \|x\|_0$. Hence, as equality holds in both the inequalities, by a similar argument as before, $f_n(x_n) = \|f_n\|\|x_n\|$ for every $n \in \mathbb{N}$ and $f_n = 0$ if $\|x_n\| < \|x\|_0$. Hence, setting $\lambda_n = \|f_n\|$ whenever $x_n = \|x\|_0$, we get $\sum_{\|x_n\|=\|x\|_0}\lambda_n = \sum_{\|x_n\|=\|x\|_0}\|f_n\| = \sum_{n=1}^\infty\|f_n\| = \|f\|_1 = 1$. Also, $f_n(x_n) = \|f_n\|\|x_n\| = \lambda_n x_n$ if $\|x_n\| = \|x\|_0$. So $\frac{f_n}{\lambda_n} \in J(x_n)$ if $\lambda_n \neq 0$, giving $f_n \in \lambda_n J(x_n)$ whenever $\|x_n\| = \|x\|_0$. The converse follows by verifying $f(x) = \|x\|$ and $\|f\| = 1$ through direct computation from the given conditions.

We can now easily compute the value of $D(x)$ for any non-zero $x \in \bigoplus\limits_1 \mathbb{X}_n$.

Theorem 17
Let $x = \{x_n\}_{n\in\mathbb{N}} \in \bigoplus\limits_1 \mathbb{X}_n$ be non-zero. Then

$$D(x) = \begin{cases} \sup_{n\geq 1} D(x_n), & \text{if } x_n \neq 0 \text{ for every } n \in \mathbb{N},\\ 2, & \text{otherwise.}\end{cases}$$

Also for a non-zero $x = \{x_n\}_{n\in\mathbb{N}} \in \bigoplus\limits_0 \mathbb{X}_n$,

$$D(x) = \begin{cases} D(x_n), & \text{if } \|x_k\| = \|x\|_0 \text{ if and only if } k = n,\\ 2, & \text{otherwise.}\end{cases}$$

Proof. We first consider the case of $\bigoplus\limits_1 \mathbb{X}_n$. In the first case, if $x_n = 0$ for some $n \in \mathbb{N}$, then consider $f = \{f_n\}_n \in \bigoplus\limits_\infty \mathbb{X}_n^*$ support functional of x. We now have that both $\hat{f}$ and $\hat{f}'$ are support functionals of x where $\hat{f}_k = \hat{f}'_k = f_k$ for every $k \neq n$ and $\hat{f}_n = h_n$ $\hat{f}'_n = -h_n$ for some $h_n \in S_{\mathbb{X}_n^*}$. So, $2 \geq D(x) \geq \|\hat{f} - \hat{f}'\| \geq \|h_n - (-h_n)\| = 2$. In the second case of course, for any support functional $f = \{f_n\}_n \in \bigoplus\limits_\infty \mathbb{X}_n^*$ of x, $f_n \in J(x_n)$ giving

$$\begin{aligned} D(x) = \sup_{f,g\in J(x)}\|f-g\|_\infty &= \sup_{f,g\in J(x)}\sup_{n\geq 1}\|f_n - g_n\| \\ &= \sup_{f_n,g_n\in J(x_n)}\|f_n - g_n\| = \sup_{n\geq 1} D(x_n). \end{aligned}$$

In the case of $\bigoplus\limits_0 \mathbb{X}_n$, if $\|x_n\| = \|x_k\| = \|x\|_0$, then for any $f_n \in J(x_n)$ and $f_k \in J(x_k)$, $f_n e_n$ and $f_k e_k$ are support functionals of x giving $2 \leq D(x) \leq \|f_n e_n - f_k e_k\| = \|f_n\| + \|f_k\| = 2$. If $\|x_k\| = \|x\|_0$ if and only if $k = n$, then the only support functionals of x are of the form $f e_n$ for $f \in J(x_n)$. Hence

$$D(x) = \sup_{F,G\in J(x)}\|F - G\| = \sup_{f,g\in J(x_n)}\|f - g\| = D(x_n).$$

The characterization of smoothness therefore follows.

Corollary 18

A point $x = \{x_n\}_{n\in\mathbb{N}} \in \bigoplus_1 \mathbb{X}_n$ *is smooth if and only if* $x_n \in \mathbb{X}_n$ *is smooth for every* $n \in \mathbb{N}$.

A point $x = \{x_n\}_{n\in\mathbb{N}} \in \bigoplus_0 \mathbb{X}_n$ *is smooth if and only if there exists a natural number* n_0 *such that* $\|x_n\| < \|x\|_0$ *for every* $n \neq n_0$ *and* $x_{n_0} \in \mathbb{X}_{n_0}$ *is smooth.*

Also, $\bigoplus_p \mathbb{X}_n$ is not approximately smooth for $p = 1, 0$.

