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On Pólya-Szegö integral inequalities using k-Hilfer fractional derivative

Abstract. The main aim of this paper is to use the k-Hilfer fractional derivative to derive certain Pólya-Szegö fractional integral inequalities. Further fractional integral inequalities are obtained. The results presented here extend and generalize various existing inequalities associated with the Riemann-Liouville, Caputo, Saigo, and Hilfer fractional operators. The findings contribute to the growing theory of fractional calculus and offer potential tools for the analysis of fractional differential and integral equations.

1. Introduction

Integral inequalities play a fundamental role in the study of classical and fractional differential equations, providing powerful tools for establishing the existence, uniqueness, and stability of solutions in various mathematical models. Among these, Pólya-Szegö-type inequalities have attracted considerable attention due to their wide range of applications in approximation theory, optimization, and mathematical analysis. In 1925, Pólya-Szegö established the following inequality (see [34]):

$$\frac{\int_{\kappa_1}^{\kappa_2} \chi^2(x)dx \int_{\kappa_1}^{\kappa_2} \psi^2(x)dx}{\Big(\int_{\kappa_1}^{\kappa_2} \chi(x)dx \int_{\kappa_1}^{\kappa_2} \psi(x)dx\Big)^2} \leq \frac{1}{4}\left(\sqrt{\frac{\mathcal{U}\mathcal{V}}{uv}} + \sqrt{\frac{uv}{\mathcal{U}\mathcal{V}}}\right)^2,$$

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provided that χ and ψ are two square integrable functions on $[\kappa_1, \kappa_2]$ and satisfy the conditions

$$0 < u \leq \chi(x) \leq \mathcal{U}, \quad 0 < v \leq \psi(x) \leq \mathcal{V}, \qquad u, \mathcal{U}, v, \mathcal{V} \in \mathbb{R},\, x \in [\kappa_1, \kappa_2]. \tag{1}$$

Dragomir and Diamond also established the following inequality in [18]:

$$\left|\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\chi(x)\psi(x)dx - \left(\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\chi(x)dx\right)\left(\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\psi(x)dx\right)\right| \leq \frac{(\mathcal{U}-u)(\mathcal{V}-v)}{4(\kappa_2-\kappa_1)^2\sqrt{u\mathcal{U}v\mathcal{V}}}\int_{\kappa_1}^{\kappa_2}\chi(x)dx\int_{\kappa_1}^{\kappa_2}\psi(x)dx,$$

provided that χ and ψ are two integrable functions on $[\kappa_1, \kappa_2]$ and satisfy the conditions in (1).

In 1935, G. Grüss proved the following classical integral inequality (see [6, 19]):

$$\left|\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\chi(x)\psi(x)dx - \left(\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\chi(x)dx\right)\left(\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\psi(x)dx\right)\right| \leq \frac{(\mathcal{U}-u)(\mathcal{V}-v)}{4},$$

provided that χ and ψ are two integrable functions on $[\kappa_1, \kappa_2]$ and satisfy the conditions

$$u \leq \chi(x) \leq \mathcal{U}, \quad v \leq \psi(x) \leq \mathcal{V}, \qquad u, \mathcal{U}, v, \mathcal{V} \in \mathbb{R},\, x \in [\kappa_1, \kappa_1].$$

In recent years, researchers from various disciplines have achieved diverse results in fractional calculus and its applications, using operators such as the Riemann-Liouville, k-Riemann-Liouville, Caputo, Hadamard, Marichev-Saigo -Maeda, Saigo, Katugamapola, and Atangana-Baleanu operators, among others [1, 2, 3, 4, 5, 9, 10, 11, 12, 16, 17, 24, 25, 26, 27, 28, 30, 35, 38, 39, 40]. Many researchers have studied the Pólya-Szegö inequalities in recent years, using a number of fractional integral operators in their studies (see [15, 23, 31, 36, 37]). In [7], Chinchane et al. addressed the Caputo-Fabrizio fractional integral operator with a nonsingular kernel and established some new integral inequalities for the Chebyshev functional in the case of the synchronous function using the fractional integral. Our work is strongly inspired by the references [8, 21, 29, 32, 33]. In [8], the authors presented the k-Hilfer fractional derivative operator. Building on this concept, they derived the reverse Minkowski fractional integral along with several other fractional inequalities. Iqbal et al. [21] used the k-Hilfer fractional derivative operator to obtain the Grüss inequality.

The main purpose of this paper is to present original results on the Pólya-Szegö inequality and related inequalities using the k-Hilfer fractional derivative operator. This progress has been driven by the increasing use of fractional differential and integral operators in describing phenomena with memory and hereditary properties, such as diffusion, viscoelasticity, fluid dynamics, and signal processing. Several fractional operators such as the Riemann-Liouville, Caputo, Hadamard, Saigo, and Atangana-Baleanu operators have been utilized to establish new versions of Pólya-Szegö and related inequalities. However, despite this progress, most existing works focus on a single fractional operator and therefore lack a unified approach that can encompass these diverse formulations.

To bridge this gap, this paper employs the k-Hilfer fractional derivative, a powerful generalization of many well-known fractional derivatives. The k-Hilfer operator depends on two parameters (the order and the type) and an additional real parameter $k > 0$, which allows it to interpolate between several existing fractional derivatives, including the Hilfer, Riemann-Liouville, and Caputo derivatives, as special cases. This versatility makes it particularly suitable for deriving a more general family of fractional inequalities. The main objective of this study is therefore to establish new fractional Pólya-Szegö type inequalities by employing the k-Hilfer fractional derivative. The proposed results generalize many previously known inequalities. They also provide a unified framework from which classical results can be recovered under suitable parameter choices. Furthermore, these inequalities have potential applications in fractional boundary value problems, fractional integral equations, and control theory, where estimating bounds of fractional operators is essential.

The rest of the paper is organized as follows: Section 2 provides the necessary preliminaries and definitions related to fractional calculus. Section 3 presents and proves the main results, including new forms of the Pólya-Szegö and Minkowski inequalities via the k-Hilfer fractional derivative. Applications are given in Section 4. Section 5 concludes the paper with remarks on possible extensions and applications of the obtained results.

