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Local convergence analysis of a bi-parametric iterative method in $\mathbb{R}$ or $\mathbb{C}$

Abstract. In this manuscript, we accelerate the local convergence of a third-order biparametric iterative method in $\mathbb{R}$ or $\mathbb{C}$ by assuming that the first-order Fréchet derivative satisfies the Generalized continuity condition. We extend this analysis by using the Hölder continuity condition, which allows us to solve more numerical problems. Our study also shows the sizes of the convergence balls, the smallest error bounds that can be computed, and the fact that the answer is unique. Several math tests show that this third-order method gives better results than the midpoint method established by I.K. Argyros and S. George [4]. This method solves problems that earlier studies have not been able to solve.

1. Introduction

This manuscript examines the local convergence of a biparametric iterative technique, resulting in a locally unique solution x^* of

$$F(x) = 0, \tag{1}$$

where $F\colon D \subseteq X \to Y$ is a Fréchet differentiable operator, X and Y are $\mathbb{R}$ or $\mathbb{C}$, and D is a subset of the set X, which is convex. There are numerous problems in applied science and engineering that will be boiled down to equation (1). A lot of optimization problems, such as many integral equations, boundary value problems, and so on, can be turned into equations of the form (1). Generally, iterative

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techniques are used for these equations. Newton's algorithm is a well-known and widely used technique for working out (1), which is written below.

$$x_{k+1} = x_k - \frac{F(x_k)}{F'(x_k)}, \qquad k = 0, 1, 2, \ldots.$$

Certain conventional third-order convergent techniques like Chebyshev's, Super-Halley's, and the Halley's techniques are produced by putting $\lambda = 0$, $\lambda = 1$ and $\lambda = \frac{1}{2}$ accordingly in

$$x_{k+1} = x_k - (1 + \frac{1}{2}(1 - \lambda H_F(x_k))^{-1} H_F(x_k))[F'(x_k)]^{-1} F(x_k),$$

where $H_F(x_k) = F'(x_k)^{-1} F''(x_k) F'(x_k)^{-1} F(x_k)$. $F''(x_k)$ is the double Fréchet derivative of $F(x)$ determined at x_k. One problem with these approaches is that they need computing F'' and this might not be bounded or difficult to calculate.

The domain of convergence is crucial for ensuring the stable behavior of an iterative scheme from a numerical perspective. The analysis of local convergence for iterative schemes relies on information surrounding the result and also determines the convergence sphere's radius [9]. Many researchers [5–7, 9] have examined various techniques like Chebyshev-Halley-type schemes, including modified Halley-like, deformed Halley, and improved Chebyshev-Halley methods under local convergence study. The local convergence of Newton-type, Jarratt-type, and Weerakoon-type methods, among others, was examined within a Banach space framework in references [1, 3, 8, 10, 12–14, 16–18]. The main objective of this study is to enhance the usefulness of a third-order technique in $\mathbb{R}$ or $\mathbb{C}$ by applying the generalized continuity condition only to the first-order Fréchet derivative.

The authors in [14] derive a biparametric iterative technique with third-order convergence for approximating the solutions of nonlinear equations. The technique is given by

$$\begin{aligned} y_k &= x_k - \alpha \frac{F(x_k)}{F'(x_k)}, \\ x_{k+1} &= y_k - \Big(\beta + \gamma \frac{F(y_k)}{F(x_k)} + \delta \Big(\frac{F(y_k)}{F(x_k)}\Big)^2\Big) \frac{F(x_k)}{F'(x_k)}, \end{aligned} \tag{2}$$

where $\beta = \frac{(\alpha-1)^2(\alpha^2\delta-\alpha-1)}{\alpha^2}$, $\gamma = \frac{2\alpha^3\delta-2\alpha^2\delta+1}{\alpha^2}$ and α, δ are arbitrary parameters.

