Paradigm

Ann. Univ. Paedagog. Crac. Stud. Math. 25 (2025), 53-75
DOI: 10.2478/aupcsm-2025-0007

FOLIA 427

Annales Universitatis Paedagogicae Cracoviensis
Studia Mathematica 25 (2025)

Mohamed Dilmi and Bilal Basti

Asymptotic behavior of generalized self-similar solutions for a nonlinear hybrid problem of porous medium equations

Abstract. The present paper investigates the asymptotic behavior of positive generalized self-similar solutions for a nonlinear hybrid problem involving n^{th}-order derivative porous medium equations. We provide sufficient conditions for the existence and uniqueness of weak solutions that have compact support and dynamic characteristics. Furthermore, we establish the behavior of these solutions by examining a specific set of variables and their signs, which must meet certain conditions to determine whether the solutions exist globally or locally in time.

1. Introduction

Partial differential equations (PDEs) model and analyze several phenomena across various disciplines, including physics, mechanics, and chemistry. This discussion focuses on a well-established class of equations, a powerful tool for describing diffusion phenomena. Specifically, we focus on a nonlinear PDE termed the nonlinear porous media equation, distinguished by its n^{th}-order derivative, which can be expressed as follows:

$$\frac{\partial \omega}{\partial t} = \frac{\partial^n \omega^m}{\partial x^n}, \qquad m \in (1, \infty),\, n \geq 3. \tag{1}$$

Here, the scalar function $\omega = \omega(x,t)$ is nonnegative and depends on the spatial variables $x \in [0, \infty)$, and temporal parameter $t > 0$.

AMS (2020) Subject Classification: 35A01, 35B06, 35B44, 35C06, 35G30.
Keywords and phrases: porous medium equation, generalized self-similar solution, blow-up, global existence, uniqueness.
ISSN: 2081-545X, e-ISSN: 2300-133X.

For certain types of PDEs exhibiting symmetry properties, it is possible to derive exact solutions through the application of specific transformations that convert the given PDE into an ordinary differential equation (ODE) (see, for instance, [11, 12, 13]). These solutions, obtained under such circumstances, are sometimes called self-similar solutions (see [2, 3, 6, 9, 15]). They often occupy a central role in analysing PDEs and addressing their solutions, whether locally or globally, in terms of time.

The well-known heat equation corresponds to the case where $m = 1$ and $n = 2$, but the nonlinear range for $m > 1$ actually has rather different characteristics. The porous medium equation ([1, 5, 9, 10]) can be expressed in the divergence form for the case where $n = 2$, and it can describe a large category of nonlinear parabolic equations, as referenced in [14].

We investigate the existence, uniqueness, and behavior of positive solutions to a nonlinear hybrid problem of porous medium equations (1) under the generalized self-similar form

$$\omega(x,t) = \varphi(t)u(z) \qquad \text{for } z = \frac{x}{\psi(t)} \text{ with } \varphi, \psi \in \mathbb{R}_+^*. \tag{2}$$

The non-negative functions φ and ψ, initially unknown and dependent on the time variable t, must be determined.

2. Existence and uniqueness results

Before we can analyze the existence and uniqueness of the generalized self-similar solutions, it is necessary to determine the equation satisfied by the function u in equation (2).

2.1. Basic principle for converting PDE to ODE

When expression (2) is substituted into the original PDE (1), the resulting equation should be reduced to the following functional-differential equation, (as documented in [13]),

$$\dot{\varphi}(t)u(z) - \varphi(t)\frac{\dot{\psi}(t)}{\psi^2(t)}xu'(z) = \frac{\varphi^m(t)}{\psi^n(t)}(u^m)^{(n)}(z). \tag{3}$$

By expressing x in terms of z through (2), substituting it into equation (3), and subsequently dividing the resulting expression by φ, we get the functional equation with respect to two variables t and z, as outlined below

$$\frac{\dot{\varphi}(t)}{\varphi(t)}u(z) - \frac{\dot{\psi}(t)}{\psi(t)}zu'(z) - \frac{\varphi^{m-1}(t)}{\psi^n(t)}(u^m)^{(n)}(z) = 0.$$

The above functional equation can be rewritten as

$$f_1g_1 + f_2g_2 + f_3g_3 = 0,$$

where

$$f_1 = \frac{\dot{\varphi}}{\varphi}, \quad f_2 = \frac{\dot{\psi}}{\psi}, \quad f_3 = -\frac{\varphi^{m-1}}{\psi^n},$$

and

$$g_1 = u, \quad g_2 = -zu', \quad g_3 = (u^m)^{(n)}.$$

The system of ordinary differential equations is obtained by substituting these expressions into the solution of the three-term functional equation (refer to [2, 3, 4, 13])

$$\begin{cases} \frac{\dot{\varphi}}{\varphi} = q\frac{\varphi^{m-1}}{\psi^n}, \\ \frac{\dot{\psi}}{\psi} = p\frac{\varphi^{m-1}}{\psi^n}, \\ (u^m)^{(n)} = qu - pzu', \end{cases} \tag{4}$$

where q and p are arbitrary constants.

The functions φ and ψ are required to be conditionally determined because the precedent system of ODEs (4) relies on several unknown parameters.

2.2. Necessary definitions and preliminary results

In this section, we investigate the existence and uniqueness of positive solutions to the following problem involving porous medium equations of n^{th}-order derivative

$$\begin{cases} \frac{\partial \omega}{\partial t} = \frac{\partial^n \omega^m}{\partial x^n}, & 0 \le x < \infty, \\ & \forall m > 1,\ n \ge 3, \\ \omega(0,t) = \lambda\varphi(t), & \lambda \ge 0,\ t > 0, \\ \omega(b\psi(t),t) = 0,\ \frac{\partial^k \omega^m}{\partial x^k}(b\psi(t),t) = 0,\ k = 1,\dots,(n-1), & b > 0,\ \forall m > 1 \\ \omega(x,0) = u(x), & \forall x \in [0, b\psi(t)), \end{cases} \tag{5}$$

under the generalized self-similar form

$$\omega(x,t) = \varphi(t)u(z) \qquad \text{with } z = \frac{x}{\psi(t)}.$$

In consideration of the preceding part (system (4)), we investigate the following problem

$$\begin{cases} (u^m)^{(n)}(z) = qu(z) - pzu'(z), & z \ge 0,\ m \in (1,\infty),\ n \ge 3, \\ u(0) = \lambda, & \lambda \ge 0, \\ u(b) = 0,\ (u^m)^{(k)}(b) = 0,\ k = 1,\dots,(n-1), & b > 0, \end{cases} \tag{6}$$

where p and q are arbitrary real constants.

As in [6], it is necessary to define the notion of weak solutions for (6).

Definition 1 (Nontrivial Weak Solution)
A function u will be said to be a weak solution to the problem (6) if:

1. The function u is continuous, bounded and nonnegative on the interval $[0, b)$ and vanishes on $[b, \infty)$.

2. The function u^m has $n - 1$ continuous derivatives with respect to z on the interval $[0, \infty)$.

3. The function u satisfies the following identity

$$-\int_0^\infty (u^m)^{(n-1)}(s)v'(s)ds = \int_0^\infty psu(s)v'(s)ds + (p+q)\int_0^\infty u(s)v(s)ds$$

for any $v \in C_0^1(0,\infty)$.