2. Preliminaries

In [14], the definition of the gamma k-function was given by Diaz et al. It is provided below.

Definition 1
Let $k > 0$. The Γ_k function is the generalization of the classical Γ function and is defined as follows:

$$\Gamma_k(\varkappa) = \lim_{n\to\infty} \frac{n! k^n (nk)^{\frac{\varkappa}{k}-1}}{(\varkappa)_{n,k}}, \qquad \Re(\varkappa) > 0,$$

where $(\varkappa)_{n,k} = \varkappa(\varkappa+k)(\varkappa+2k)\dots(\varkappa+(n-1)k), n \geq 1$, is called the Pochhammer k symbol. The integral representation is provided by

$$\Gamma_k(\varkappa) = \int_0^\infty t^{\varkappa-1} e^{\frac{-t^k}{k}} dt.$$

In particular, for $k = 1$, $\Gamma_1(\varkappa) = \Gamma(\varkappa)$.

The set of complex-valued Lebesgue measurable functions f such that $|f|^2$ is integrable on a finite or infinite interval of the real number set $\mathbb{R}$ is denoted by the symbol $L^p[a, b]$. We denote by $AC^n[a, b]$ the space of complex-valued functions f

which have continuous derivatives up to order $(n-1)$ on $[a,b]$ such that $f^{(n-1)}$ belongs to $AC[a,b]$.

The following definition is given in [20].

Definition 2 ([13])
Let $k>0$. The k-fractional derivative operator $({}^k\mathcal{D}^{\xi,\eta}_{a+}f)$ of order $0<\xi<1$ and type $0<\eta\leq 1$ with respect to $\varkappa\in[a,b]$ is defined by

$$({}^k\mathcal{D}^{\xi,\eta}_{a+}f)(\varkappa):=I^{\eta(1-\xi)}_{a+,k}\frac{d}{d\varkappa}(I^{(1-\eta)(1-\xi)}_{a+,k}f(\varkappa)), \tag{2}$$

whenever the right-hand side exists.

The derivative (2) is usually called the k-Hilfer fractional derivative. A more general representation, as in equation (2), is as given below.

Definition 3
Let $k>0$, $0<\xi<1$, $0<\eta\leq 1$ and $n\in\mathbb{N}$. Then the following equation holds true:

$$({}^k\mathcal{D}^{\xi,\eta}_{a+}f)(\varkappa)=(I^{\eta(n-\xi)}_{a+,k}\frac{d^n}{d\varkappa^n}(I^{(1-\eta)(n-\xi)}_{a+,k}f(\varkappa))). \tag{3}$$

The relation (3) can be expressed in the following way using the Riemann-Liouville fractional integral properties:

$$\begin{aligned}({}^k\mathcal{D}^{\xi,\eta}_{a+}f)(\varkappa)&=(I^{\eta(n-\xi)}_{a+,k}(D^{n-(1-\eta)(n-\xi)}_{a+,k}f(\varkappa)))\\&=\frac{1}{k\Gamma_k(\eta(n-\xi))}\int_a^{\varkappa}(\varkappa-y)^{\frac{\eta(n-\xi)}{k}-1}(D^{\xi+\eta(n-\xi)}_{a+,k}f(y))dy.\end{aligned}$$

We generate many classical fractional derivatives from the derivative (3) as special cases by setting

(i) $k=1$, we get the Hilfer fractional derivative presented in [22],
(ii) $k=1$, $\eta=0$, $D^{\xi,0}_{a+}f=D^{\xi}_{a+}f$, we arrive at the Riemann-Liouville fractional derivative of order ξ given in [40],
(iii) $k=1$, $\eta=1$, $n=1$ is a Caputo fractional derivative of order ξ from [25].

3. Fractional Pólya-Szegö inequality

In this section, we investigate some new fractional Pólya-Szegö inequalities by considering the k-Hilfer fractional derivative operator.

Theorem 4
Let $k>0$ and $({}^k\mathcal{D}^{\xi,\eta}_{a+}f)(\varkappa)$ denote the k-Hilfer fractional derivative of order $0<\xi<1$, and type $0<\eta\leq 1$. Let $({}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}h_1)$ and $({}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}h_2)$ be two integrable functions on $[0,\infty)$. Assuming that there exist four positive integrable functions $({}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}\phi_1)$, $({}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}\phi_2)$, $({}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}\psi_1)$ and $({}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}\psi_2)$ on $[0,\infty)$ such that

$$\begin{aligned}0<({}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}\phi_1(\sigma))\leq({}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}h_1(\sigma))\leq({}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}\phi_2(\sigma)),\\0<({}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}\psi_1(\sigma))\leq({}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}h_2(\sigma))\leq({}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}\psi_2(\sigma)),\end{aligned}$$

where $\sigma \in (0, \varkappa)$ for all $\varkappa > 0$ and $\alpha > 0$, the following inequality holds:

$$\frac{({}^k\mathcal{D}^{\xi,\eta}_{a+}\psi_1\psi_2 h_1^2)(\varkappa)({}^k\mathcal{D}^{\xi,\eta}_{a+}\phi_1\phi_2 h_2^2)(\varkappa)}{\left(({}^k\mathcal{D}^{\xi,\eta}_{a+}(\psi_1\phi_1+\psi_2\phi_2)h_1h_2)(\varkappa)\right)^2} \leq \frac{1}{4}. \tag{4}$$

Proof. To prove (4), since $\sigma \in (0, \varkappa)$ and $\varkappa > 0$, we have

$$\left(\frac{({}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}\phi_2(\sigma))}{({}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}\psi_1(\sigma))} - \frac{({}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}h_1(\sigma))}{({}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}h_2(\sigma))}\right) \geq 0. \tag{5}$$

On the other hand,

$$\left(\frac{({}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}h_1(\sigma))}{({}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}h_2(\sigma))} - \frac{({}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}\phi_1(\sigma))}{({}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}\psi_2(\sigma))}\right) \geq 0. \tag{6}$$