The Taylor series process is used to demonstrate the convergence study, which is based on higher-order derivatives, but the execution of this method requires the calculation of the first-order Fréchet derivative. The technique's application is limited by these strategies. For example, let the function F be defined on $D=[\frac{-1}{2}, \frac{5}{2}]$ by

$$F(x) = \begin{cases} x^3 \log(x^2) + x^5 - x^4, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

Since F''' is not bounded on D. Hence, the method outlined in [14] is not applicable for this mentioned issue. Furthermore, the previous work [14] provides no information on the radii of convergence balls. The examination of the convergence domain of an iterative technique could be conducted with the use of an effective tool known as local convergence analysis. In the subsequent part, we will investigate the local convergence of method (2) by using the concept that emphasizes F' as the only primary focus. Our investigation does not include any higher-order derivatives of F in any way. With the intention of enhancing the significance of the approach (2), we make the specific assumption that the first-order Fréchet derivative falls on the Lipschitz continuity class. Likewise, it offers error boundaries that may be computed, the uniqueness of the solution, and the convergence ball's radius.

Here is how this manuscript is laid out: In order to start the introduction, visit Section 1. In Section 2, the analysis of local convergence of technique (2) is presented. The numerical implementations of our results are detailed in Section 3. The final portion presents the conclusion.

2. Local Convergence Analysis

In this portion, the local convergence study of method (2) will be covered. Assume that H(c, ϵ) and $\overline{H}$(c, ϵ) are open and closed balls in X, respectively. where $\epsilon > 0$ is the radius and center c. We take BL(Y, X) as the set of bounded linear operators from Y to X. We present the study of local convergence of the method (2) in the theorem-2.2.

2.1. Study of local convergence of technique (2) under generalized continuity conditions

Some real functions must be introduced which are defined on the interval $L = [0, +\infty)$ or a subset of it. Suppose:

- (C_1) There exists a continuous and nondecreasing function $\varphi_0 \colon L \to L$ such that the equation $1 - \varphi_0(t) = 0$ has a smallest solution in the interval $L \setminus \{0\}$. We shall denote such a solution by s and define the interval $L_0 = [0, s)$.
- (C_2) There exists a continuous and nondecreasing function $\varphi \colon L_0 \to L$. Define the function $h_1 \colon L_0 \to L$ by

$$h_1(t) = \frac{\displaystyle\int_0^1 \varphi((1-\tau)t)\, d\tau + |1-\alpha|\left(1 + \int_0^1 \varphi_0(t\tau)\, d\tau\right)}{1 - \varphi_0(t)}.$$

- (C_3) The equation $1 - h_1(t) = 0$ has a smallest solution in the interval $L_0 \setminus \{0\}$. We shall denote such solution by s_1. Let us consider the functions with domain L_0 and range L defined by

$$p(t) = \int_0^1 \varphi_0(\tau t) d\tau,$$

$$\sqrt{q(t)} = \frac{\left(1 + \displaystyle\int_0^1 \varphi_0(\tau h_1(t)t) d\tau\right) h_1(t)}{1 - p(t)},$$

$$\overline{\varphi}(t) = \begin{cases} \varphi((1+h_1(t))t) \\ or \\ \varphi_0 + \varphi_0(h_1(t)t) \end{cases}$$

and

$$\begin{aligned} h_2(t) \\ &= \frac{\displaystyle\int_0^1 \varphi((1-\tau)h_1(t)t)d\tau h_1(t)}{1-\varphi_0(h_1(t)t)} + \frac{\bar{\varphi}(t) + \left(1 + \displaystyle\int_0^1 \varphi_0(\tau h_1(t)t)d\tau\right)h_1(t)}{(1-\varphi_0(t))(1-\varphi_0(h_1(t)t))} \\ &+ \frac{|1-\gamma|\left(1+\displaystyle\int_0^1 \varphi_0(\tau\, h_1(t)t)d\tau\right)h_1(t)}{1-\varphi_0(t)} \\ &+ \frac{(|\beta|+|\delta|q(t))\left(1+\displaystyle\int_0^1 \varphi_0(\tau t)d\tau\right)}{1-\varphi_0(t)}. \end{aligned}$$

(C_4) The equation $1-h_2(t)=0$ has a smallest solution in the interval $L_0 - \backslash\{0\}$. We shall denote such a solution by s_2. The parameter s^* is defined by

$$s^* = \min\{s_m\}, \qquad m = 1, 2, \tag{3}$$

is shown to be a radius of convergence in Theorem 2.2. Set $L^* = [0, s^*)$. Then, it follows by (C_1)–(C_4) and (3) that for each $t \in L^*$

$$0 \le \varphi_0(t) < 1, \quad 0 \le \varphi_0(h_1(t)t) < 1, \tag{4}$$

$$0 \le p(t) < 1, \quad 0 \le q(t) < 1$$

and

$$0 \le h_m(t) < 1. \tag{5}$$

Next, we relate the functions φ_0 and φ to the ones appearing on the method (2).