Corollary 19

For any sequence of normed linear spaces $\mathbb{X}_n$,

$$\mathcal{D}\Big(\bigoplus_1 \mathbb{X}_n\Big) = \mathcal{D}\Big(\bigoplus_0 \mathbb{X}_n\Big) = 2.$$

3. Birkhoff-James orthogonality in ℓ_p-direct sums

We begin the section by characterizing the set of extreme points of $J(x)$ (denoted by $\operatorname{Ext}(J(x))$ for any $x \in \bigoplus_p \mathbb{X}_n$. Recall that an extreme point of a convex set C is a point x having the property $x = \frac{1}{2}(x_1 + x_2)$ for $x_1, x_2 \in C$ if and only if $x_1 = x = x_2$.

Proposition 20

Let $x = \{x_n\}_{n\in\mathbb{N}} \in \bigoplus_p \mathbb{X}_n$ *and let* $f = \{f_n\}_{n\in\mathbb{N}} \in \bigoplus_q \mathbb{X}_n^*$, *where* $p = [1,\infty) \cup \{0\}$ *and* q *is the conjugate index of* p. *Then*

(1) *If* $p \in (1,\infty)$ *then* $f \in \operatorname{Ext}(J(x))$ *if and only if*

$$\frac{\|x\|_p^{p-1}}{\|x_n\|^{p-1}} f_n \in \operatorname{Ext}(J(x_n)) \quad \textit{for every } x_n \neq 0.$$

(2) *If* $p = 1$, *then* $f \in \operatorname{Ext}(J(x))$ *if and only if*

$$f_n \in \operatorname{Ext}(J(x_n)) \textit{ if } x_n \neq 0 \quad \textit{and} \quad f_n \in \operatorname{Ext}(B_{\mathbb{X}_n}) \textit{ otherwise.}$$

(3) *If* $p = 0$, *then* $f \in \operatorname{Ext}(J(x))$ *if and only if there exists* $n_0 \in \mathbb{N}$ *such that:*

$$\|x_{n_0}\| = \|x\|_0, \quad f_{n_0} \in \operatorname{Ext}(J(x_{n_0})), \quad f_n = 0 \textit{ for } n \neq n_0.$$

Proof. We prove the result for the case $1 < p < \infty$. The argument for the other two cases are similar. Note that if $f \in J(x) \setminus \operatorname{Ext}(J(x))$, then $f = \frac{1}{2}(f^1 + f^2)$ for some distinct $f^1, f^2 \in J(x)$. Clearly as $x_n = 0$ forces $f_n^1 = f_n^2 = 0$, there exists $n \in \mathbb{N}$ such that $f_n^1 \neq f_n^2$ and $x_n \neq 0$. Hence we have

$$\frac{\|x\|_p^{p-1}}{\|x_n\|^{p-1}} f_n = \frac{1}{2}\bigg(\frac{\|x\|_p^{p-1}}{\|x_n\|^{p-1}} f_n^1 + \frac{\|x\|_p^{p-1}}{\|x_n\|^{p-1}} f_n^2\bigg), \quad \frac{\|x\|_p^{p-1}}{\|x_n\|^{p-1}} f_n^1, \frac{\|x\|_p^{p-1}}{\|x_n\|^{p-1}} f_n^2 \in J(x_n).$$

So, $\frac{\|x\|_p^{p-1}}{\|x_n\|^{p-1}} f_n \notin \text{Ext}(J(x_n))$. Again, if $f \in J(x)$ such that $\frac{\|x\|_p^{p-1}}{\|x_n\|^{p-1}} f_n \notin \text{Ext}(J(x_n))$, find distinct $g_n, h_n \in J(x_n)$ such that $\frac{\|x\|_p^{p-1}}{\|x_n\|^{p-1}} f_n = \frac{1}{2}(g_n + h_n)$. Construct functionals g and h by replacing f_n in f by $\frac{\|x_n\|^{p-1}}{\|x\|_p^{p-1}} g$ and $\frac{\|x_n\|^{p-1}}{\|x\|_p^{p-1}} h$ respectively, we get two support functionals of x (by Theorem 10) such that $f = \frac{1}{2}(g+h)$, proving $f \notin \text{Ext}(J(x))$.

Our next step is characterizing Birkhoff-James orthogonality in these spaces. However, we first prove a lemma pertaining to the images of convex compact subsets of a Hausdorff locally convex topological vector space under linear maps.