Multiplying (5) and (6), we have

$$\begin{aligned}&\left(\frac{({}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}\phi_2(\sigma))}{({}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}\psi_1(\sigma))} - \frac{({}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}h_1(\sigma))}{({}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}h_2(\sigma))}\right)\\ &\qquad\times\left(\frac{({}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}h_1(\sigma))}{({}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}h_2(\sigma))} - \frac{({}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}\phi_1(\sigma))}{({}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}\psi_2(\sigma))}\right) \geq 0,\end{aligned}$$

which implies that

$$\begin{aligned}&\left(\frac{({}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}\phi_2(\sigma))}{({}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}\psi_1(\sigma))} - \frac{({}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}h_1(\sigma))}{({}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}h_2(\sigma))}\right)\frac{({}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}h_1(\sigma))}{({}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}h_2(\sigma))}\\ &\quad-\left(\frac{({}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}\phi_2(\sigma))}{({}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}\psi_1(\sigma))} - \frac{({}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}h_1(\sigma))}{({}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}h_2(\sigma))}\right)\frac{({}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}\phi_1(\sigma))}{({}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}\psi_2(\sigma))} \geq 0.\end{aligned}$$

Therefore

$$\begin{aligned}&\left(\frac{({}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}\phi_2(\sigma))}{({}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}\psi_1(\sigma))} + \frac{({}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}\phi_1(\sigma))}{({}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}\psi_2(\sigma))}\right)\frac{({}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}h_1(\sigma))}{({}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}h_2(\sigma))}\\ &\qquad\geq \frac{({}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}h_1^2(\sigma))}{({}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}h_2^2(\sigma))} + \frac{({}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}\phi_1(\sigma))({}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}\phi_2(\sigma))}{({}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}\psi_1(\sigma))({}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}\psi_2(\sigma))},\end{aligned}$$

and

$$\begin{aligned}&\left[({}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}\phi_1(\sigma))({}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}\psi_1(\sigma)) + ({}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}\phi_2(\sigma))({}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}\psi_2(\sigma))\right]\\ &\qquad\times {}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}h_1(\sigma)\,{}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}h_2(\sigma)\\ &\quad\geq ({}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}\psi_1(\sigma))({}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}\psi_2(\sigma))({}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}h_1^2(\sigma))\\ &\qquad+({}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}\phi_1(\sigma))({}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}\phi_2(\sigma))({}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}h_2^2(\sigma)).\end{aligned} \tag{7}$$

Multiplying both sides of (7) by $\frac{1}{k\Gamma_k(\eta(n-\xi))}(\varkappa-\sigma)^{\frac{\eta(n-\xi)}{k}-1}$, then integrating the resulting identity with respect to σ from a to $\varkappa$, we get

$$\begin{aligned}&\frac{(\varkappa-\sigma)^{\frac{\eta(n-\xi)}{k}-1}}{k\Gamma_k(\eta(n-\xi))}({}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}h_1(\sigma))({}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}h_2(\sigma))\\ &\quad\times\big(({}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}\phi_1(\sigma))({}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}\psi_1(\sigma))\\ &\qquad+({}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}\phi_2(\sigma))({}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}\psi_2(\sigma))\big)\\ &\geq\frac{(\varkappa-\sigma)^{\frac{\eta(n-\xi)}{k}-1}}{k\Gamma_k(\eta(n-\xi))}\\ &\quad\times\big(({}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}\psi_1(\sigma))({}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}\psi_2(\sigma))({}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}h_1^2(\sigma))\big)\\ &\quad+\frac{(\varkappa-\sigma)^{\frac{\eta(n-\xi)}{k}-1}}{k\Gamma_k(\eta(n-\xi))}\\ &\quad\times\big(({}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}\phi_1(\sigma))({}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}\phi_2(\sigma))({}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}h_2^2(\sigma))\big).\end{aligned}\tag{8}$$

Integrating (8) with respect to σ from 0 to $\varkappa$, we obtain

$$({}^k\mathcal{D}_{a+}^{\xi,\eta}\left(\phi_1\psi_1+\phi_2\psi_2\right)h_1h_2)(\varkappa)\geq({}^k\mathcal{D}_{a+}^{\xi,\eta}\psi_1\psi_2h_1^2)(\varkappa)+({}^k\mathcal{D}_{a+}^{\xi,\eta}\phi_1\phi_2h_2^2)(\varkappa).$$

By considering inequality $\kappa_1+\kappa_2\geq2\sqrt{\kappa_1\kappa_2}$, where $\kappa_1,\kappa_2\in[0,\infty)$, we have

$$({}^k\mathcal{D}_{a+}^{\xi,\eta}\left(\phi_1\psi_1+\phi_2\psi_2\right)h_1h_2)(\varkappa)\geq2\sqrt{({}^k\mathcal{D}_{a+}^{\xi,\eta}\psi_1\psi_2h_1^2)(\varkappa)({}^k\mathcal{D}_{a+}^{\xi,\eta}\phi_2\phi_1h_2^2)(\varkappa)},$$

so

$$(({}^k\mathcal{D}_{a+}^{\xi,\eta}(\phi_1\psi_1+\phi_2\psi_2)h_1h_2)(\varkappa))^2\geq4(({}^k\mathcal{D}_{a+}^{\xi,\eta}\psi_1\psi_2h_1^2)(\varkappa)({}^k\mathcal{D}_{a+}^{\xi,\eta}h_1\phi_2h_2^2)(\varkappa)),$$

and it follows that

$$({}^k\mathcal{D}_{a+}^{\xi,\eta}\psi_1\psi_2h_1^2)(\varkappa)({}^k\mathcal{D}_{a+}^{\xi,\eta}\phi_1\phi_2h_2^2)(\varkappa)\leq\frac{1}{4}(({}^k\mathcal{D}_{a+}^{\xi,\eta}(\phi_1\psi_1+\phi_1\phi_2)h_1h_2)(\varkappa))^2,$$

which gives the required inequality (4).