(C_5) There exists an invertible function $M = F'(x^*) \in B(X,Y)$ and a solution $x^* \in D$ of the equation $F(x) = 0$ such that for all $v \in D$,

$$\|F'(x^*)^{-1}(F'(v) - F'(x^*))\| \le \varphi_0(\|v - x^*\|).$$

Define the region $D_0 = D \cap H(x^*, s)$.

(C_6) $\|F'(x^*)^{-1}(F'(v_2) - F'(v_1))\| \le \varphi(\|v_2 - v_1\|)$ for all $v_2, v_1 \in D_0$ and

(C_7) $\overline{H}(x^*, s^*) \subset D$.

Remark 2.1

Some choices for M can be $M = I$, the identity function or $M = F'(\tilde{x})$, where $\tilde{x} \in D$ is an auxiliary point other than x^* or $M = F'(x^*)$. Under the last choice, it follows that x^* is a simple solution of the equation $F(x) = 0$. But (C_1)-(C_7) do not necessarily imply that x^* is simple. Therefore, the method (2) can be used to find solutions of multiplicity greater than one.

The main local convergence analysis result follows next.

Theorem 2.2
Suppose that (C_1)-(C_7) *hold. Then, for* $x_0 \in H(x^*, s^*) \setminus \{x^*\}$ *the sequence* $\{x_k\}$ *converges to* x^*. *Moreover, the following assertions hold:*

$$\|y_k - x^*\| \leq h_1(\|x_k - x^*\|)(\|x_k - x^*\|) \leq \|x_k - x^*\| < s^* \tag{6}$$

and

$$\|x_{k+1} - x^*\| \leq h_2(\|x_k - x^*\|)\|x_k - x^*\| \leq \|x_k - x^*\| < s^*. \tag{7}$$

Proof. Assertions (6) and (7) are shown using induction. Let $v_0 \in H(x^*, s^*)\setminus\{x^*\}$. Using ($C_1$), ($C_5$), (3) and (4), we have in turn

$$\|F'(x^*)^{-1}(F'(v_0) - F'(x^*))\| \leq \varphi_0(\|v_0 - x^*\|) \leq \varphi_0(s^*) < 1. \tag{8}$$

It follows by (8) and the Banach Lemma [2, 11, 15, 19] on invertible functions that $F'(v_0)$ is invertible and

$$\|F'(v_0)^{-1}F'(x^*)\| \leq \frac{1}{1 - \varphi_0(\|v_0 - x^*\|)}. \tag{9}$$

In particular, estimate (9) holds if $v_0 = x_0$, since $x_0 \in H(x^*, s^*) \setminus \{x^*\}$. Thus, the iterate y_0 is well-defined by the first substep of the method (2), and we can write in turn

$$y_0 - x^* = x_0 - x^* - F'(x_0)^{-1}F(x_0) + |1 - \alpha|F'(x_0)^{-1}F(x_0). \tag{10}$$

We need two estimates:

$$\begin{aligned} x_0 - x^* - F'(x_0)^{-1}F(x_0) = {} & [F'(x_0)^{-1} - F'(x^*)] \\ & \times \left[\int_0^1 F'(x^*)^{-1}(F'(x^* + \tau(x_0 - x^*)) - F'(x_0))d\tau(x_0 - x^*)\right]. \end{aligned}$$

Leading by conditions (C_6) and (9) to

$$\|x_0 - x^* - F'(x_0)^{-1}F(x_0)\| \leq \frac{\displaystyle\int_0^1 \varphi((1-\tau)\|x_0 - x^*\|)d\tau\|x_0 - x^*\|}{1 - \varphi_0(\|x_0 - x^*\|)}. \tag{11}$$

Moreover, we can write

$$F(x_0) = F(x_0) - F(x^*) = \int_0^1 F'(x^* + \tau(x_0 - x*))d\tau(x_0 - x^*).$$