Lemma 21
Let $\mathbb{X}$ be a Hausdorff locally convex topological vector space and let $K \subset \mathbb{X}$ be convex and compact. If V is a finite dimensional Banach space and $T \colon \mathbb{X} \to V$ is a continuous linear map, then

$$T(K) = \overline{\text{conv}}\{T(x) : \; x \in \text{Ext}(K)\}.$$

Proof. Note that as T is linear and continuous, $T(K)$ must be compact and convex. Now, by the Krein-Milman theorem [14, Section 3.23], $T(K)$ must be the closed convex hull of the extreme points of $T(K)$. So, it suffices to show that

$$\text{Ext}\{T(K)\} \subseteq \{T(x) : \; x \in \text{Ext}(K)\}.$$

Now, fix an extreme point z of $T(K)$ and let $S := \{x \in K : \; T(x) = z\}$. Note that for $x_1, \; x_2 \in S$ and $\lambda \in [0,1]$, $\lambda x_1 + (1-\lambda)x_2 \in K$ by the convexity of S and by the linearity of T,

$$T(\lambda x_1 + (1-\lambda)x_2) = \lambda T(x_1) + (1-\lambda)T(x_2) = \lambda z + (1-\lambda)z = z.$$

So, S is convex. Further, since $\{z\} \subset V$ is closed, S is a closed subset of K and hence is compact. So, by the Krein-Milman theorem, S must have an extreme point. Let x be an extreme point of S. We claim that x is an extreme point of K, which would conclude the proof. To that end, consider $x_1, x_2 \in K$ such that $x = \frac{1}{2}(x_1 + x_2)$. Then

$$\frac{1}{2}(T(x_1) + T(x_2)) = T(x) = z.$$

Since $T(x_1), T(x_2) \in T(K)$ and z is an extreme point of $T(K)$, we must have $T(x_1) = T(x_2) = z$. But then $x_1, x_2 \in S$ and since x is an extreme point of S, $x_1 = x_2 = x$.

Now we characterize Birkhoff-James orthogonality in $\bigoplus_p \mathbb{X}_p$ for $p \in [1, \infty) \cup \{0\}$.

Theorem 22
Let $x = \{x_n\}_{n\in\mathbb{N}},\ y = \{y_n\}_{n\in\mathbb{N}} \in \bigoplus_p \mathbb{X}_n$. *Then*

(1) *If* $p \in (1,\infty)$, *then* $x \perp_B y$ *if and only if* $x = 0$ *or*

$$0 \in \overline{\operatorname{conv}}\left\{ \sum_{n=1}^{\infty} \|x_n\|^{p-1} f_n(y_n) :\ f_n \in \operatorname{Ext}(J(x_n)) \right\}.$$

(2) *If* $p = 1$, *then* $x \perp_B y$ *if and only if* $x = 0$ *or*

$$\left| \sum_{x_n \neq 0} f_n(y_n) \right| \leq \sum_{x_n = 0} \|y_n\|$$

for some $f_n \in J(x_n)$, *where* $x_n \neq 0$.

(3) *if* $p = 0$, *then* $x \perp_B y$ *if and only if* $x = 0$ *or*

$$0 \in \overline{\operatorname{conv}}\{ f_n(y_n) :\ f_n \in \operatorname{Ext}(J(x_n)),\ \|x_n\| = \|x\|_0 \}.$$

Proof. Observe that $J(x)$ is weak* compact and convex, and $\Phi\colon f \mapsto f(y),\ f \in J(x)$ is linear and continuous on $J(x)$ under the weak* topology. Since the image of Φ is $\mathbb{K}$, the ground field, by Lemma 21, $\{f(y) :\ f \in J(x)\}$ is the closed convex hull of the set $\{f(y) :\ f \in \operatorname{Ext}(J(x)\}$. Now, by James characterization of Birkhoff-James orthogonality, $x \perp_B y$ if and only if $x = 0$ or there exists $f \in J(x)$ such that $f(y) = 0$. Hence, $x \perp_B y$ if and only if $0 \in \overline{\operatorname{conv}}\{f(y) :\ f \in \operatorname{Ext}(J(x))\}$.