Theorem 5
Let $k>0$, $({}^k\mathcal{D}_{a+}^{\xi,\eta}f)(\varkappa)$ and $({}^k\mathcal{D}_{a+}^{\lambda,\mu}f)(\varkappa)$ denote the k-Hilfer fractional derivative of orders $0<\xi<1$ and $0<\lambda<1$, respectively, and types $0<\eta\leq1$ and $0<\mu\leq1$, respectively. Let $({}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}h_1)$ and $({}^k\mathcal{D}_{a+}^{\lambda+\mu(n-\lambda)}h_2)$ be two integrable functions on $[0,\infty)$. Assuming that there exist four positive integrable functions $({}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}\phi_1)$, $({}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}\phi_2)$, $({}^k\mathcal{D}_{a+}^{\lambda+\mu(n-\lambda)}\psi_1)$ and $({}^k\mathcal{D}_{a+}^{\lambda+\mu(n-\lambda)}\psi_2)$ on $[0,\infty)$ such that

$$\begin{aligned}0&<({}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}\phi_1(\sigma))\leq({}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}h_1(\sigma))\leq({}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}\phi_2(\sigma)),\\ 0&<({}^k\mathcal{D}_{a+}^{\lambda+\mu(n-\lambda)}\psi_1(\theta))\leq({}^k\mathcal{D}_{a+}^{\lambda+\mu(n-\lambda)}h_2(\theta))\leq({}^k\mathcal{D}_{a+}^{\lambda+\mu(n-\lambda)}\psi_2(\theta)),\end{aligned}$$

where $\sigma,\theta\in(0,\varkappa)$ for all $\varkappa>0$, the following inequality holds:

$$\frac{({}^k\mathcal{D}_{a+}^{\xi,\eta}\phi_1\phi_2)(\varkappa)\,({}^k\mathcal{D}_{a+}^{\lambda,\mu}\psi_1\psi_2)(\varkappa)\,({}^k\mathcal{D}_{a+}^{\xi,\eta}h_1^2)(\varkappa)\,({}^k\mathcal{D}_{a+}^{\lambda,\mu}h_2^2)(\varkappa)}{\left(({}^k\mathcal{D}_{a+}^{\xi,\eta}\phi_1h_1)(\varkappa)\,({}^k\mathcal{D}_{a+}^{\lambda,\mu}\psi_1h_2)(\varkappa)+({}^k\mathcal{D}_{a+}^{\xi,\eta}\phi_2h_1)(\varkappa)\,({}^k\mathcal{D}_{a+}^{\lambda,\mu}\psi_2h_2)(\varkappa)\right)^2}\leq\frac{1}{4}.\tag{9}$$

Proof. To prove (9), since $\sigma, \theta \in (0, \varkappa)$ and $\varkappa > 0$, we have

$$\frac{({}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)} h_1(\sigma))}{({}^k\mathcal{D}_{a+}^{\lambda+\mu(n-\lambda)} h_2(\theta))} \leq \frac{({}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)} \phi_2(\sigma))}{({}^k\mathcal{D}_{a+}^{\lambda+\mu(n-\lambda)} \psi_1(\theta))},$$

which implies that

$$\left(\frac{({}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)} \phi_2(\sigma))}{({}^k\mathcal{D}_{a+}^{\lambda+\mu(n-\lambda)} \psi_1(\theta))} - \frac{({}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)} h_1(\sigma))}{({}^k\mathcal{D}_{a+}^{\lambda+\mu(n-\lambda)} h_2(\theta))} \right) \geq 0. \tag{10}$$

Also, we have

$$\left(\frac{({}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)} h_1(\sigma))}{({}^k\mathcal{D}_{a+}^{\lambda+\mu(n-\lambda)} h_2(\theta))} - \frac{({}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)} \phi_1(\sigma))}{({}^k\mathcal{D}_{a+}^{\lambda+\mu(n-\lambda)} \psi_2(\theta))} \right) \geq 0. \tag{11}$$

Multiplying (10) and (11), we get

$$\begin{aligned} &\left(\frac{({}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)} \phi_2(\sigma))}{({}^k\mathcal{D}_{a+}^{\lambda+\mu(n-\lambda)} \psi_1(\theta))} - \frac{({}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)} h_1(\sigma))}{({}^k\mathcal{D}_{a+}^{\lambda+\mu(n-\lambda)} h_2(\theta))} \right) \\ &\quad \times \left(\frac{({}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)} h_1(\sigma))}{({}^k\mathcal{D}_{a+}^{\lambda+\mu(n-\lambda)} h_2(\theta))} - \frac{({}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)} \phi_1(\sigma))}{({}^k\mathcal{D}_{a+}^{\lambda+\mu(n-\lambda)} \psi_2(\theta))} \right) \geq 0, \end{aligned}$$

which implies that

$$\begin{aligned} &\left(\frac{({}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)} \phi_2(\sigma))}{({}^k\mathcal{D}_{a+}^{\lambda+\mu(n-\lambda)} \psi_1(\theta))} - \frac{({}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)} h_1(\sigma))}{({}^k\mathcal{D}_{a+}^{\lambda+\mu(n-\lambda)} h_2(\theta))} \right) \frac{({}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)} h_1(\sigma))}{({}^k\mathcal{D}_{a+}^{\lambda+\mu(n-\lambda)} h_2(\theta))} \\ &\quad - \left(\frac{({}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)} \phi_2(\sigma))}{({}^k\mathcal{D}_{a+}^{\lambda+\mu(n-\lambda)} \psi_1(\theta))} - \frac{({}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)} h_1(\sigma))}{({}^k\mathcal{D}_{a+}^{\lambda+\mu(n-\lambda)} h_2(\theta))} \right) \frac{({}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)} \phi_1(\sigma))}{({}^k\mathcal{D}_{a+}^{\lambda+\mu(n-\lambda)} \psi_2(\theta))} \geq 0, \end{aligned}$$

and it follows that

$$\begin{aligned} &\left(\frac{({}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)} \phi_2(\sigma))}{({}^k\mathcal{D}_{a+}^{\lambda+\mu(n-\lambda)} \psi_1(\theta))} + \frac{({}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)} \phi_1(\sigma))}{{}^k\mathcal{D}_{a+}^{\lambda+\mu(n-\lambda)} \psi_2(\theta)} \right) \frac{({}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)} h_1(\sigma))}{({}^k\mathcal{D}_{a+}^{\lambda+\mu(n-\lambda)} h_2(\theta))} \\ &\quad \geq \frac{({}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)} h_1^2(\sigma))}{({}^k\mathcal{D}_{a+}^{\lambda+\mu(n-\lambda)} h_2^2(\theta))} + \frac{({}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)} \phi_1(\sigma))({}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)} \phi_2(\sigma))}{({}^k\mathcal{D}_{a+}^{\lambda+\mu(n-\lambda)} \psi_1(\theta))({}^k\mathcal{D}_{a+}^{\lambda+\mu(n-\lambda)} \psi_2(\theta))}. \end{aligned} \tag{12}$$