So,

$$F'(x^*)^{-1}F(x_0) = \int_0^1 \left[F'(x^*)F'(x^* + \tau(x_0 - x^*)) - F'(x^*) + F'(x^*)\right]d\tau.$$

Consequently,

$$\|F'(x^*)^{-1}F(x_0)\| \le \left(1+\int_0^1 \varphi_0(\tau\|x_0-x^*\|)d\tau\right)\|x_0-x^*\|. \tag{12}$$

Summing up (11), (12) and using (10), (3) and (5) for $m=1$, we get

$$\begin{aligned}
&\|y_0-x^*\|\\
&\le \int_0^1 \frac{\varphi((1-\tau)\|x_0-x^*\|)d\tau + |1-\alpha|\left(1+\int_0^1 \varphi_0(\tau\|x_0-x^*\|)d\tau\right)}{1-\varphi_0(\|x_0-x^*\|)}\|x_0-x^*\|\\
&\le h_1(\|x_0-x^*\|)\|x_0-x^*\| \le \|x_0-x^*\| < s^*.
\end{aligned}$$

Thus, the iterate $y_0 \in H(x^*, s^*)$ and the assertion (6) holds if $k=0$.

Next, we shall establish the existence of the iterate if we just show the invertibility of $F(x_0)$. Indeed, by (3), (4) for $x_0 \neq x^*$ (otherwise we stop the iteration), we have

$$\begin{aligned}
&\|(x_0-x^*)^{-1}F'(x^*)^{-1}F(x_0)\|\\
&\qquad \le \|x_0-x^*\|^{-1}\|F'(x^*)^{-1}(F(x_0)-F(x^*)-F'(x^*)(x_0-x^*))\|\\
&\qquad \le \|x_0-x^*\|^{-1}\int_0^1 \varphi_0(\tau\|x_0-x^*\|)d\tau\|x_0-x^*\|\\
&\qquad = \int_0^1 \varphi_0(\tau\|x_0-x^*\|)d\tau = p_0 < 1,
\end{aligned}$$

so $F(x_0)\neq 0$ and

$$\|F(x_0)^{-1}F'(x^*)\| \le \frac{1}{\|x_0-x^*\|(1-p_0)}. \tag{13}$$

We need two more estimates. First, with y_0 replacing x_0 in (12), we have

$$\begin{aligned}
&\|F'(x^*)^{-1}F(y_0)\|\\
&\quad\le \left(1+\int_0^1 \varphi_0(\tau\|y_0-x^*\|)d\tau\right)\|y_0-x^*\|\\
&\quad\le \left(1+\int_0^1 \varphi_0(\tau h_1(\|x_0-x^*\|))\|x_0-x^*\|d\tau\right)h_1(\|x_0-x^*\|)\|x_0-x^*\|.
\end{aligned} \tag{14}$$

Hence we obtain

$$\begin{aligned}
&\left\|\frac{F(y_0)}{F(x_0)}\right\|^2 = \left\|\frac{F'(x^*)^{-1}F(y_0)}{F'(x^*)^{-1}F(x_0)}\right\|^2\\
&\quad\le \left(\frac{\left(1+\int_0^1 \varphi_0(\tau h_1(\|x_0-x^*\|))\|x_0-x^*\|d\tau\right)h_1(\|x_0-x^*\|)}{1-p_0}\right)^2 = q_0.
\end{aligned} \tag{15}$$

It follows from the second substep that we can write in turn

$$\begin{aligned} x_1 - x^* &= y_0 - x^* - F'(y_0)^{-1}F(y_0) + (F'(y_0)^{-1} - F'(x_0)^{-1})F(y_0) \\ &\quad + |1-\gamma|F'(x_0)^{-1}F(y_0) - \Big(|\beta| + |\delta|\Big(\frac{F(y_0)}{F(x_0)}\Big)^2\Big)F(x_0)^{-1}F(x_0). \end{aligned} \tag{16}$$