We now shall be mainly concerned with proving the existence and uniqueness of a positive solution for problem (6) which satisfies the conditions

$$u(0) = \lambda, \quad u > 0 \text{ on } (0,b) \quad \text{and} \quad u \equiv 0 \text{ on } [b,\infty),$$

where b is an arbitrary positive number.

In order to establish the existence of a nontrivial weak solution to the problem (6) with compact support, we will initially address the prerequisites necessary for parameters p and q.

Theorem 2
Assume that $u = u(z)$ is a nontrivial weak solution of (6) with compact support on $[0,b]$ and that u is positive in a left neighborhood of b. Then, the following conditions hold

a. *n is even and*
   i. *$p > 0$ or*
   ii. *$p = 0$ and $q > 0$.*

b. *n is odd and*
   i. *$p < 0$ or*
   ii. *$p = 0$ and $q < 0$.*

Proof. Let us consider that u represents a nontrivial weak solution to the problem (6) with compact support. It follows that there exists some $\varepsilon > 0$ such that

$$u \begin{cases} > 0 & \text{in } (b-\varepsilon, b), \\ = 0 & \text{in } [b,\infty), \end{cases} \qquad b > 0.$$

Iteratively integrating the first equation of (6) $(n-1)$ times, starting from z to b, yields

$$\begin{aligned} (-1)^{n-1}(u^m)'(z) = {} & \frac{pz}{(n-3)!}\int_z^b (s-z)^{n-3}u(s)ds \\ & + \frac{(n-1)p+q}{(n-2)!}\int_z^b (s-z)^{n-2}u(s)ds. \end{aligned} \tag{7}$$

The continuity of u and u' to the left of b ensures the existence of $z^* \in (b-\varepsilon, b)$ such that $u'(z^*) < 0$. Applying the Mean Value Theorem on the interval $(b-\varepsilon, b)$, we obtain

$$\exists z^* \in (b-\varepsilon, b) \text{ such that } u'(z^*) = \frac{u(b)-u(b-\varepsilon)}{b-(b-\varepsilon)} = -\frac{u(b-\varepsilon)}{\varepsilon} < 0.$$

a. If n is even, then the left-hand side of (7) at $z = z^*$ is positive for $m \in (1, \infty)$. Consequently, both p and $(n-1)p+q$ cannot be negative. Furthermore, when $p = 0$, it implies that $q > 0$. Since u is positive and tends to zero as z approaches b, the Mean Value Theorem allows us to establish the existence of $b-\varepsilon \leq a < b$, for which it holds that $u'(z) < 0$ for all $z \in (a, b)$. Let us consider $p < 0$, for all $z \in (a, b)$, this implies that $(n-1)p + q > 0$, this implies that $q > 0$. It results from (7) that

$$-mu'(z)u^{m-1}(z) = \frac{pz}{(n-3)!}\int_z^b (s-z)^{n-3}u(s)ds + \frac{(n-1)p+q}{(n-2)!}\int_z^b (s-z)^{n-2}u(s)ds \leq \Big(\frac{pz}{(n-3)!} + \frac{(n-1)p+q}{(n-2)!}(b-z)\Big)\int_z^b (s-z)^{n-3}u(s)ds.$$

In fact, we will use

$$\forall z \in [a, b)\ \forall s \in (z, b), \text{ we have } u(s) < u(z),$$

to get

$$-mu'(z)u^{m-2}(z) \leq \Big(\frac{pz}{(n-3)!} + \frac{(n-1)p+q}{(n-2)!}(b-z)\Big)\int_z^b (s-z)^{n-3}\frac{u(s)}{u(z)}ds \leq \frac{b^{n-2}}{n-2}\Big(\frac{pz}{(n-3)!} + \frac{(n-1)p+q}{(n-2)!}(b-z)\Big).$$

If we let z tend to b in the last inequality we get a contradiction, which implies that $p > 0$.

b. If n is odd, we would use the same techniques as in the previous case to prove that u is a solution defined on a compact support only when p is a negative coefficient, or when $p = 0$ and q takes a strictly negative value.

Therefore, nontrivial weak solutions with compact support for the boundary value problem (6) exist only in the particular cases:

$$p > 0, \text{ or } (p = 0 \text{ and } q > 0) \text{ if } n \text{ is even},$$

or

$$p < 0 \text{ or } (p = 0 \text{ and } q < 0) \text{ if } n \text{ is odd}.$$

The proof is complete.

Following the previous theorem, if we set $p = 0$, we find that the boundary value problem (6) admits nontrivial weak solutions with compact support when

$q > 0$ and n is even, or $q < 0$ and n is odd. In these cases, the following proposition presents a family of new exact solutions of the form

$$u(z) = C(b - z)^{\alpha}, \tag{8}$$

where C and α are constants to be determined.

Proposition 3
Let $p = 0$, $m > 1$, and $n \geq 3$. The boundary value problem (6) admits an exact solution of the form

$$u(z) = \Big[(-1)^n q \frac{\Gamma(\frac{mn}{m-1} - n + 1)}{\Gamma(\frac{mn}{m-1} + 1)}\Big]^{\frac{1}{m-1}} (b - z)^{\frac{n}{m-1}},$$

where b is determined by the boundary condition $u(0) = \lambda$.

Proof. Assume a solution of the form (8). Then we have

$$\begin{aligned}(u^m)^{(n)}(z) &= C^m \frac{d^n}{dz^n}(b - z)^{m\alpha} \\ &= C^m(-1)^n(m\alpha)(m\alpha - 1)\cdots(m\alpha - n + 1)(b - z)^{m\alpha - n}.\end{aligned}$$

Using the property of Gamma function,

$$\Gamma(\beta + 1) = \beta\Gamma(\beta) \qquad \text{for all } \beta > 0,$$

we obtain

$$\Gamma(m\alpha + 1) = (m\alpha)(m\alpha - 1)\cdots(m\alpha - n + 1)\Gamma(m\alpha - n + 1),$$

and thus,

$$(u^m)^{(n)}(z) = C^m(-1)^n \frac{\Gamma(m\alpha + 1)}{\Gamma(m\alpha - n + 1)}(b - z)^{m\alpha - n}.$$

Substituting into the first equation of problem (6) yields

$$C^m(-1)^n \frac{\Gamma(m\alpha + 1)}{\Gamma(m\alpha - n + 1)}(b - z)^{m\alpha - n} = qC(b - z)^{\alpha}.$$

We determine that the expression is valid if

$$\alpha = \frac{n}{m - 1} \quad \text{and} \quad C = \Big[(-1)^n q \frac{\Gamma(m\alpha - n + 1)}{\Gamma(m\alpha + 1)}\Big]^{\frac{1}{m-1}}.$$

Therefore, u as defined in (8) satisfies the first equation of problem (6). Moreover, at $z = 0$, we obtain

$$u(0) = Cb^{\alpha} = \lambda \qquad \text{for any } \lambda > 0,$$

and at $z = b$, we observe that $u(b) = 0$ is automatically satisfied. Further, $(u^m)^{(k)}(b) = 0$ for $k = 1, \dots, n - 1$, are satisfied since

$$(u^m)^{(k)}(z) = C^m(-1)^k \frac{\Gamma(m\alpha + 1)}{\Gamma(m\alpha - k + 1)}(b - z)^{m\alpha - k}$$

vanishes at $z = b$ whenever $m\alpha - k > 0$, which holds for $m > 1$ and $n \geq 3$. Consequently, we can rewrite the solution as

$$u(z) = C(b - z)^{\frac{n}{m-1}},$$

where b is explicitly determined by

$$b = \lambda^{\frac{m-1}{n}} \left[(-1)^n q \frac{\Gamma(m\alpha - n + 1)}{\Gamma(m\alpha + 1)}\right]^{-\frac{1}{n}} \qquad \text{for any } \lambda > 0.$$

This completes the proof.