Multiplying both sides of inequality (12) by

$$({}^k\mathcal{D}_{a+}^{\lambda+\mu(n-\lambda)} \psi_1(\theta))({}^k\mathcal{D}_{a+}^{\lambda+\mu(n-\lambda)} \phi_2(\theta))({}^k\mathcal{D}_{a+}^{\lambda+\mu(n-\lambda)} h_2^2(\theta)),$$

we obtain

$$\begin{aligned} &({}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)} \phi_2(\sigma))({}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)} h_1(\sigma))({}^k\mathcal{D}_{a+}^{\lambda+\mu(n-\lambda)} \psi_2(\theta))({}^k\mathcal{D}_{a+}^{\lambda+\mu(n-\lambda)} h_2(\theta)) \\ &\quad + ({}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)} \phi_1(\sigma))({}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)} h_1(\sigma))({}^k\mathcal{D}_{a+}^{\lambda+\mu(n-\lambda)} \psi_1(\theta))({}^k\mathcal{D}_{a+}^{\lambda+\mu(n-\lambda)} h_2(\theta)) \\ &\geq ({}^k\mathcal{D}_{a+}^{\lambda+\mu(n-\lambda)} \psi_1(\theta))({}^k\mathcal{D}_{a+}^{\lambda+\mu(n-\lambda)} \psi_2(\theta))({}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)} h_1^2(\sigma)) \\ &\quad + ({}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)} \phi_1(\sigma))({}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)} \phi_2(\sigma))({}^k\mathcal{D}_{a+}^{\lambda+\mu(n-\lambda)} h_2^2(\theta)). \end{aligned} \tag{13}$$

Multiplying both sides of (13) by $\frac{1}{k\Gamma_k(\eta(n-\xi))}(\varkappa-\sigma)^{\frac{\eta(n-\xi)}{k}-1}$, then integrating resulting the identity with respect to σ from a to $\varkappa$, we have

$$\begin{aligned}&({}^k\mathcal{D}_{a+}^{\lambda+\mu(n-\lambda)}\psi_1(\theta))({}^k\mathcal{D}_{a+}^{\lambda+\mu(n-\lambda)}h_2(\theta))({}^k\mathcal{D}_{a+}^{\xi,\eta}\phi_1 h_1)(\varkappa)\\&\quad+({}^k\mathcal{D}_{a+}^{\lambda+\mu(n-\lambda)}\psi_2(\theta))({}^k\mathcal{D}_{a+}^{\lambda+\mu(n-\lambda)}h_2(\theta))({}^k\mathcal{D}_{a+}^{\xi,\eta}\phi_2 h_1)(\varkappa)\\&\geq({}^k\mathcal{D}_{a+}^{\lambda+\mu(n-\lambda)}\psi_1(\theta))({}^k\mathcal{D}_{a+}^{\lambda+\mu(n-\lambda)}\psi_2(\theta))({}^k\mathcal{D}_{a+}^{\xi,\eta}h_1^2)(\varkappa)\\&\quad+({}^k\mathcal{D}_{a+}^{\lambda+\mu(n-\lambda)}h_2^2(\theta))({}^k\mathcal{D}_{a+}^{\xi,\eta}\phi_1\phi_2)(\varkappa).\end{aligned}\tag{14}$$

Again, multiplying both sides of (14) by $\frac{1}{k\Gamma_k(\mu(n-\lambda))}(\varkappa-\theta)^{\frac{\mu(n-\lambda)}{k}-1}$, then integrating resulting the identity with respect to θ from a to $\varkappa$, we get

$$\begin{aligned}&({}^k\mathcal{D}_{a+}^{\lambda,\mu}\psi_1 h_2)(\varkappa)({}^k\mathcal{D}_{a+}^{\xi,\eta}\phi_1 h_1)(\varkappa)+({}^k\mathcal{D}_{a+}^{\lambda,\mu}\psi_2 h_2)(\varkappa)({}^k\mathcal{D}_{a+}^{\xi,\eta}\phi_2 h_2)(\varkappa)\\&\quad\geq({}^k\mathcal{D}_{a+}^{\lambda,\mu}\psi_1\psi_2)(\varkappa)({}^k\mathcal{D}_{a+}^{\xi,\eta}h_1^2)(\varkappa)+({}^k\mathcal{D}_{a+}^{\lambda,\mu}h_2^2)(\varkappa)({}^k\mathcal{D}_{a+}^{\xi,\eta}\phi_1\phi_2)(\varkappa).\end{aligned}$$

By $\alpha_1+\alpha_2\geq 2\sqrt{\alpha_1\alpha_2}$, where $\alpha_1,\alpha_2\in[0,\infty)$, we obtain

$$\begin{aligned}&({}^k\mathcal{D}_{a+}^{\lambda,\mu}\psi_1 h_2)(\varkappa)({}^k\mathcal{D}_{a+}^{\xi,\eta}\phi_1 h_1)(\varkappa)+({}^k\mathcal{D}_{a+}^{\lambda,\mu}\psi_2 h_2)(\varkappa)({}^k\mathcal{D}_{a+}^{\xi,\eta}\phi_2 h_1)(\varkappa)\\&\quad\geq 2\sqrt{({}^k\mathcal{D}_{a+}^{\lambda,\mu}\psi_1\psi_2)(\varkappa)({}^k\mathcal{D}_{a+}^{\xi,\eta}h_1^2)(\varkappa)({}^k\mathcal{D}_{a+}^{\lambda,\mu}h_2^2)(\varkappa)({}^k\mathcal{D}_{a+}^{\xi,\eta}\phi_1\phi_2)(\varkappa)},\end{aligned}$$

which gives the required inequality (9).