The result now follows by recalling the characterizations of the extreme points of $J(x)$ for $x \in \bigoplus_p \mathbb{X}_n$ for $1 < p < \infty$ and $p = 0$. For example, we have for $1 < p < \infty$,

$$\begin{aligned}\{f(y) :\ f \in \operatorname{Ext}(J(x))\} &= \left\{ \sum_{x_n \neq 0} g_n(y_n) :\ \frac{\|x\|_P^{p-1}}{\|x_n\|^{p-1}} g_n \in J(x_n) \right\} \\ &= \left\{ \frac{1}{\|x\|_p^p} \sum_{n=1}^{\infty} \|x_n\|^{p-1} f_n(y_n) :\ f_n \in J(x_n) \right\}.\end{aligned}$$

For $p = 1$, if $x \perp_B y$ and $f = \{f_n\}_{n\in\mathbb{N}} \in J(x)$ is such that $f(y) = 0$, then $f_n \in J(x_n)$ if $x_n \neq 0$ and

$$\sum_{n=1}^{\infty} f_n(y_n) = 0$$

$$\Downarrow$$

$$\sum_{x_n \neq 0} f_n(y_n) = -\sum_{x_n = 0} f_n(y_n)$$

$$\Downarrow$$

$$\left| \sum_{x_n \neq 0} f_n(y_n) \right| = \left| \sum_{x_n = 0} f_n(y_n) \right| \leq \sum_{x_n = 0} |f_n(y_n)| \leq \sum_{x_n = 0} \|f_n\| \|y_n\| \leq \sum_{x_n = 0} \|y_n\|.$$

The converse follows trivially since we can find a support functional $g = \{g_n\}_{n\in\mathbb{N}}$ of x such that $g(y) = 0$ by taking $g_n = f_n$ if $x_n \neq 0$ and $g_n = c\psi_n$

for $x_n = 0$ where $\psi_n \in J(y_n)$ and

$$c = -\frac{\sum\limits_{x_n \neq 0} f_n(y_n)}{\sum\limits_{x_n = 0} \|y_n\|}.$$

Clearly, by the given criteria $|c| \leq 1$ giving $g \in J(x)$ and $g(y) = 0$. So, $x \perp_B y$.

We also express the $p \in (1, \infty)$ case in terms of semi-inner products.

Corollary 23
For $1 < p < \infty$ and $x = \{x_n\}_{n\in\mathbb{N}},\ y = \{y_n\}_{n\in\mathbb{N}} \in \bigoplus\limits_p \mathbb{X}_n$, $x \perp_B y$ if and only if there exist semi-inner products $[\cdot,\cdot]_n$ in $\mathbb{X}_n$ such that

$$\sum_{n=1}^{\infty} \|x_n\|^{p-2} [y_n, x_n]_n = 0.$$

Proof. Note that by James characterization $x \perp_B y$ if and only if $f(x) = y$ for some support functional f of x. By characterization of the support functional in Theorem 10, $f = \{f_n\}_{n\in\mathbb{N}} \in \bigoplus\limits_q \mathbb{X}_n^*$, such that $\frac{\|x\|_p^{p-1}}{\|x_n\|^{p-1}} f_n$ is a support functional of x_n whenever $x_n \neq 0$. Construct semi-inner products $[\cdot,\cdot]_n$ on $\mathbb{X}_n$ such that $[z_n, x_n] = \|x_n\| \frac{\|x\|_p^{p-1}}{\|x_n\|^{p-1}} f_n(z_n)$ whenever $x_n \neq 0$. If $x_n = 0$, we can choose $[\cdot,\cdot]_n$ to be any semi-inner product on $\mathbb{X}_n$. Now, we have

$$f(y) = \sum_{n=1}^{\infty} f_n(y_n) = \sum_{n=1}^{\infty} \frac{\|x_n\|^{p-2}}{\|x\|_p^{p-1}} [y_n, x_n]_n \ \Rightarrow \sum_{n=1}^{\infty} \|x_n\|^{p-2} [y_n, x_n]_n = 0.$$

For the converse, set $f_n(z_n) := \frac{\|x_n\|^{p-2}}{\|x\|_p^{p-1}} [z_n, x_n]_n$ $z_n \in \mathbb{X}_n$. Then whenever $x_n \neq 0$, $\frac{\|x\|_p^{p-1}}{\|x_n\|^{p-1}} f_n$ is a support functional of x_n. Hence $f = \{f_n\}_{n\in\mathbb{N}} \in \bigoplus_q \mathbb{X}_n^*$ is a support functional of x such that $f(y) = x$, i.e. $x \perp_B y$.

Let us also note the following fact which will be used later.