The following lemma is for the case n is even, then p is positive.

Lemma 4

Let n be even, p positive, $a \in [0, b)$, and let u be a nontrivial weak solution to the problem (6) on $[a, b)$, then we separate the following cases:

1. *If $(n-1)p + q > 0$, then $u'(z) < 0$.*
2. *We have*
   $u'(0) > 0$, when $(n-1)p + q < 0$,
   $u'(0) = 0$, when $(n-1)p + q = 0$,
   $u'(0) < 0$, when $(n-1)p + q > 0$.
3. *Assume $(n-1)p + q < 0$, and there exists $z_0 \in [a, b)$ such that $u'(z_0) = 0$, then we get $(u^m)^{(n)}(z_0) < 0$ and u has a maximum at $z_0 \leq \frac{(n-1)p+q}{p+q} b$.*

Proof. If n is even, equation (7) gives

$$-(u^m)'(z) = \frac{pz}{(n-3)!} \int_z^b (s-z)^{n-3} u(s) ds + \frac{(n-1)p + q}{(n-2)!} \int_z^b (s-z)^{n-2} u(s) ds.$$

1. If $(n-1)p + q > 0$, then $-(u^m)'(z) > 0$, that implies that $u'(z) < 0$.
2. At $z = 0$ we have

$$m u^{m-1}(0) u'(0) = -\frac{(n-1)p + q}{(n-2)!} \int_0^b s^{n-2} u(s) ds. \tag{9}$$

   The sign of $u'(0)$ is then determined by (9).
3. If $(n-1)p + q < 0$, according to (6), we have $(u^m)^{(n)}(z_0) < 0$ when $u'(z_0) = 0$, also

$$\begin{aligned} 0 &= \frac{pz_0}{(n-3)!} \int_{z_0}^b (s-z_0)^{n-3} u(s) ds + \frac{(n-1)p + q}{(n-2)!} \int_{z_0}^b (s-z_0)^{n-2} u(s) ds \\ &\geq \left(\frac{pz_0}{(n-3)!} + \frac{(n-1)p + q}{(n-2)!}(b - z_0)\right) \int_{z_0}^b (s-z)^{n-3} u(s) ds, \end{aligned}$$

   this implies that $z_0 \leq \frac{(n-1)p+q}{p+q} b$. The proof is complete.

The following lemma is for the case n is odd, then p is negative.

Lemma 5
Let n be odd, p negative, $a \in [0, b)$, and let u be a nontrivial weak solution to the problem (6) *on $[a, b)$, then we separate the following cases:*

1. *If $(n-1)p + q < 0$, then $u'(z) < 0$.*
2. *We have*
   $u'(0) > 0$, *when* $(n-1)p + q > 0$,
   $u'(0) = 0$, *when* $(n-1)p + q = 0$,
   $u'(0) < 0$, *when* $(n-1)p + q < 0$.
3. *Assume $(n-1)p + q > 0$, and there exists $z_0 \in [a, b)$ such that $u'(z_0) = 0$, then we get $(u^m)^{(n)}(z_0) < 0$ and u has a maximum at $z_0 \leq \frac{(n-1)p+q}{p+q} b$.*

Proof. We use the same method for n is odd.

We already know that weak solutions are bounded and continuous. However, in the following lemmas, we provide explicit estimates for the solution (Lemma 6) and establish a result concerning its sign on the interval $[0, b)$ (Lemma 7). We consider two distinct possibilities:

(A) $u(z) \to \infty$ as $z \to z_1$, where $z_1 \in [0, b)$;
(B) $u(z)$ can be extended continuously back to $z = 0$.

We begin by ruling out possibility (A).

Lemma 6
Let u be a nontrivial weak solution to the problem (6) *on (a, b), for some $a \in [0, b)$. Then u is bounded on (a, b) and*

$$\sup_{a<z<b} u(z) \leq \left[\frac{b^n}{n!} \max\left\{n|p|, \frac{(m-1)}{m}|2np + q|\right\}\right]^{\frac{1}{m-1}} \qquad \text{for } n \geq 3.$$

Proof. We will establish this lemma by considering the following two cases:

a. For n is even, p is positive.
   i. Suppose that $(n-1)p + q > 0$ then $u'(z) < 0$, we have

$$\begin{aligned} -mu'(z)u^{m-1}(z) &= \frac{pz}{(n-3)!} \int_z^b (s-z)^{n-3} u(s) ds \\ &\quad + \frac{(n-1)p+q}{(n-2)!} \int_z^b (s-z)^{n-2} u(s) ds, \end{aligned} \tag{10}$$

which implies that

$$-mu'(z)u^{m-2}(z) \leq \frac{p}{(n-2)!} z(b-z)^{n-2} + \frac{(n-1)p+q}{(n-1)!}(b-z)^{n-1},$$

then

$$\frac{m}{m-1}u^{m-1}(z) \leq \frac{pz(b-z)^{n-1}}{(n-1)!} + \frac{(np+q)(b-z)^n}{n!}$$
$$\leq \frac{2np+q}{n!}b^n.$$

Thus

$$\sup_{a<z<b} u(z) \leq \Big[\frac{b^n(2np+q)(m-1)}{mn!}\Big]^{\frac{1}{m-1}}. \tag{11}$$

ii. Now, we suppose that $(n-1)p+q \leq 0$. By (10), we have

$$-mu'(z)u^{m-1}(z) \leq \frac{pz}{(n-3)!}\int_z^b (s-z)^{n-3}u(s)ds.$$

We put $M = \sup_{a\leq z\leq b} u(z)$, where $M \neq 0$. Then, we have

$$-mu'(z)u^{m-1}(z) \leq \frac{pzM}{(n-3)!}\int_z^b (s-z)^{n-3}ds,$$

this implies that

$$-mu'(z)u^{m-1}(z) \leq \frac{pbM}{(n-2)!}(b-z)^{n-2}.$$

Integrating both sides from z to b, we deduce that

$$u^m(z) \leq \frac{pM}{(n-1)!}b^n.$$

Now, take the supremum over $z \in (a,b)$, we get

$$M^m \leq \frac{pM}{(n-1)!}b^n.$$

As $M > 0$, we divide both sides by M, we find

$$M^{m-1} \leq \frac{pn}{n!}b^n.$$

Therefore

$$\sup_{a\leq z\leq b} u(z) \leq \Big(\frac{pn}{n!}b^n\Big)^{\frac{1}{m-1}}. \tag{12}$$

b. For the case n is odd, p is negative.

i. If $(n-1)p+q<0$, then $u'(z)<0$. Consequently

$$mu'(z)u^{m-2}(z) \geq \frac{p}{(n-2)!}z(b-z)^{n-2} + \frac{(n-1)p+q}{(n-1)!}(b-z)^{n-1},$$