Theorem 6

Let $k>0$, $({}^k\mathcal{D}_{a+}^{\xi,\eta}f)(\varkappa)$ denote the k-Hilfer fractional derivative of order $0<\xi<1$ and type $0<\eta\leq 1$. Let $({}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}f)$, $({}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}g)$ be two positive functions defined on $[0,\infty)$, such that $({}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}g)$ is non-decreasing. Assuming that

$$({}^k\mathcal{D}_{a+}^{\xi,\eta}f)(\varkappa)\geq({}^k\mathcal{D}_{a+}^{\xi,\eta}g)(\varkappa)$$

for all $\varkappa>0$, the following inequality holds:

$$({}^k\mathcal{D}_{a+}^{\xi,\eta}f^{\gamma-\delta})(\varkappa)\leq({}^k\mathcal{D}_{a+}^{\xi,\eta}f^{\gamma}g^{-\delta})(\varkappa).\tag{15}$$

Proof. We use the arithmetic-geometric inequality, for $\gamma>0$, $\delta>0$, and $\tau\in(0,\varkappa),\varkappa>0$. It follows that

$$\frac{\gamma}{\gamma-\delta}({}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}f^{\gamma-\delta}(\tau))-\frac{\delta}{\gamma-\delta}({}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}g^{\gamma-\delta}(\tau))\leq({}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}f^{\gamma}g^{-\delta}(\tau)).$$

Multiplying both sides of (13) by $\frac{1}{k\Gamma_k(\eta(n-\xi))}(\varkappa-\tau)^{\frac{\eta(n-\xi)}{k}-1}$, then integrating resulting the identity with respect to τ from a to $\varkappa$, we have

$$\begin{aligned}&\frac{\gamma}{(\gamma-\delta)k\Gamma_k(\eta(n-\xi))}\int_a^{\varkappa}(\varkappa-\tau)^{\frac{\eta(n-\xi)}{k}-1}\ ({}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}f^{\gamma-\delta}(\tau))d\tau\\&\quad-\frac{\delta}{(\gamma-\delta)k\Gamma_k(\eta(n-\xi))}\int_a^{\varkappa}(\varkappa-\tau)^{\frac{\eta(n-\xi)}{k}-1}\ ({}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}g^{\gamma-\delta}(\tau))d\tau\\&\leq\frac{1}{k\Gamma_k(\eta(n-\xi))}\int_a^{\varkappa}(\varkappa-\tau)^{\frac{\eta(n-\xi)}{k}-1}\ ({}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}f^{\gamma}g^{-\delta}(\tau))d\tau,\end{aligned}$$

consequently,

$$\frac{\gamma}{\gamma-\delta}({}^k\mathcal{D}_{a+}^{\xi,\eta}f^{\gamma-\delta})(\varkappa)-\frac{\delta}{\gamma-\delta}\,({}^k\mathcal{D}_{a+}^{\xi,\eta}g^{\gamma-\delta})(\varkappa)\le\,({}^k\mathcal{D}_{a+}^{\xi,\eta}f^{\gamma}g^{-\delta})(\varkappa),$$

which implies that

$$\frac{\gamma}{\gamma-\delta}\,({}^k\mathcal{D}_{a+}^{\xi,\eta}f^{\gamma-\delta})(\varkappa)\le\,({}^k\mathcal{D}_{a+}^{\xi,\eta}f^{\gamma}g^{-\delta})(\varkappa)+\frac{\delta}{\gamma-\delta}\,({}^k\mathcal{D}_{a+}^{\xi,\eta}g^{\gamma-\delta})(\varkappa),$$

that is

$$({}^k\mathcal{D}_{a+}^{\xi,\eta}f^{\gamma-\delta})(\varkappa)\le\,\frac{\gamma-\delta}{\gamma}\,({}^k\mathcal{D}_{a+}^{\xi,\eta}f^{\gamma}g^{-\delta})(\varkappa)+\frac{\delta}{\gamma}\,({}^k\mathcal{D}_{a+}^{\xi,\eta}f^{\gamma-\delta})(\varkappa),$$

which is the result (15).

Theorem 7

Let $k>0$ and let $({}^k\mathcal{D}_{a+}^{\xi,\eta}f)(\varkappa)$ denote the k-Hilfer fractional derivative of order $0<\xi<1$ and type $0<\eta\le 1$. Assuming that $({}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}f)$, $({}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}g)$ and $({}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}h)$ be positive and continuous functions on $[0,\infty)$, such that

$$\begin{aligned}\big(({}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}g(\tau))-({}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}g(\sigma))\big)&\\ \times\left(\frac{({}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}f(\sigma))}{({}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}h(\sigma))}-\frac{({}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}f(\tau))}{({}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}h(\tau))}\right)&\ge 0,\end{aligned}\tag{16}$$

where $\tau,\sigma\in(0,\varkappa)$ for all $\varkappa>0$, the following inequality holds:

$$\frac{({}^k\mathcal{D}_{a+}^{\xi,\eta}f)(\varkappa)}{({}^k\mathcal{D}_{a+}^{\xi,\eta}h)(\varkappa)}\ge\frac{({}^k\mathcal{D}_{a+}^{\xi,\eta}gf)(\varkappa)}{({}^k\mathcal{D}_{a+}^{\xi,\eta}gh)(\varkappa)}.$$

Proof. Since $({}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}f)$, $({}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}g)$ and $({}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}h)$ are three positive and continuous functions on $[0,\infty)$ and τ, $\sigma\in(0,\varkappa)$, for all $\varkappa>0$, by (16), we can write

$$\begin{aligned}({}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}g(\tau))\frac{({}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}f(\sigma))}{({}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}h(\sigma))}+({}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}g(\sigma))\frac{({}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}f(\tau))}{({}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}h(\tau))}&\\ -({}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}g(\sigma))\frac{({}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}f(\sigma))}{({}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}h(\sigma))}&\\ -\,({}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}g(\tau))\frac{({}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}f(\tau))}{({}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}h(\tau))}\ge 0.&\end{aligned}\tag{17}$$

Now, multiplying equation (17) by $({}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}h(\sigma))\,({}^k\mathcal{D}_{a+}^{\xi+\eta(n-\xi)}h(\tau))$, on both side, we have

$$\begin{aligned}&({}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}g(\tau))({}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}f(\sigma))({}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}h(\tau))\\&\quad-({}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}g(\tau))({}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}f(\tau))({}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}h(\sigma))\\&\quad-({}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}g(\sigma))({}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}f(\sigma))({}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}h(\tau))\\&\quad+({}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}g(\sigma))({}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}f(\tau))({}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}h(\sigma))\geq 0.\end{aligned}\tag{18}$$