Next, using (3), (5) (for $m = 1$), (13)-(15), the identity (16) can give

$$\begin{aligned} &\|x_1 - x^*\| \\ &\le \frac{\displaystyle\int_0^1 \varphi((1-\tau)h_1(\|x_0 - x^*\|)\|x_0 - x^*\|)d\tau(h_1(\|x_0 - x^*\|)\|x_0 - x^*\|)}{1 - \varphi_0(h_1(\|x_0 - x^*\|)\|x_0 - x^*\|)} \\ &\quad + \frac{\overline{\varphi_0}\Big(1 + \displaystyle\int_0^1 \varphi_0(\tau h_1(\|x_0 - x^*\|)\|x_0 - x^*\|)d\tau\Big)h_1(\|x_0 - x^*\|)\|x_0 - x^*\|}{(1 - \varphi_0(\|x_0 - x^*\|))(1 - \varphi_0(h_1(\|x_0 - x^*\|)\|x_0 - x^*\|))} \\ &\quad + \frac{|1-\gamma|\Big(1 + \displaystyle\int_0^1 \varphi_0(\tau h_1(\|x_0 - x^*\|)\|x_0 - x^*\|)d\tau\Big)h_1(\|x_0 - x^*\|)\|x_0 - x^*\|}{1 - \varphi_0(\|x_0 - x^*\|)} \\ &\quad + \frac{(|\beta| + |\delta|q_0)\Big(1 + \displaystyle\int_0^1 \varphi_0(\tau\|x_0 - x^*\|)d\tau\Big)\|x_0 - x^*\|}{1 - \varphi_0(\|x_0 - x^*\|)} \\ &\le h_2(\|x_0 - x^*\|)\|x_0 - x^*\| \le \|x_0 - x^*\| < s^*. \end{aligned}$$

Where we also used the estimates

$$\|F'(x^*)^{-1}(F'(y_0) - F'(x_0))\| \le \varphi(\|y_0 - x_0\|) \le \varphi(\|x_0 - x^*\| + \|y_0 - x^*\|) \le \overline{\varphi_0}$$

or

$$\begin{aligned} &\|F'(x^*)^{-1}(F'(y_0) - F'(x_0))\| \\ &\qquad \le \|F'(x^*)^{-1}(F'(y_0) - F'(x^*))\| + \|F'(x^*)^{-1}(F'(x_0) - F'(x^*))\| \\ &\qquad \le \varphi_0(\|y_0 - x^*\|) + \varphi_0(\|x_0 - x^*\|) \le \overline{\varphi_0}. \end{aligned}$$

Hence the iterate $y_1 \in H(x^*, s^*)$ and the assertion (7) holds if $k = 0$. Simply exchange x_0, y_0, x_1 by x_k, y_k, x_{k+1} respectively in the preceding calculations to complete the induction for (6) and (7). Notice that $c = h_2(\|x_0 - x^*\|) \in [0, 1)$. Then from (7), we obtain

$$\|x_{k+1} - x^*\| \le c\|x_k - x^*\| \le c^{k+1}\|x_0 - x^*\| \le s^*.$$

Hence, we conclude that $\lim_{k\to\infty} x_k = x^*$ and the iterate $x_{k+1} \in H(x^*, s^*)$.

The isolation of x^* is provided in the next result.

PROPOSITION 2.3
Suppose that (C_5) *holds in* $H(x^*, s_3)$ *for some* $s_3 > 0$ *and there exists* $s_4 \geq s_3$ *such that*

$$\int_0^1 \varphi_0(\tau s_4)d\tau < 1.$$

Define the region $D_1 = D \cap \overline{H}(x^*, s_4)$. *Then,* x^* *is the unique solution of the equation* $F(x) = 0$ *in the region* D_1.

Proof. Suppose there exists a solution $z^* \in D_1$ of the equation $F(x) = 0$ such that $z^* \neq x^*$. Then, define the linear function

$$M_1 = \int_0^1 F'(x^* + \tau(z^* - x^*))d\tau.$$

Using (C_5), (8) and (9) we get in turn

$$\|F'(x^*)^{-1}(M_1 - F'(x^*))\| \leq \int_0^1 \varphi_0(\tau\|z^* - x^*\|)d\tau \leq \int_0^1 \varphi_0(\tau s_4)d\tau < 1.$$

Hence, by (10), the function M_1 is invertible. Finally, from the identity

$$z^* - x^* = M_1^{-1}(F(z^*) - F(x^*)) = M_1^{-1}(0) = 0,$$

we deduce that $z^* = x^*$.

REMARK 2.4

(i) The limit point s^* can be replaced by s in (C_7).