After an integration from z to b, we get

$$-\frac{m}{m-1}u^{m-1}(z) \geq \frac{pz(b-z)^{n-1}}{(n-1)!} + \frac{(np+q)(b-z)^{n}}{n!},$$

then

$$\sup_{a<z<b} u(z) \leq \Big[-\frac{b^n(2np+q)(m-1)}{mn!}\Big]^{\frac{1}{m-1}}. \tag{13}$$

ii. Assume now $(n-1)p+q \geq 0$. For this case, we have from (10) that

$$mu'(z)u^{m-1}(z) \geq \frac{pz}{(n-3)!}\int_z^b (s-z)^{n-3}u(s)ds \qquad \text{for all } z \in (a,b).$$

Let $M = \sup_{a\leq z\leq b} u(z)$, where $M \neq 0$. Then, we get

$$mu'(z)u^{m-1}(z) \geq \frac{pbM}{(n-2)!}(b-z)^{n-2}. \tag{14}$$

The integration of (14) starting with z to b give us

$$-u^{m-1}(z) \geq \frac{pM}{(n-1)!}b^n,$$

this means that

$$\sup_{a<z<b} u(z) \leq \Big[-\frac{pb^n n}{n!}\Big]^{\frac{1}{m-1}}. \tag{15}$$

We observe that the bounds of inequalities (11), (12), (13), and (15) are independent of the chosen value a. Consequently, the function u cannot be unlimited when z approaches the boundary points 0 or b.

The following lemma explains probability (B).

Lemma 7
Suppose u is a nontrivial weak solution solution to problem (6) in the left neighborhood of $z=b$.

a. *Assume that n is even and $p>0$. Then*

1. *If $np+q>0$, then $u(z)>0$ on $[0,b)$.*
2. *If $np+q=0$, then $u(z)>0$ on $(0,b)$ and $u(0)=0$.*

b. *Assume that n is odd and $p < 0$. Then*
   1. *If $np + q < 0$, then $u(z) > 0$ on $[0, b)$.*
   2. *If $np + q = 0$, then $u(z) > 0$ on $(0, b)$ and $u(0) = 0$.*

Proof. Let u be a nontrivial weak solution solution to problem (6) on $[0, b)$. After the integration of (7) starting with z to b, we get

$$(-1)^n u^m(z) = \frac{pz}{(n-2)!} \int_z^b (s-z)^{n-2} u(s)ds + \frac{np+q}{(n-1)!} \int_z^b (s-z)^{n-1} u(s)ds. \quad (16)$$

Hence the lemma.

Proposition 8
Suppose $u(z; b_1)$ and $u(z; b_2)$ are solutions for the boundary value problem (6) on $(0, b_1)$ and $(0, b_2)$ respectively.

a. *Let n be an even, $p > 0$, and $np+q > 0$. If $b_1 > b_2$, then $u(z; b_1) > u(z; b_2)$ everywhere on $(0, b_2)$.*
b. *Let n be an odd, $p < 0$ and $np + q < 0$. If $b_1 > b_2$, then $u(z; b_1) > u(z; b_2)$ everywhere on $(0, b_2)$.*

Proof. Denote $u(z; b_i)$ by $u_i(z)$ for $i = 1, 2$. Suppose the Proposition 8 is not true. Then there exists $z \in (0, b_2)$ such that $u_1(z) = u_2(z)$ and $u_1(s) > u_2(s)$ on (z, b_2). It follows from (16) that

$$(-1)^n u_i^m(z) = \frac{pz}{(n-2)!} \int_z^{b_i} (s-z)^{n-2} u_i(s)ds + \frac{np+q}{(n-1)!} \int_z^{b_i} (s-z)^{n-1} u_i(s)ds, \ i = 1, 2.$$

Thus

$$\begin{aligned} 0 = & \frac{pz}{(n-2)!} \int_z^{b_2} (s-z)^{n-2} (u_1(s) - u_2(s))ds \\ & + \frac{np+q}{(n-1)!} \int_z^{b_2} (s-z)^{n-1} (u_1(s) - u_2(s))ds \\ & + \frac{pz}{(n-2)!} \int_{b_2}^{b_1} (s-z)^{n-2} u_1(s)ds + \frac{np+q}{(n-1)!} \int_{b_2}^{b_1} (s-z)^{n-1} u_1(s)ds. \end{aligned}$$

a. If $p > 0$ and $np + q > 0$, the right side of the expression is nonnegative, we then have a contradiction.
b. If $p < 0$ and $np + q < 0$, the right side of the expression is negative, we therefore have a contradiction.

Hence the proposition.

Proposition 9

Suppose $u_k(z;b_k)$ and $u_{k+2}(z;b_{k+2})$ are two solutions for the boundary value problem (6) for $n=k$ on $(0,b_k)$ and $n=k+2$ on $(0,b_{k+2})$ respectively. If $b_{k+2}>b_k$, then $u_{k+2}(z;b_{k+2})>u_k(z;b_k)$ everywhere on $(0,b_k)$ in the following cases:

a. *n is even, $p>0$, and $np+q>0$.*
b. *n is odd, $p<0$, and $np+q<0$.*

Proof. Suppose the Proposition 8 is not true. Then there exists $z\in(0,b_k)$ such that $u_k(z)=u_{k+2}(z)$ and $u_k(s)>u_{k+2}(s)$ on (z,b_k). It follows from (16) that

$$(-1)^j u_j^m(z)=\frac{pz}{(j-2)!}\int_z^{b_j}(s-z)^{j-2}u_j(s)ds+\frac{jp+q}{(j-1)!}\int_z^{b_j}(s-z)^{j-1}u_j(s)ds,\qquad j=k,k+2.$$

Thus, for $j=k$ and $j=k+2$, we subtract the expressions to obtain

$$\begin{aligned}0=&\frac{pz}{(k-2)!}\int_z^{b_k}(s-z)^{k-2}u_k(s)ds-\int_z^{b_k}\frac{pz}{k!}(s-z)^k u_{k+2}(s)ds\\&+\frac{kp+q}{(k-1)!}\int_z^{b_k}(s-z)^{k-1}u_k(s)ds-\int_z^{b_k}\frac{(k+2)p+q}{(k+1)!}(s-z)^{k+1}u_{k+2}(s)ds\\&+\frac{pz}{k!}\int_{b_k}^{b_{k+2}}(s-z)^k u_{k+2}(s)ds+\frac{(k+2)p+q}{(k+1)!}\int_{b_{k+2}}^{b_k}(s-z)^{k+1}u_{k+2}(s)ds.\end{aligned}\tag{17}$$

Case 1: n even, $p>0$ and $np+q>0$. From (17), we deduce

$$\begin{aligned}0\geq&\Big(\frac{pz}{(k-2)!}+\frac{pz}{k!}(b_k)^2\Big)\int_z^{b_k}(s-z)^{k-2}(u_k(s)-u_{k+2}(s))ds\\&+\Big(\frac{kp+q}{(k-1)!}+\frac{(k+2)p+q}{(k+1)!}(b_k)^2\Big)\int_z^{b_k}(s-z)^{k-1}(u_k(s)-u_{k+2}(s))ds\\&+\frac{pz}{(k)!}\int_{b_k}^{b_{k+2}}(s-z)^k u_{k+2}(s)ds+\frac{(k+2)p+q}{(k+1)!}\int_{b_k}^{b_{k+2}}(s-z)^{k+1}u_{k+2}(s)ds,\end{aligned}$$

this implies that

$$0\geq\frac{pz}{k!}\int_{b_k}^{b_{k+2}}(s-z)^k u_{k+2}(s)ds+\frac{(k+2)p+q}{(k+1)!}\int_{b_k}^{b_{k+2}}(s-z)^{k+1}u_{k+2}(s)ds.$$