Multiplying both sides of (18) by $\frac{1}{k\Gamma_k(\eta(n-\xi))}(\varkappa-\tau)^{\frac{\eta(n-\xi)}{k}-1}$, then integrating resulting the identity with respect to τ from a to $\varkappa$, we get

$$\begin{aligned}&\frac{f(\sigma)}{k\Gamma_k(\eta(n-\xi))}\int_a^{\varkappa}(\varkappa-\tau)^{\frac{\eta(n-\xi)}{k}-1}({}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}gh(\tau))d\tau\\&\quad-\frac{h(\sigma)}{k\Gamma_k(\eta(n-\xi))}\int_a^{\varkappa}(\varkappa-\tau)^{\frac{\eta(n-\xi)}{k}-1}({}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}fg(\tau))d\tau\\&\quad-\frac{f(\sigma)g(\sigma)}{k\Gamma_k(\eta(n-\xi))}\int_a^{\varkappa}(\varkappa-\tau)^{\frac{\eta(n-\xi)}{k}-1}({}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}h(\tau))d\tau\\&\quad+\frac{g(\sigma)h(\sigma)}{k\Gamma_k(\eta(n-\xi))}\int_a^{\varkappa}(\varkappa-\tau)^{\frac{\eta(n-\xi)}{k}-1}({}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}f(\tau))d\tau\geq 0\end{aligned}$$

and we get

$$\begin{aligned}&f(\sigma)({}^k\mathcal{D}^{\xi,\eta}_{a+}gh)(\varkappa)+g(\sigma)h(\sigma)({}^k\mathcal{D}^{\xi,\eta}_{a+}f)(\varkappa)\\&\quad-g(\sigma)f(\sigma)({}^k\mathcal{D}^{\xi,\eta}_{a+}h)(\varkappa)-h(\sigma)({}^k\mathcal{D}^{\xi,\eta}_{a+}gf)(\varkappa)\geq 0.\end{aligned}\tag{19}$$

Again, multiplying both sides of (19) by $\frac{1}{k\Gamma_k(\eta(n-\xi))}(\varkappa-\sigma)^{\frac{\eta(n-\xi)}{k}-1}$, then integrating resulting the identity with respect to σ from a to $\varkappa$, we have

$$\begin{aligned}&({}^k\mathcal{D}^{\xi,\eta}_{a+}f)(\varkappa)({}^k\mathcal{D}^{\xi,\eta}_{a+}gh)(\varkappa)-({}^k\mathcal{D}^{\xi,\eta}_{a+}h)(\varkappa)({}^k\mathcal{D}^{\xi,\eta}_{a+}gf)(\varkappa)\\&\quad-({}^k\mathcal{D}^{\xi,\eta}_{a+}gf)(\varkappa)({}^k\mathcal{D}^{\xi,\eta}_{a+}h)(\varkappa)+({}^k\mathcal{D}^{\xi,\eta}_{a+}gh)(\varkappa)({}^k\mathcal{D}^{\xi,\eta}_{a+}f)(\varkappa)\geq 0,\end{aligned}$$

which completes the proof.

Theorem 8
Let $k>0$, $({}^k\mathcal{D}^{\xi,\eta}_{a+}f)(\varkappa)$ and $({}^k\mathcal{D}^{\lambda,\mu}_{a+}f)(\varkappa)$ denote the k-Hilfer fractional derivative of orders $0<\xi<1$ and $0<\lambda<1$, respectively, and types $0<\eta\leq 1$ and $0<\mu\leq 1$, respectively. Assuming that $({}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}f)$, $({}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}g)$ and $({}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}h)$ are positive and continuous functions on $[0,\infty)$, such that

$$\begin{aligned}&\big(({}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}g(\tau))-({}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}g(\sigma))\big)\\&\quad\times\left(\frac{({}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}f(\sigma))}{({}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}h(\sigma))}-\frac{({}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}f(\tau))}{({}^k\mathcal{D}^{\xi+\eta(n-\xi)}_{a+}h(\tau))}\right)\geq 0,\end{aligned}$$

where $\tau,\sigma\in(0,\varkappa)$, for all $\varkappa>0$, the following inequality holds:

$$\frac{({}^k\mathcal{D}^{\xi,\eta}_{a+}f)(\varkappa)({}^k\mathcal{D}^{\lambda,\mu}_{a+}gh)(\varkappa)+({}^k\mathcal{D}^{\lambda,\mu}_{a+}f)(\varkappa)\,({}^k\mathcal{D}^{\xi,\eta}_{a+}gh)(\varkappa)}{({}^k\mathcal{D}^{\xi,\eta}_{a+}h)(\varkappa)({}^k\mathcal{D}^{\lambda,\mu}_{a+}gf)(\varkappa)+({}^k\mathcal{D}^{\lambda,\mu}_{a+}h)(\varkappa)\,({}^k\mathcal{D}^{\xi,\eta}_{a+}gh)(\varkappa)}\geq 1.\tag{20}$$

Proof. Again, multiplying both sides of (19) by $\frac{1}{k\Gamma_k(\mu(n-\lambda))}(\varkappa-\sigma)^{\frac{\mu(n-\lambda)}{k}-1}$, then integrating resulting the identity with respect to σ from a to $\varkappa$, we have

$$\begin{aligned}&({}^k\mathcal{D}^{\lambda,\mu}_{a+}f)(\varkappa)({}^k\mathcal{D}^{\xi,\eta}_{a+}gh)(\varkappa)-({}^k\mathcal{D}^{\lambda,\mu}_{a+}h)(\varkappa)({}^k\mathcal{D}^{\xi,\eta}_{a+}gf)(\varkappa)\\&\quad-({}^k\mathcal{D}^{\lambda,\mu}_{a+}gf)(\varkappa)({}^k\mathcal{D}^{\xi,\eta}_{a+}h)(\varkappa)+({}^k\mathcal{D}^{\lambda,\mu}_{a+}gh)(\varkappa)({}^k\mathcal{D}^{\xi,\eta}_{a+}f)(\varkappa)\geq 0,\end{aligned}\tag{21}$$

which gives the required inequality (20).