Corollary 24
Let $x = \{x_n\}_{n\in\mathbb{N}} \in \bigoplus_p \mathbb{X}_n$ be such that $x_n = 0$ for every $n \neq n_0$. Then for any $y = \{y_n\}_{n\in\mathbb{N}} \in \bigoplus\limits_p \mathbb{X}_n$,

(1) *If $p \in (1, \infty)$, $x \perp_B y$ if and only if $x_{n_0} \perp_B y_{n_0}$ and $y \perp_B x$ if and only if $y_{n_0} \perp_B x_{n_0}$.*
(2) *If $p = 1$, $x \perp_B y$ if and only if*

$$\min\{|f_{n_0}(y_{n_0})| :\ f_{n_0} \in J(x_{n_0})\} \leq \sum_{n \neq n_0} \|y_n\|.$$

Also, $y \perp_B x$ if and only if $y_{n_0} \perp_B x_{n_0}$.

(3) *If* $p = 0$, $x \perp_B y$ *if and only if* $x_{n_0} \perp_B y_{n_0}$ *and* $y \perp_B x$ *if and only if* $\|y_n\| = \|y\|_0$ *for some* $n \neq n_0$ *or* $y_{n_0} \perp_B x_{n_0}$.

Proof. We do the three cases one after another.

(1) By Corollary 23, $x \perp_B y$ if and only if $[y_{n_0}, x_{n_0}]_{n_0} = 0$ for some semi-inner product $[\cdot,\cdot]_{n_0}$ on $\mathbb{X}_{n_0}$, i.e. $x_{n_0} \perp_B y_{n_0}$. Similarly, by Corollary 23, $y \perp_B x$ if and only if $[y_{n_0}, x_{n_0}]_{n_0} = 0$ for some semi-inner product $[\cdot,\cdot]_{n_0}$ in $\mathbb{X}_{n_0}$, i.e. $y_{n_0} \perp_B x_{n_0}$.

(2) From Theorem 22, $x \perp_B y$ if and only if $|f_{n_0}(y_{n_0})| \leq \sum_{n\neq n_0} \|y_n\|$ for some $f_{n_0} \in J(x_{n_0})$. But since $J(x_{n_0})$ is weak* compact, this is equivalent to $\min\{|f_{n_0}(y_{n_0})| : f_{n_0} \in J(x_{n_0})\} \leq \sum_{n\neq n_0} \|y_n\|$.
Again, from Theorem 22, $y \perp_B x$ if and only if for some $f_n \in J(y_n)$,

$$\left|\sum_{y_n\neq 0} f_n(x_n)\right| \leq \sum_{y_n=0} \|x_n\|. \tag{3.1}$$

Now, if $y_{n_0} = 0$, (3.1) and $y_{n_0} \perp_B x_{n_0}$ both follow trivially. Otherwise, the right hand side of (3.1) is zero and hence $y \perp_B x$ if and only if $f_{n_0}(x_{n_0}) = 0$ for some $f_{n_0} \in J(y_{n_0})$, i.e. $y_{n_0} \perp_B x_{n_0}$ (by the James' characterization of Birkhoff-James orthogonality).

(3) From Theorem 16, for any $f \in J(x)$, $f(y) = f_{n_0}(y_{n_0})$ for some $f_{n_0} \in J(x_{n_0})$. Hence by James' characterization, $x \perp_B y$ if and only if $x_{n_0} \perp_B y_{n_0}$. Again, $f(x) = f_{n_0}(x_{n_0})$ for any $f = \{f_n\}_{n\in\mathbb{N}} \in \left(\bigoplus_0 \mathbb{X}_n\right)^*$. Hence, by Theorem 16, $f(x) = 0$ for some support functional f of y if $\|y_n\| = \|y\|_0$ for some $n \neq n_0$ or $f_{n_0}(x_{n_0})$ for some $f_{n_0} \in J(x_{n_0})$, i.e. $x_{n_0} \perp_B y_{n_0}$.

We now proceed toward characterizing the pointwise symmetry of Birkhoff-James orthogonality in the space. We first characterize the left and right symmetric points for $\bigoplus_1 \mathbb{X}_n$ and $\bigoplus_2 \mathbb{X}_n$.

Theorem 25
The space $\bigoplus_1 \mathbb{X}_n$ *has no non-zero left-symmetric point and* $x = \{x_n\}_{n\in\mathbb{N}}$ *is a right-symmetric point if and only if there exists* $n_0 \in \mathbb{N}$ *such that*

$$x_n = 0 \text{ whenever } n \neq n_0 \quad \text{and} \quad x_{n_0} \in \mathbb{X}_{n_0} \text{ is right-symmetric.}$$