The right side of the expression is nonnegative, we then have a contradiction.
Case 2: n odd, $p < 0$ and $np + q < 0$. From (17), we derive

$$0 \leq \Big(\frac{k(k+1)(k-1)+(b_k)^2}{k!}\Big) pz \int_z^{b_k} (s-z)^{k-2}(u_k(s)-u_{k+2}(s))ds$$
$$+ \Big(\frac{kp+q}{(k-1)!} + \frac{(k+2)p+q}{(k+1)!}(b_k)^2\Big) \int_z^{b_k} (s-z)^{k-1}(u_k(s)-u_{k+2}(s))ds$$
$$+ \frac{pz}{k!}\int_{b_k}^{b_{k+2}} (s-z)^k u_{k+2}(s)ds + \frac{(k+2)p+q}{(k+1)!}\int_{b_k}^{b_{k+2}} (s-z)^{k+1} u_{k+2}(s)ds.$$

Then, we get

$$0 \leq \frac{pz}{k!}\int_{b_k}^{b_{k+2}} (s-z)^k u_{k+2}(s)ds + \frac{(k+2)p+q}{(k+1)!}\int_{b_k}^{b_{k+2}} (s-z)^{k+1} u_{k+2}(s)ds,$$

the right side of the expression is negative, resulting in a contradiction.

2.3. Main Theorems

In this section, we present and prove the principal theorems that form the foundation of our analysis. This involves considering two distinct cases:

Case 1 : n is even, $p > 0$, and $np + q > 0$.
Case 2 : n is odd, $p < 0$, and $np + q < 0$.

Let $E = C([0, b], \mathbb{R}^+)$ be the Banach space of continuous, non-negative functions on $[0, b]$, equipped with the norm

$$\|u\|_\infty = \sup_{z\in[0,b]} |u(z)|.$$

THEOREM 10
Problem (6) *admits at least one positive solution in* E.

Proof. To establish the result, we reformulate problem (6) as a fixed-point problem. Define the operator $\mathcal{A}$ as follows

$$\mathcal{A}(u)(z) = \left(\frac{(-1)^n pz}{(n-2)!}\int_z^b (s-z)^{n-2}u(s)ds + \frac{(-1)^n(np+q)}{(n-1)!}\int_z^b (s-z)^{n-1}u(s)ds\right)^{\frac{1}{m}}.$$

We will demonstrate that $\mathcal{A}$ satisfies the assumptions of Schauder's fixed-point theorem [7] through the following steps, which ensures the existence of at least one fixed point.
Step 1: $\mathcal{A}$ maps a closed, convex, bounded subset into itself.

$$B_R = \{u \in E : \ \|u\|_\infty \leq R\},$$

where $R > 0$ is a constant to be determined. We show that $\mathcal{A}$ maps B_R into itself, i.e. $\|\mathcal{A}(u)\|_\infty \leq R$ for all $u \in B_R$.

From the definition of $\mathcal{A}$, we have

$$|\mathcal{A}(u)(z)| \leq \left(\frac{|p|bR}{(n-2)!} \int_z^b (s-z)^{n-2}\, ds + \frac{|np+q|R}{(n-1)!} \int_z^b (s-z)^{n-1} ds \right)^{\frac{1}{m}}.$$

Evaluating the integrals

$$\int_z^b (s-z)^{n-2}\, ds = \frac{(b-z)^{n-1}}{n-1} \leq \frac{b^{n-1}}{n-1}, \qquad \int_z^b (s-z)^{n-1}\, ds = \frac{(b-z)^n}{n} \leq \frac{b^n}{n}.$$

Thus

$$|\mathcal{A}(u)(z)| \leq \Big(\frac{|p|bR}{(n-2)!} \frac{b^{n-1}}{n-1} + \frac{|np+q|R}{(n-1)!} \frac{b^n}{n} \Big)^{\frac{1}{m}} \leq \kappa R^{\frac{1}{m}},$$

where

$$\kappa = b^{\frac{n}{m}} \Big(\frac{n|p| + |np+q|}{n!} \Big)^{\frac{1}{m}}.$$

To ensure $\|\mathcal{A}(u)\|_\infty \leq R$, we require $R \geq \kappa^{\frac{m}{m-1}}$. Therefore, choosing

$$R \geq b^{\frac{n}{m-1}} \Big(\frac{n|p| + |np+q|}{n!} \Big)^{\frac{1}{m-1}}$$

guarantees that $\mathcal{A}$ maps B_R into itself.

Step 2: $\mathcal{A}$ is Continuous.To show that $\mathcal{A}$ is continuous, we prove that if $(u_k)_{k\in\mathbb{N}} \to u$ in E, then $\mathcal{A}(u_k) \to \mathcal{A}(u)$ in E. This follows from the continuity of the integrals and the $\frac{1}{m}$-power function.

Step 3: $\mathcal{A}$ is Compact. To show that $\mathcal{A}$ is compact, we prove that $\mathcal{A}$ maps bounded sets to relatively compact sets using the Arzelà-Ascoli theorem. This requires showing that $\mathcal{A}(B_R)$ is uniformly bounded and equicontinuous.

a. Uniformly Bounded. We already showed that $\|\mathcal{A}(u)\|_\infty \leq R$ for $u \in B_R$, then $\mathcal{A}(B_R)$ is uniformly bounded.

b. Equicontinuous. To prove the equicontinuity of $\mathcal{A}$, we show that for all $z_1, z_2 \in [0, b]$,

$$|\mathcal{A}(u)(z_1) - \mathcal{A}(u)(z_2)| \to 0, \quad \text{when} \quad z_1 \to z_2.$$

Let

$$\begin{aligned} Fu(z) = \frac{(-1)^n pz}{(n-2)!} &\int_z^b (s-z)^{n-2} u(s) ds \\ &+ \frac{(-1)^n (np+q)}{(n-1)!} \int_z^b (s-z)^{n-1} u(s) ds. \end{aligned} \tag{18}$$

Then,

$$|\mathcal{A}(u)(z_1) - \mathcal{A}(u)(z_2)| = |(Fu(z_1))^{\frac{1}{m}} - (Fu(z_2))^{\frac{1}{m}}|.$$

Since $|F_1^\alpha - F_2^\alpha| \leq |F_1 - F_2|^\alpha$ for any $F_1, F_2 \geq 0$ and $0 < \alpha \leq 1$, we have

$$|\mathcal{A}(u)(z_1) - \mathcal{A}(u)(z_2)| \leq |Fu(z_1) - Fu(z_2)|^{\frac{1}{m}}.$$