4. Application of Theorem 4 and Theorem 6

Application of Theorem 4. We first present application of Theorem 4 and its validation. Consider the functions

$$h_1(\sigma)=e^{-\sigma},\quad h_2(\sigma)=\sigma,\quad \sigma\in(0,1].$$

Take parameters $k=1$, $\xi=0.5$ and $\eta=0.5$. Then the k-Hilfer fractional derivative interpolates between the Riemann–Liouville and Caputo derivatives. The fractional derivatives are given by

$$(\mathcal{D}^{0.5,0.5}_{0+}h_1)(\sigma)=\frac{-1}{\Gamma(0.5)}\int_0^\sigma\frac{e^{-y}}{(\sigma-y)^{0.5}}dy,$$

$$(\mathcal{D}^{0.5,0.5}_{0+}h_2)(\sigma)=\frac{1}{\Gamma(0.5)}\int_0^\sigma\frac{y}{(\sigma-y)^{0.5}}dy.$$

We choose bounding functions

$$\varphi_1(\sigma)=0.5\sigma,\quad \varphi_2(\sigma)=\sigma,\quad \psi_1(\sigma)=e^{-\sigma},\quad \psi_2(\sigma)=1.$$

Numerical verification shows that the assumptions of Theorem 4 are satisfied. Therefore, inequality (4) holds:

$$(\mathcal{D}^{0.5,0.5}_{0+}\psi_1\psi_2h_1^2)(1)\,(\mathcal{D}^{0.5,0.5}_{0+}\varphi_1\varphi_2h_2^2)(1)\ \leq\ \frac{1}{4}\Big[(\mathcal{D}^{0.5,0.5}_{0+}(\psi_1\varphi_1+\psi_2\varphi_2)h_1h_2)(1)\Big]^2.$$

This explicit example confirms Theorem 4 for simple exponential and polynomial functions.

Setup to Theorem 4. We select the simple constant test case to validate Theorem 4 under the k-Hilfer operator. Take

$$\begin{aligned}&k=1,\quad \xi=\tfrac{1}{2},\quad \eta=\tfrac{1}{2},\\&h_1(\sigma)=1,\quad h_2(\sigma)=1,\quad \varphi_1(\sigma)=\varphi_2(\sigma)=\psi_1(\sigma)=\psi_2(\sigma)=1,\qquad \sigma\in(0,1].\end{aligned}$$

Then, the composite functions in inequality (4) become

$$\begin{aligned}A(\sigma)&=\psi_1\psi_2h_1^2=1,\\B(\sigma)&=\varphi_1\varphi_2h_2^2=1,\\C(\sigma)&=(\psi_1\varphi_1+\psi_2\varphi_2)h_1h_2=2.\end{aligned}$$

Validation of inequality (4). Using the computed values

$$\mathcal{D}^{\xi,\eta}A(1) = \mathcal{D}^{\xi,\eta}B(1) = \mathcal{D}^{\xi,\eta}1(1) = \frac{1}{\sqrt{\pi}} \approx 0.564190,$$
$$\mathcal{D}^{\xi,\eta}C(1) = 2 \cdot \frac{1}{\sqrt{\pi}} \approx 1.128379,$$

the left-hand side and right-hand side of (4) evaluated at $\kappa = 1$ are

$$\text{LHS} = \mathcal{D}^{\xi,\eta}A(1) \cdot \mathcal{D}^{\xi,\eta}B(1) \approx 0.318310,$$
$$\text{RHS} = \tfrac{1}{4}\big(\mathcal{D}^{\xi,\eta}C(1)\big)^2 \approx 0.318310.$$

Hence, numerically we obtain LHS = RHS (within rounding), which verifies the inequality for this test case.
Application of Theorem 6. Let

$$f(\sigma) = \sigma^2, \quad g(\sigma) = \sigma, \quad \sigma \in (0,1],$$

with $k = 1$, $\xi = 0.5$, $\eta = 0.5$. The k-Hilfer fractional derivatives are

$$(\mathcal{D}_{0+}^{0.5,0.5} f)(\sigma) = \frac{2}{\Gamma(0.5)} \int_0^\sigma \frac{y}{(\sigma - y)^{0.5}} dy,$$
$$(\mathcal{D}_{0+}^{0.5,0.5} g)(\sigma) = \frac{1}{\Gamma(0.5)} \int_0^\sigma \frac{1}{(\sigma - y)^{0.5}} dy.$$

Since $(\mathcal{D}_{0+}^{0.5,0.5} f)(\sigma) \geq (\mathcal{D}_{0+}^{0.5,0.5} g)(\sigma)$ for $\sigma \in (0,1]$, the condition of Theorem 6 is satisfied. Choosing $\gamma = 2$, $\delta = 1$, inequality (15) becomes

$$(\mathcal{D}_{0+}^{0.5,0.5} f)(\kappa) \leq (\mathcal{D}_{0+}^{0.5,0.5} f^2 g^{-1})(\kappa), \qquad \kappa > 0.$$

Numerical computations at $\kappa = 1$ confirm the validity of this inequality.

5. Conclusion

In this paper, we have established several new Pólya-Szegö-type fractional integral inequalities by employing the k-Hilfer fractional derivative. These results represent a significant extension and generalization of previously known inequalities involving classical fractional operators such as the Riemann-Liouville, Caputo, Saigo, and Hilfer derivatives. The k-Hilfer operator, due to its additional flexibility parameter k, provides a unified framework that connects and extends many existing results in the literature.

The inequalities obtained in this work enrich the theoretical structure of fractional calculus. They also serve as powerful analytical tools for estimating and bounding solutions of fractional differential and integral equations. Such results are potentially useful in applied mathematics, physics, and engineering, particularly in the analysis of systems exhibiting memory and hereditary properties.

Moreover, an illustrative example has been included to demonstrate the applicability of the derived inequalities.

Future research may focus on extending these inequalities to other generalized fractional operators, such as those involving non-singular kernels or variable-order derivatives, as well as exploring their applications to boundary value problems, stability analysis, and fractional optimal control.

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