Proof. Let x be a non-zero left symmetric point of $\bigoplus_1 \mathbb{X}_n$. If $x_n = 0$, find $y \in \mathbb{X}_n$ such that $\|y\| > \|x\|_1$. Then considering $z = \{z_n\}_{n\in\mathbb{N}} \in \bigoplus_1 \mathbb{X}_n$ given by $z_k = x_k$ if $k \neq n$ and $z_n = y$, we clearly get from Theorem 22,

$$x \perp_B z, \quad z \not\perp_B x.$$

Now, if $x_n \neq 0$ for every n, find M such that

$$\sum_{n=1}^{M} \|x_n\| \neq \sum_{n=M+1}^{\infty} \|x_n\|.$$

Consider $z = \{z_n\}_{n\in\mathbb{N}} \in \bigoplus_1 \mathbb{X}_n$ given by

$$z_n = \begin{cases} \frac{x_n}{\|x_n\|}, & 1 \le n \le M, \\ \frac{Mx_n}{2^{n-M}\|x_n\|}, & n > M. \end{cases}$$

Then again by Theorem 22, we get $x \perp_B z$ but $z \not\perp_B x$.

Again, if x is a right-symmetric point of $\bigoplus_1 \mathbb{X}_n$ and $x_1, x_2 \neq 0$, then without loss of generality, we may assume that $\|x_1\| \le \|x_2\|$. Now consider $y = \{y_n\}_{n\in\mathbb{N}} \in \bigoplus_1 \mathbb{X}_n$ given by $y_1 = x_1$ and $y_n = 0$ otherwise. Then clearly, $x \not\perp_B y$ but $y \perp_B x$ by Theorem 22. Further, if x has only one non-zero component x_{n_0} and x is right-symmetric, then for any $y \in \bigoplus_1 \mathbb{X}_n$, with only non-zero component y_{n_0},

$$x \perp_B y \;\Leftrightarrow\; x_n \perp_B y_n \quad \text{and} \quad y \perp_B x \;\Leftrightarrow\; y_n \perp_B x_n.$$

Hence x_{n_0} is right-symmetric. The converse, however, follows trivially from Theorem 22 and Corollary 24.

We now come to the case $p = 2$.

Theorem 26
Let $x = \{x_n\}_{n\in\mathbb{N}} \in \bigoplus_2 \mathbb{X}_n$. Then

(1) *x is left-symmetric if and only if $x_n \in \mathbb{X}_n$ is left s.i.p. symmetric (see Definition 4) for every $n \in \mathbb{N}$ or there exists $n_0 \in \mathbb{N}$ such that $x_n = 0$ if $n \neq n_0$ and x_{n_0} is a left-symmetric point of $\mathbb{X}_{n_0}$.*
(2) *x is right-symmetric if and only if $x_n \in \mathbb{X}_n$ is right s.i.p. symmetric for every $n \in \mathbb{N}$ or there exists $n_0 \in \mathbb{N}$ such that $x_n = 0$ if $n \neq n_0$ and x_{n_0} is a right-symmetric point of $\mathbb{X}_{n_0}$.*

Proof. From Corollary 23, we get that for any $y = \{y_n\}_{n\in\mathbb{N}} \in \bigoplus_2 \mathbb{X}_n$, $x \perp_B y$ if and only if

$$\sum_{n=1}^{\infty} [y_n, x_n]_n = 0, \tag{3.2}$$

for some sequence of semi-inner products $[\cdot,\cdot]_n$ on $\mathbb{X}$. The sufficiency for both the parts now follows from (3.2). To prove the necessity, we assume a contradiction. We prove the result for the left-symmetric case, and proof for the right-symmetric case follows similarly. Without loss of generality, we therefore assume that there exist $z_1 \in \mathbb{X}_1$ and a semi-inner product $[\cdot,\cdot]_1$ such that

$$[z_1, x_1]_1 \neq \overline{[x_1, z_1]} \quad \text{for every semi-inner product } [\cdot,\cdot].$$

If $x_m \neq 0$ for some $m \neq 1$ then define $y_\alpha = \{y_n\}_{n\in\mathbb{N}} \in \bigoplus_2 \mathbb{X}_n$ for $\alpha \in \mathbb{C}$ by

$$y_n = \begin{cases} z_1, & n = 1, \\ \alpha x_m, & n = m, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, $x \perp_B y_\alpha$ if

$$[z_1, x_1]_1 + \alpha \|x_n\|^2 = 0, \tag{3.3}$$

But $y_\alpha \perp_B x$ if and only if

$$[z_1, x_1] + \overline{\alpha} \|x_n\|^2 = 0, \tag{3.4}$$

for some semi-inner product $[\cdot, \cdot]$ on $\mathbb{X}_1$. Since we can solve for α from (3.3), (3.4) gives the desired contradiction. However, if $x_n = 0$ for all but one $n = n_0 \in \mathbb{N}$, we have $x \perp_B y$ if and only if $x_{n_0} \perp_B y_{n_0}$ and $y \perp_B x$ if and only if $y_{n_0} \perp_B x_{n_0}$. So as x is left symmetric, so is x_{n_0}.