Moreover,

$$\begin{aligned}
&|Fu(z_1) - Fu(z_2)| \\
&\quad \leq \frac{|p|}{(n-2)!}\left| z_1 \int_{z_1}^{b} (s-z_1)^{n-2} u(s)ds - z_2 \int_{z_2}^{b} (s-z_2)^{n-2} u(s)ds \right| \\
&\qquad + \frac{|np+q|}{(n-1)!}\left| \int_{z_1}^{b} (s-z_1)^{n-1} u(s)ds - \int_{z_2}^{b} (s-z_2)^{n-1} u(s)ds \right|.
\end{aligned}$$

Using the inequalities

$$\begin{aligned}
&\left| z_1 \int_{z_1}^{b} (s-z_1)^{n-2} u(s)ds - z_2 \int_{z_2}^{b} (s-z_2)^{n-2} u(s)ds \right| \\
&\quad \leq |z_1 - z_2| \int_{z_1}^{b} (s-z_1)^{n-2} |u(s)| ds \\
&\qquad + z_2 \left| \int_{z_1}^{b} (s-z_1)^{n-2} u(s)ds - \int_{z_2}^{b} (s-z_2)^{n-2} u(s)ds \right| \\
&\quad \leq \frac{R}{n-1}\left(|z_1 - z_2|(b-z_1)^{n-1} + z_2|(b-z_1)^{n-1} - (b-z_2)^{n-1}|\right)
\end{aligned}$$

and

$$\left| \int_{z_1}^{b} (s-z_1)^{n-1} u(s)ds - \int_{z_2}^{b} (s-z_2)^{n-1} u(s)ds \right| \leq \frac{R}{n}|(b-z_1)^n - (b-z_2)^n|,$$

we obtain

$$\begin{aligned}
&|\mathcal{A}(u)(z_1) - \mathcal{A}(u)(z_2)| \\
&\quad \leq \Big(\frac{|p|R}{(n-1)!}\left(|z_1 - z_2|(b-z_1)^{n-1} + z_2|(b-z_1)^{n-1} - (b-z_2)^{n-1}|\right) \\
&\qquad + \frac{|np+q|R}{n!}|(b-z_1)^n - (b-z_2)^n| \Big)^{\frac{1}{m}}.
\end{aligned}$$

The right-hand side of this inequality approaches zero as $z_1 \to z_2$. Hence,

$$\lim_{z_1 \to z_2} \|\mathcal{A}(u(z_1)) - \mathcal{A}(u(z_2))\|_\infty = 0.$$

That what strongly supports the continuity of the operator $\mathcal{A}$. On the other hand, we can use the algebraic identity

$$a^{n-1} - b^{n-1} = (a-b)\sum_{k=0}^{n-2} a^{n-2-k}b^k$$

to show that

$$\begin{aligned}|(b-z_1)^{n-1} - (b-z_2)^{n-1}| &\leq |z_1 - z_2|\sum_{k=0}^{n-2}(b-z_1)^{n-2-k}(b-z_2)^k\\ &\leq (n-1)b^{n-1}|z_1 - z_2|\end{aligned}$$

and

$$|(b-z_1)^n - (b-z_2)^n| \leq nb^n|z_1 - z_2|.$$

Thus,

$$|\mathcal{A}(u)(z_1) - \mathcal{A}(u)(z_2)| \leq M|z_1 - z_2|^{\frac{1}{m}},$$

where

$$M = \Big(\frac{R}{(n-1)!}[|p|(b^{n-1} + (n-1)b^n) + b^n|np+q|]\Big)^{\frac{1}{m}}.$$

Here, $\mathcal{A}(B_R)$ is equicontinuous, because for all $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that for all $u \in B_R$ and all $z_1, z_2 \in [0, b]$, with $|z_1 - z_2| < \delta(\varepsilon)$, we have

$$|\mathcal{A}(u)(z_1) - \mathcal{A}(u)(z_2)| < \varepsilon.$$

In this case, $\delta(\varepsilon) = (\frac{\varepsilon}{M})^m$.

By Steps 1–3 and the Arzelà-Ascoli theorem, $\mathcal{A}$ is continuous, compact, and satisfies the assumptions of Schauder's fixed-point theorem [7], ensuring that $\mathcal{A}$ has at least one fixed point in E, which solves problem (6). This completes the proof.

After establishing the existence of at least one fixed point of problem (6) in E, we proceed to demonstrate its uniqueness by employing the monotonicity and concavity properties of $\mathcal{A}$ on the convex ball B_R, following the approach of Guo and Lakshmikantham [8].

Lemma 11

Let $\mathcal{A} : E \to E$ be the operator defined by

$$\mathcal{A}(u)(z) = (Fu(z))^{\frac{1}{m}} \qquad \textit{for } m > 1,$$

where F, given by (18)*, is linear in u and defined via integrals with strictly positive kernels. Then*

1. *$\mathcal{A}$ is monotone increasing on E; that is, if $u(z) \leq v(z)$ for all $z \in [0, b]$, then $\mathcal{A}(u)(z) \leq \mathcal{A}(v)(z)$ for all $z \in [0, b]$.*

2. *$\mathcal{A}$ is strictly monotone on E; that is, if $u(z) \leq v(z)$ for all $z \in [0, b]$ and $u \neq v$, then $\mathcal{A}(u)(z) < \mathcal{A}(v)(z)$ for all $z \in [0, b]$.*

Proof. Let $u, v \in E$, and suppose $u(z) \leq v(z)$ for all $z \in [0, b]$. Since the operator $Fu(z)$ is defined via integrals that are linear in u, it follows that

$$Fu(z) \leq Fv(z) \qquad \text{for all } z \in [0, b].$$

Taking the $\frac{1}{m}$-power (which is monotone increasing for $m > 1$), we obtain

$$\mathcal{A}(u)(z) \leq \mathcal{A}(v)(z).$$

Thus, $\mathcal{A}$ is monotone increasing.

Now suppose $u \neq v$, with $u(z) \leq v(z)$ for all $z \in [0, b]$. Then there exists some $z_0 \in [0, b]$ such that $u(z_0) < v(z_0)$. By continuity, there exists a neighborhood of z_0 where $u(s) < v(s)$. Since the kernels in the definition of $Fu(z)$ are strictly positive for all $z \in [0, b]$, and $u < v$ on a set of positive measure, we conclude that

$$Fu(z) < Fv(z) \qquad \text{for all } z \in [0, b].$$

Taking the $\frac{1}{m}$-power again yields

$$\mathcal{A}(u)(z) < \mathcal{A}(v)(z) \qquad \text{for all } z \in [0, b].$$

This completes the proof.