(ii) Under all the conditions (C_1)–(C_7), one can take $s_3 = s^*$ in Proposition 2.3.

3. Numerical Examples

Here, we analyze the radii of convergence for the method (2) by applying it to several well-known numerical problems. We present examples with the generalized continuity condition for different values of α and δ. Furthermore, we compare these results with the convergence radii achieved by the third-order midpoint method, which was introduced by I.K. Argyros and S. George in [4], highlighting the differences in performance. This technique is

$$y_k = x_k - \frac{F(x_k)}{F'(x_k)},$$

$$x_{k+1} = x_k - \frac{F(x)}{F'(\frac{x_k+y_k}{2})}. \tag{17}$$

We denote η as aradius of convergence of method (17) and $s^* = \min\{s_1, s_2\}$ as a radius of convergence of method (2). Our method (2) has a larger radius of the convergence ball than method (17) for $\alpha = 1$ and $\delta = 0$.

Example 3.1 ([18])
Let $X = Y = \mathbb{R}$. We define F on $D = [\frac{-1}{2}, \frac{5}{2}]$ by

$$F(x) = \begin{cases} x^3 \log(x^2) + x^5 - x^4, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

We have $x^* = 1$ and $\varphi_0(t) = \varphi(t) = 96.6628t$. The radius s^* is evaluated by the functions 'h_1' and 'h_2' in the following Table 1.

Table 1: Parameters of Example 3.1

Several Radii of Method (2)				
α	δ	s_1	s_2	$s^* = \min\{s_1, s_2\}$
1	0	0.006897	0.003269	0.003269
1	0.5	0.006897	0.003170	0.003170
1	1	0.006897	0.003086	0.003086
1	1.5	0.006897	0.003013	0.003013
1	2	0.006897	0.002949	0.002949
0.8	1	0.005173	0.001503	0.001503
0.9	1	0.006007	0.002305	0.002305
1.1	1	0.006007	0.002322	0.002322
1.2	1	0.005173	0.001644	0.001644
1.3	1	0.004389	0.001038	0.001038

From Table 1, we get the largest radius of convergence of method (2) for $\alpha = 1$ and $\delta = 0$, that is, $s^* = \min\{s_1, s_2\} = 0.003269$, which is greater than the radius of convergence of method (17) $= \eta = 0.002500$. The graph of the functions 'h_1' and 'h_2' is shown in the Fig. 1 for $\alpha = 1$ and $\delta = 0$.

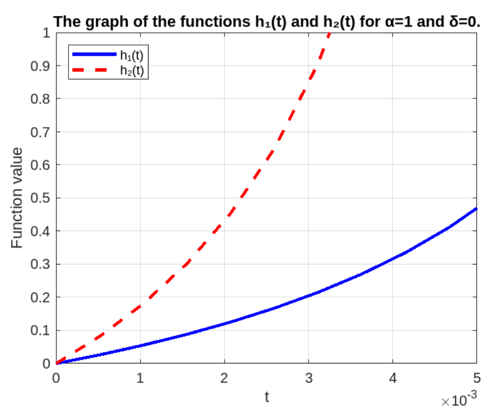

Fig. 1: Figure of Example 3.1

Example 3.2 ([18])
Let $X = Y = \mathbb{R}$. We define F on $D = [-1, 1]$ by $F(x) = \sin(x)$. We have $x^* = 0$ and $\varphi_0(t) = \varphi(t) = t$. The radius s^* is evaluated by the functions 'h_1' and 'h_2' in the following Table 2.

Table 2: Parameters of Example 3.2

Several Radii of Method (2)				
α	δ	s_1	s_2	$s^* = \min\{s_1, s_2\}$
1	0	0.666667	0.315972	0.315972
1	0.5	0.666667	0.306390	0.306390
1	1	0.666667	0.298297	0.298297
1	1.5	0.666667	0.291282	0.291282
1	2	0.666667	0.285085	0.285085
0.8	1	0.500000	0.145283	0.145283
0.9	1	0.580645	0.222771	0.222771
1.1	1	0.580645	0.224421	0.224421
1.2	1	0.500000	0.158885	0.158885
1.3	1	0.424242	0.100304	0.100304

From Table 2, we get the largest radius of convergence of method (2) for $\alpha = 1$ and $\delta = 0$, that is, $s^* = \min\{s_1, s_2\} = 0.315972$, which is less than the radius of convergence of method (17) $= \eta = 0.333333$. The graph of the functions 'h_1' and 'h_2' is shown in the Fig. 2 for $\alpha = 1$ and $\delta = 0$.