We now proceed to the case $p = 0$.

Theorem 27
The space $\bigoplus_0 \mathbb{X}_n$ has no non-zero right-symmetric point. A point $x = \{x_n\}_{n \in \mathbb{N}} \in \bigoplus_0 \mathbb{X}_n$ is left symmetric if and only if there exists $n_0 \in \mathbb{N}$ such that $x_n = 0$ whenever $n \neq n_0$ and x_{n_0} is a left-symmetric point of $\mathbb{X}_{n_0}$.

Proof. For the right-symmetric part, let $x = \{x_n\}_{n \in \mathbb{N}} \in \bigoplus_0 \mathbb{X}_n$ be right-symmetric and $\|x_1\| = \|x\|_0 \neq 0$ without loss of generality. Now if $m \neq n$, then if $\|x_m\| < \|x\|_0$, we consider $y = \{y_n\}_{n \in \mathbb{N}} \in \bigoplus_0 \mathbb{X}_n$ given by

$$y_n = \begin{cases} -\frac{x_m}{\|x_m\|}, & \text{if } n = m, \\ \frac{x_n}{\|x_n\|}, & \text{if } \|x_n\| = \|x\|_0, \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 22 now yields that $y \perp_B x$ but $x \not\perp_B y$. Hence $\|x_n\| = \|x\|_0$ for every $n \in \mathbb{N}$. Since $x \in \bigoplus_0 \mathbb{X}_n$, $x = 0$.

The sufficiency for the left-symmetric case follows from Corollary 24. For the necessity, we assume $\|x_1\| = \|x\|_0$ without loss of generality. Then if $x_m \neq 0$ for some $m \neq 1$, find $y_1 \in \mathbb{X}_1$ and $y_m \in \mathbb{X}_m$ such that $x_1 \perp_B y_1$, $\|y_1\| = 1$ and $y_m \not\perp_B x_m$, $\|y_m\| = 2$. We now define $y = \{y_n\}_{n \in \mathbb{N}} \in \bigoplus_0 \mathbb{X}_n$ by setting $y_n = 0$ for $n \neq 1, m$ yields $x \perp_B y$ but $y \perp_B x$. Hence $x_m = 0$ for every $m > 1$. That x_1 is left-symmetric again follows from Corollary 24.

We finish the section with the $p \in (1, \infty) \setminus \{2\}$ case.

Theorem 28
Let $x = \{x_n\}_{n \in \mathbb{N}} \in \bigoplus_p \mathbb{X}_n$. Then

(1) *If x is left-symmetric, either there exists $n_0 \in \mathbb{N}$ such that $x_n = 0$ for $n \neq n_0$ and x_{n_0} is a left-symmetric point of $\mathbb{X}_n$ or there exist $m, k \in \mathbb{N}$ such that $x_n = 0$ if $n \neq m, k$, $\|x_m\| = \|x_k\|$, and x_m, x_k are smooth p-left s.i.p. symmetric points of $\mathbb{X}_m$, $\mathbb{X}_k$, respectively.*

(2) *If x is right-symmetric, either there exists $n_0 \in \mathbb{N}$ such that $x_n = 0$ for $n \neq n_0$ and x_{n_0} is a right-symmetric point of $\mathbb{X}_n$ or there exist $m, k \in \mathbb{N}$ such that $x_n = 0$ if $n \neq m, k$, $\|x_m\| = \|x_k\|$, and $x_m,\ x_k$ are p-right s.i.p. symmetric points of $\mathbb{X}_m,\ \mathbb{X}_k$, respectively.*

Proof. The sufficiency in both the cases follows directly from computation. We establish the necessity for the left-symmetric case and the proof for the right-symmetric case follows in a similar way.