Lemma 12
The operator $\mathcal{A} : E \to E$, defined by $\mathcal{A}(u)(z) = (Fu(z))^{\frac{1}{m}}$ with $m > 1$, is strictly concave on E. That is, for any $u, v \in E$ with $u \neq v$, and any $\beta \in (0, 1)$, the following strict inequality holds

$$\mathcal{A}(\beta u + (1 - \beta)v)(z) > \beta\mathcal{A}(u)(z) + (1 - \beta)\mathcal{A}(v)(z) \qquad \textit{for all } z \in [0, b].$$

Proof. The functional F defined by

$$Fu(z) = \frac{(-1)^n pz}{(n-2)!} \int_z^b (s - z)^{n-2} u(s) ds + \frac{(-1)^n (np + q)}{(n-1)!} \int_z^b (s - z)^{n-1} u(s) ds$$

is linear in u. Suppose $Fu(z) = Fv(z)$ for all $z \in [0, b]$. Subtracting the expressions yields

$$\frac{(-1)^n pz}{(n-2)!} \int_z^b (s-z)^{n-2} [u(s)-v(s)] ds + \frac{(-1)^n (np + q)}{(n-1)!} \int_z^b (s-z)^{n-1} [u(s)-v(s)] ds = 0.$$

Since the kernels $(s - z)^{n-2}$ and $(s - z)^{n-1}$ are strictly positive and linearly independent, it follows that $u(s) = v(s)$ for all $s \in [0, b]$. Thus, F is

Moreover, by the linearity of F, we have for any $\beta \in [0,1]$,

$$F(\beta u + (1-\beta)v)(z) = \beta Fu(z) + (1-\beta)Fv(z).$$

Now consider the function $\sigma \mapsto \sigma^{\frac{1}{m}}$, which is strictly concave on $\sigma > 0$ when $m > 1$, since its second derivative $\frac{1}{m}(\frac{1}{m}-1)\sigma^{\frac{1}{m}-2}$ is negative. Therefore, if $Fu(z) \neq Fv(z)$ for all $z \in [0,b]$, and $\beta \in (0,1)$, we have

$$[\beta Fu(z) + (1-\beta)Fv(z)]^{\frac{1}{m}} > \beta[Fu(z)]^{\frac{1}{m}} + (1-\beta)[Fv(z)]^{\frac{1}{m}} \qquad \text{for all } z \in [0,b].$$

That is,

$$\mathcal{A}(\beta u + (1-\beta)v)(z) > \beta\mathcal{A}(u)(z) + (1-\beta)\mathcal{A}(v)(z) \qquad \text{for all } z \in [0,b].$$

Equality holds if and only if $u = v$. The proof is complete.

Lemma 13
Let $\mathcal{A}$ be a continuous, strictly increasing, and strictly concave operator on a convex subset $B_R \subset E$. If $\mathcal{A}$ has a fixed point in B_R, then it is unique.

Proof. Assume, for contradiction, that $u, v \in B_R$ are two distinct fixed points of $\mathcal{A}$, i.e.

$$\mathcal{A}(u) = u, \quad \mathcal{A}(v) = v, \quad u \neq v.$$

Without loss of generality, suppose $u(z) < v(z)$ for all $z \in [0,b]$. Define the midpoint

$$w = \frac{u+v}{2}.$$

Since B_R is convex and $u, v \in B_R$, it follows that $w \in B_R$. Clearly

$$u(z) < w(z) < v(z) \qquad \text{for all } z \in [0,b].$$

By the strict concavity of $\mathcal{A}$, we have

$$\mathcal{A}(w) > \frac{1}{2}\mathcal{A}(u) + \frac{1}{2}\mathcal{A}(v) = \frac{1}{2}u + \frac{1}{2}v = w.$$

Because $\mathcal{A}(w) > w$ and $\mathcal{A}$ is strictly increasing, it follows that

$$\mathcal{A}(\mathcal{A}(w)) > \mathcal{A}(w).$$

Define the sequence $(w_\mu)_{\mu\in\mathbb{N}}$ by

$$w_0 = w, \quad w_{\mu+1} = \mathcal{A}(w_\mu) \text{ for all } \mu \geq 0.$$

By induction, w_μ is strictly increasing

$$w_0 < w_1 = \mathcal{A}(w_0) < w_2 = \mathcal{A}(w_1) < \cdots .$$

Moreover, since $u < w_\mu < v$ for all μ, the sequence is bounded above by v. Hence, w_μ converges to some limit $\bar{w} \in B_R$ with

$$u < \bar{w} \leq v.$$

By continuity of $\mathcal{A}$,

$$\mathcal{A}(\bar{w}) = \lim_{\mu\to\infty} \mathcal{A}(w_\mu) = \lim_{\mu\to\infty} w_{\mu+1} = \bar{w}.$$

Thus, $\bar{w}$ is a new fixed point of $\mathcal{A}$ strictly between u and v, contradicting the assumption that u and v are the only fixed points.

Even, if there exists a third fixed point between u and v, that contradicts the definition of concavity of $\mathcal{A}$, because if we let $\bar{w} = \beta u + (1-\beta)v$ for some $\beta \in (0,1)$, we get

$$\mathcal{A}(\bar{w}) > \beta\mathcal{A}(u) + (1-\beta)\mathcal{A}(v) = \beta u + (1-\beta)v = \bar{w} = \mathcal{A}(\bar{w}).$$

Therefore, the fixed point of $\mathcal{A}$ is unique.

Theorem 14
Problem (6) *has a unique positive solution in E.*

Proof. By Theorem 10, the operator $\mathcal{A}$ admits at least one positive fixed point in the closed ball $B_R \subset E$. Lemmas 11 and 12 show that $\mathcal{A}$ is strictly increasing and strictly concave, respectively. Then, applying Lemma 13, which guarantees the uniqueness of the fixed point for a continuous, strictly increasing, and strictly concave operator on a convex set, we conclude that the fixed point of $\mathcal{A}$ is unique. Therefore, problem (6) has exactly one positive solution in E.

We are currently focused on studying the behavior and stability of the solutions. To facilitate this, we designate the positive part of η as $(\eta)_+$, defined as η if $\eta > 0$, and as zero otherwise.

Theorem 15 (Blow-up and global existence of generalized self-similar solutions)
Let φ and ψ be two positive real functions of time t, that satisfy the following condition

$$\varphi(0) = \psi(0) = 1. \tag{19}$$

Then, for any $u \in E = C([0,b],\mathbb{R}_+)$ problem (5) *admits a generalized self-similar solution, which can be expressed as*

$$\omega(x,t) = \varphi(t)u(z) \qquad \text{for } z = \frac{x}{\psi(t)},\ x \in \mathbb{R}_+,\ t > 0,$$

if the function u presents a nonnegative solution to problem (6) *on* $[0,b]$ *and consistently satisfies at each point*

$$(u^m)^{(n)}(z) = qu(z) - pzu'(z).$$

1. *If $np + q(1-m) > 0$, the functions φ and ψ are given by*

$$\begin{cases} \varphi(t) &= (1+[np+q(1-m)]t)^{\frac{q}{np+q(1-m)}} \\ \psi(t) &= (1+[np+q(1-m)]t)^{\frac{p}{np+q(1-m)}} \end{cases} \qquad \text{for all } t > 0.$$

2. *If $np+q(1-m)=0$ the functions φ and ψ are given by*

$$\begin{cases}\varphi(t) &= e^{qt}\\ \psi(t) &= e^{pt}\end{cases} \qquad \textit{for all } t>0.$$

In each case, when considering either 1 or 2. Problem (5) admits a global generalized self-similar solution that is defined for all $t>0$.
Furthermore, if $q<0$,

$$\lim_{t\to+\infty}\omega(x,t)=0 \qquad \textit{for any } x\in\mathbb{R}_+.$$

3. *If $np+q(1-m)<0$ the functions φ and ψ are given by*

$$\begin{cases}\varphi(t) &= (1+[np+q(1-m)]t)_+^{\frac{q}{np+q(1-m)}}\\ \psi(t) &= (1+[np+q(1-m)]t)_+^{\frac{p}{np+q(1-m)}}\end{cases} \qquad 0<t<T. \tag{20}$$