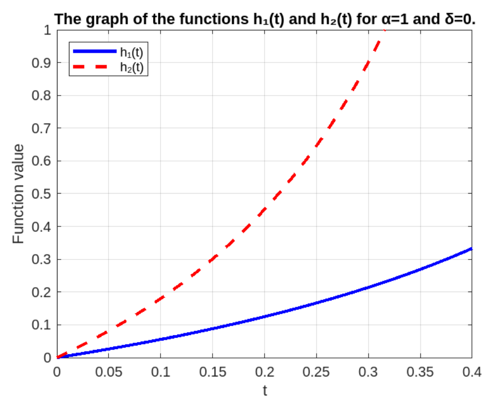

Fig. 2: Figure of Example-3.2

Example 3.3 ([18])
Let $X = Y = \mathbb{R}^3$. We define F on $D = \overline{H}(0, 1)$ for $(x_1, x_2, x_3)^t$ by

$$F(x) = \left(e^{x_1}, \frac{e-1}{2}x_2^2 + x_2, x_3\right)^t.$$

Let $x^* = (0, 0, 0)^t$, $\varphi_0(t) = (e-1)t$ and $\varphi(t) = e^{\frac{1}{e-1}}t$ then the radius s^* is evaluated by the functions 'h_1' and 'h_2' in the following Table 3.

From Table 3, we get the largest radius of convergence of method (2) for $\alpha = 1$ and $\delta = 0$, that is, $s^* = \min\{s_1, s_2\} = 0.179451$, which is greater than the radius of convergence of method (17) $= \eta = 0.085221$. The graph of the functions 'h_1' and 'h_2' is shown in the Fig. 3 for $\alpha = 1$ and $\delta = 0$.

Table 3: Parameters of Example 3.3

Several Radii of Method (2)				
α	δ	s_1	s_2	$s^* = \min\{s_1, s_2\}$
1	0	0.382692	0.179451	0.179451
1	0.5	0.382692	0.174002	0.174002
1	1	0.382692	0.169392	0.169392
1	1.5	0.382692	0.165392	0.165392
1	2	0.382692	0.161857	0.161857
0.8	1	0.287264	0.082814	0.082814
0.9	1	0.333459	0.126795	0.126795
1.1	1	0.333459	0.127737	0.127737
1.2	1	0.287264	0.090584	0.090584
1.3	1	0.243834	0.057255	0.057255

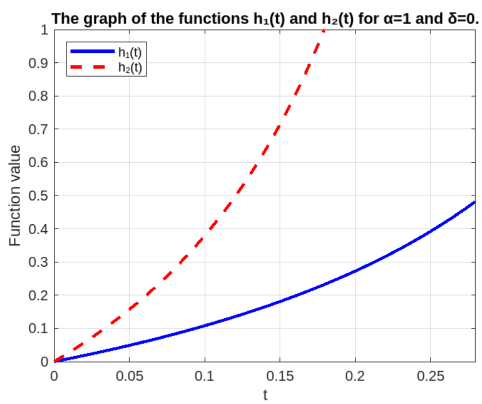

Fig. 3: Figure of Example 3.3

4. Conclusion

The local convergence analysis of a third-order bi-parametric iterative technique (2) is based on a single condition that the first-order Fréchet derivative satisfied the generalized continuity condition. This makes the method more useful. In this study, we examined the radius of convergence balls for Method (2) using the existence and uniqueness theorem. A number of numerical experiments were performed to investigate the effectiveness of the method. The findings indicate that Method (2) is most effective with $\alpha = 1$ and $\delta = 0$, resulting in the largest radius of convergence. It is easy to observe this outcome in the Cartesian graph. Furthermore, a comparison with Method (17) shows that Method (2) consistently provides a larger radius of convergence, not only for $\alpha = 1$ and $\delta = 0$, but also for various other values of α and δ. These results demonstrate that Method (2) is more efficient in many cases and would be an effective approach for solving nonlinear problems.

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