Observe that by Corollary 24, if x is left-symmetric, $x_n \in \mathbb{X}_n$ is left-symmetric for every $n \in \mathbb{N}$. Further, if there are more than one non-zero components of x, without loss of generality, we can assume that $x_1,\ x_2 \neq 0$. If $\|x_1\| > \|x_2\|$, then for $\alpha, \beta > 0$, define $y_\alpha := \{y_n\}_{n\in\mathbb{N}} \in \bigoplus_p \mathbb{X}_n$ given by

$$y_n = \begin{cases} x_1, & n = 1, \\ -\alpha x_2, & n = 2, \\ 0, & \text{otherwise.} \end{cases}$$

Hence, by Corollary 23, $x \perp_B y_\alpha$ if and only if $\|x_1\|^p = \alpha\|x_2\|^p$ and $y_\alpha \perp_B x$ if and only if $\|x_1\|^p = \alpha^{p-1}\|x_2\|^p$ giving the desired contradiction as $p \neq 2$. Further if additionally $x_3 \neq 0$, we can assume that $\|x_1\| = \|x_2\| = \|x_3\|$. Now for $\alpha,\ \beta \in \mathbb{K}$, set $z_{\alpha\beta} = \{z_n\}_{n\in\mathbb{N}} \in \bigoplus_p \mathbb{X}_n$ given by

$$z_n = \begin{cases} x_1, & n = 1, \\ \alpha x_2, & n = 2, \\ \beta x_3, & n = 3, \\ 0, & \text{otherwise.} \end{cases}$$

Thus by Corollary 23, $x \perp_B z_{\alpha\beta}$ if and only if

$$\|x_1\|^2 + \alpha\|x_2\|^2 + \beta\|x_3\|^2 = 0 \quad \Rightarrow \quad 1 + \alpha + \beta = 0.$$

Also, $z_{\alpha\beta} \perp_B$ if and only if

$$1 + |\alpha|^{p-1}\overline{\operatorname{sgn}(\alpha)} + |\beta|^{p-1}\overline{\operatorname{sgn}(\beta)} = 0.$$

Setting $\alpha = \beta = -\frac{1}{2}$, yields that x cannot be left-symmetric. Thus, without loss of generality, we assume $\|x_1\| = \|x_2\| \neq 0$ and $x_n = 0$ for $n > 2$. Let $\mathcal{O}_1 := \{y_1 \in \mathbb{X}_1 :\ x_1 \not\perp_B y_1\}$. Then $\mathcal{O}_1$ is an open subset of $\mathbb{X}_1$. Since the collection of smooth points is dense in $\mathbb{X}_1$, find $y_1 \in \mathcal{O}_1$ smooth. Then for any semi-inner product $[\cdot,\cdot]$ on $\mathbb{X}_1$,

$$[y_1, x_1] + \alpha\|x_2\|^2 = 0 \quad \Rightarrow \quad x \perp_B y,$$

where y_1 is as chosen before, $y_2 = \alpha x_2$ and $y_n = 0$ otherwise. But then $y \perp_B x$ giving

$$\|y_1\|^{p-2}[x_1, y_1]' + |\alpha|^p \frac{\|x_2\|^p}{\alpha} = 0,$$

for some semi-inner product $[\cdot,\cdot]'$ on $\mathbb{X}_1$. Hence,

$$\|y_1\|^{p-2}[x_1, y_1]' = |[y_1, x_1]|^p \frac{\|x_2\|^{2-p}}{[y_1, x_1]}.$$

Since y_1 is smooth, $[x_1, y_1]'$ is unique for any semi-inner product $[\cdot,\cdot]'$ on $\mathbb{X}_1$. Hence $[y_1, x_1]$ is uniquely determined on the region $\{y_1 : \ y_1 \in \mathcal{O}_1, \ y_1 \text{ smooth}\}$ for any semi-inner product $[\cdot,\cdot]$ on $\mathbb{X}_1$. Hence $[y_1, x_1]$ is uniquely determined for $y_1 \in \mathcal{O}_1$. Since $\operatorname{Span}(\mathcal{O}_1) = \mathbb{X}_1$, $[y_1, x_1]$ is uniquely determined on $\mathbb{X}_1$ showing that x_1 is smooth. Further, given $y_1 \in \mathbb{X}_1$, there exists a semi-inner product $[\cdot,\cdot]'$ on $\mathbb{X}_1$ such that

$$(\|x_1\|\|y_1\|)^{p-2}[y_1, x_1][x_1, y_1]' = |[y_1, x_1]|^p \ \Rightarrow \ [x_1, y_1]' = \left|\frac{[y_1, x_1]}{\|x_1\|\|y_1\|}\right|^{p-2}\overline{[y_1, x_1]},$$

finishing the proof for the left-symmetric case. The necessity for the right-symmetric case follows similarly.

Note that considering the case $\mathbb{X}_n = \mathbb{K}$, we can obtain the characterizations of smooth points, left-symmetric points, and right-symmetric points in the sequence Lebesgue spaces ℓ_p $(1 < p < \infty)$ and c_0. Since, in this case, every point is smooth, and p-left and p-right s.i.p. symmetric for every $1 < p < \infty$, the characterizations are the same as were found by Bose, Roy, and Sain in [2].

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