The moment $T=\frac{-1}{np+q(1-m)}$ represents the maximum existence time for the functions φ and ψ.
Furthermore, when $q>0$, problem (5) admits a generalized self-similar solution that blows up in a finite time. This solution remains well-defined for all $t\in(0,T)$, where T signifies the time at which the solution experiences blow-up. In fact,

$$\forall x\in\mathbb{R}_+ \lim_{t\to T^-}\omega(x,t)=+\infty \textit{ with } T=\frac{-1}{np+q(1-m)}>0.$$

Proof. First, to determine the functions φ and ψ, we need to solve the system (4) which is

$$\begin{cases}\frac{\dot\varphi}{\varphi} &= q\frac{\varphi^{m-1}}{\psi^n}\\ \frac{\dot\psi}{\psi} &= p\frac{\varphi^{m-1}}{\psi^n}\end{cases}. \tag{21}$$

then

$$\frac{\dot\psi}{\psi}=\frac{p}{q}\frac{\dot\varphi}{\varphi}.$$

The conditions stated in (19), imply that

$$\psi(t)=\varphi^{\frac{p}{q}}(t). \tag{22}$$

If we replace (22) in (21), we obtain

$$\varphi^{\frac{np+q(1-m)}{q}-1}d\varphi=qdt. \tag{23}$$

If $np+q(1-m)=0$, the functions φ and ψ are defined globally as follows

$$\begin{cases}\varphi(t) &= e^{qt}\\ \psi(t) &= e^{pt}\end{cases} \qquad \text{for all} \quad t>0.$$

If $np + q(1-m) \neq 0$, we easily find the solution of (23) in the following manner

$$\varphi(t) = (1 + [np + q(1-m)]t)_{+}^{\frac{q}{np+q(1-m)}},$$

we also get

$$\psi(t) = (1 + [np + q(1-m)]t)_{+}^{\frac{p}{np+q(m-1)}}.$$

If $np + q(1-m) > 0$, then φ and ψ are globally defined. On the other hand, φ and ψ are maximal functions if $np + q(1-m) < 0$, and well-defined in the case

$$0 < t < T = \frac{-1}{np + q(1-m)}.$$

From this theorem, it is evident that there exist two distinct time behaviors for the functions φ and ψ, with their behaviors contingent upon the parameters of similarity p and q.

For the both cases 1 and 2, i.e. $np + q(1-m) \geq 0$, the functions φ and ψ are globally defined.

Now, as u is a positive bounded function. If $q < 0$, we get

$$\lim_{t \to +\infty} \varphi(t) = 0,$$

then

$$\lim_{t \to +\infty} \omega(x,t) = \lim_{t \to +\infty} \varphi(t) u\Big(\frac{x}{\psi(t)}\Big) = 0.$$

3. If $np + q(1-m) < 0$, the functions φ and ψ, as defined in (20), are considered well-defined when

$$0 < t < T = \frac{-1}{np + q(1-m)}.$$

When $q > 0$, the value T represents the finite-time blow-up of the solution. Consequently $\lim_{t \to T^-} \varphi(t) = +\infty$, and

$$\lim_{t \to T^-} \omega(x,t) = \lim_{t \to T^-} \varphi(t) u\Big(\frac{x}{\psi(t)}\Big) = +\infty.$$

3. Conclusion

This study examined the asymptotic behavior of positive generalized self-similar solutions for a nonlinear hybrid problem involving n^{th}-order derivative porous medium equations. We established the existence and uniqueness of these solutions with compact support, revealing that the behavior of these solutions is impacted by certain parameters that must satisfy specific conditions, which determine whether their existence is global or local in a given time T.

Declarations. The authors declare no conflicts of interest.
Authors' contributions:
Author 01: Formal analysis; Investigation; Resources; Software; Writing-original draft.
Author 02: Methodology; Validation; Writing-review and editing.
All authors have read and approved the final version of the manuscript.

References

[1] Barenblatt, G.I. "On some unsteady motions of a liquid or a gas in a porous medium." *Prikl. Mat. Mekh.* 16 (1952): 67-78. Cited on 54.

[2] Basti, B., and N. Benhamidouche. "Global existence and blow-up of generalized self-similar solutions to nonlinear degenerate diffusion equation not in divergence form." *Appl. Math. E-Notes* 20 (2020): 367-387. Cited on 54 and 55.

[3] Benhamidouche, N. "Exact solutions to some nonlinear PDEs, traveling profiles method." *Electron. J. Qual. Theory Differ. Equ.* 15 (2008): 1-7. Cited on 54 and 55.

[4] Benhamidouche, N., and Y. Arioua. "New method for constructing exact solutions to non-linear PDEs." *Int. J. Nonlinear Sci.* 7 (2009): 395-398. Cited on 55.

[5] Caffarelli, L., and J.L. Vàzquez. "Viscosity solutions for the porous medium equation." *Proc. Sympos. Pure Math.* 65 (1999): 13-26. Cited on 54.

[6] Gilding, B.E., and L.A. Peletier. "On a class of similarity solution of the porous medium equation I." *J. Math. Anal. Appl.* 55 (1976): 351-364. Cited on 54 and 55.

[7] Granas, A., and J. Dugundji. *Fixed Point Theory*. New York: Springer-Verlag, 2003. Cited on 65 and 68.

[8] Guo, D., and V. Lakshmikantham. *Nonlinear Problems in Abstract Cones*. New York: Academic Press, 1988. Cited on 68.

[9] Hulshof, J., and J.L. Vàzquez. "Self-similar solutions of the second kind for the modified porous medium equation." *Eur. J. Appl. Math.* 5 (1994): 391-403. Cited on 54.

[10] Hulshof, J., and J.L. Vàzquez. "Maximal viscosity solutions of the modified porous medium equation and their asymptotic behavior." *Eur. J. Appl. Math.* 7 (1996): 453-471. Cited on 54.

[11] Ibragimov, N.H. *CRC Handbook of Lie Group Analysis of Differential Equations*. Vol.1: Symmetries, Exact Solutions and Conservation Laws. Boca Raton: CRC Press, 1994. Cited on 54.

[12] Olver, P.J. *Applications of Lie Groups to Differential Equations*. New York: Springer-Verlag, 1986. Cited on 54.

[13] Polyanin, A.D., and V.F. Zaitsev. *Handbook of Nonlinear Partial Differential Equations*. Boca Raton: Chapman & Hall/CRC, 2004. Cited on 54 and 55.

[14] Vàzquez, J.L. *The Porous Medium Equation: Mathematical Theory*. Oxford: Oxford University Press, 2007. Cited on 54.

[15] Wang, C., and Jingxue. "Shrinking self-similar solution of a nonlinear diffusion equation with nondivergence form." *J. Math. Anal. Appl.* 289 (2004): 387-404. Cited on 54.

Mohamed Dilmi
LAMDA-RO Laboratory, Department of Mathematics, Faculty of Science
Saad Dahlab University, Blida 1
Blida 09000
Algeria
E-mail: dilmi_mohamed@univ-blida.dz

Bilal Basti
Laboratory of Pure and Applied Mathematics
University Pole of Mohamed Boudiaf, Road BBA
M'sila 28000
Algeria
E-mail: bilal.basti@univ-msila.dz

Received: February 2, 2024; final version: September 18, 2025;
available online: December 29, 2025.