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Report of Meeting

21st International Conference on Functional Equations and Inequalities, Będlewo, Poland, September 14–20, 2025

The **21st International Conference on Functional Equations and Inequalities** (21st ICFEI) took place at the *Mathematical Research and Conference Center* in Będlewo, Poland, from September 14 to 20, 2025. It was organized by the *Department of Mathematics of the University of the National Education Commission, Krakow* and the *Stefan Banach International Mathematical Center*.

The Scientific Committee of the 21st ICFEI consisted of the following professors: Nicole Brillouët-Belluot (France), Janusz Brzdęk (Poland) – chairman, Jacek Chmieliński (Poland), Roman Ger (Poland), Zsolt Páles (Hungary), Dorian Popa (Romania), Ekaterina Shulman (Poland/Russia), Henrik Stetkær (Denmark), László Székelyhidi (Hungary) and Marek Cezary Zdun (Poland). Just a few weeks before the conference, we were deeply saddened to learn of the passing of Professor Roman Ger, an outstanding mathematician and a valued member of the ICFEI Scientific Committee. It was decided that the 21st ICFEI would be dedicated to his memory.

The Organizing Committee consisted of Jacek Chmieliński (chairman), Beata Deręgowska, Zbigniew Leśniak, Paweł Pasteczka, Paweł Solarz and Paweł Wójcik.

There were 44 participants present in Będlewo. In total, nine countries were represented: Poland (24 participants), Hungary (7), Croatia (3), India (3), Romania (3), and – with one participant each – Algeria, Austria, Italy and Japan.

The conference was officially opened on Monday, September 15, by Professor Jacek Chmieliński – chairman of the organizing committee.

During 19 scientific sessions, 42 talks were presented. Five of these were longer invited lectures, delivered by Professors Karol Baron, Włodzimierz Fechner, Dorota Głazowska, Diana Otrocol and PawełWójcik. The talks focused mainly on functional equations and inequalities, their stability, convexity, means, as well as related topics in real analysis, functional analysis, applications of mathematics and others. Additionally, apart from the regular talks, spontaneous contributions were presented during the *Problems and Remarks* sessions.

Some social events accompanied the conference. A picnic with a bonfire was organized on Tuesday evening, followed by a banquet on Thursday. On Wednesday afternoon, an excursion to Kórnik was arranged, including a visit to the 19th-century palace and the arboretum – the largest and oldest in Poland – with over 3,300 taxa of trees and shrubs.

The conference concluded on Saturday, September 20. Professor Zbigniew Leśniak delivered a summary of the current conference along with remarks related to previous editions.

1. Abstracts of Talks

Alina Ramona Baias *Ulam stability of a functional equation of Fibonacci type* (joint work with Dorian Popa)

In this talk we present some results on Ulam stability of the functional equation of Fibonacci type, given by

$$f(x+2) = af(x+1) + bf(x),$$

where $f\colon \mathbb{R} \to X$, $a, b \in \mathbb{C}$ and X is a Banach space over $\mathbb{C}$. Some results concerning the best Ulam constant of this equation are also provided.

Himanshu Baranwal *A framework for construction of fractals*

The talk is intended for a general mathematical audience. It establishes a perspective to the construction of fractals expanding the scope of the iterated function system theory.

References

[1] Baranwal, H., Chand, A. K. B., Petruşel, A., Wen, and C.-F. "Weakly Enriched Cyclic Contraction and Its Fractal." *Optimization*, (2025): 1-28.

[2] Barnsley, M.F. *Fractals everywhere.* Academic press, 2014.

Karol Baron Limit theorems for sequences of iterates of random-valued functions

Assume that $(\Omega, \mathcal{A}, \mathbb{P})$ is a probability space and (X, ρ) is a metric space with the σ–algebra $\mathcal{B}$ of all its Borel subsets. Let $f\colon X \times \Omega \to X$ be a function which is measurable for $\mathcal{B} \otimes \mathcal{A}$ and define its *iterates* by

$$f^0(x, \omega_1, \omega_2, \ldots) = x, \quad f^n(x, \omega_1, \omega_2, \ldots) = f\Big(f^{n-1}(x, \omega_1, \omega_2, \ldots), \omega_n\Big)$$

for $n \in \mathbb{N}$, $x \in X$ and $(\omega_1, \omega_2, \ldots) \in \Omega^{\mathbb{N}}$. We will discuss the problem of convergence of the sequence $(f^n)_{n\in\mathbb{N}}$, including the laws of large numbers. The idea is to also show its connections with functional equations, mainly with the equation

$$\varphi(x) = \int\limits_{\Omega} \varphi\Big(f(x, \omega)\Big)\mathbb{P}(d\omega) + F(x),$$

but not only.

Pál Burai *A nonlinear model of evolution of beliefs in social networks* (joint work with Paweł Pasteczka)

The main goal of this talk is to introduce a new model of evolvement of beliefs on networks. It generalizes the DeGroot model and describes the iterative process of establishing the consensus in isolated social networks in the case of nonlinear aggregation functions.

Our main tools come from mean theory and graph theory. The case, when the root set of the network (influencers, news agencies, etc.) is ergodic, is fully discussed. The other possibility (when the root contains more than one component) is partially discussed and it could be a motivation for further research.

Tomislav Burić *Application of the asymptotic expansion techniques to the compound means*

Let us consider an iterative process of two means M and N of two positive variables, in a following way:

$$\begin{aligned} &M_0(s,t) := s, && N_0(s,t) := t, \\ &M_n(s,t) := M(M_{n-1}, N_{n-1}), && N_n(s,t) := N(M_{n-1}, N_{n-1}), \quad n \geqslant 1. \end{aligned}$$

If both of these sequences converge to the same limit, this common value is called the Gaussian compound mean of s and t and is denoted by $M \otimes_g N(s,t)$. If we modify step $N_n(s,t) := N(M_n, N_{n-1})$ by using the updated mean M_n from the same iteration, if such new common limit exists, it is called the Archimedean compound mean and denoted by $M \otimes_a N(s,t)$.

In this talk we will present a complete asymptotic expansion of such Gaussian and Archimedean compound of two arbitrary means, both in symmetric and non-symmetric case, and derive an efficient algorithm for computing coefficients in this expansion. By asymptotic expansion of a mean we consider representation of mean in a form

$$M(x+s, x+t) = x \sum_{n=0}^{\infty} c_n(s,t) x^{-n}, \quad x \to \infty,$$

and our new approach is based on solving functional equations in terms of such power series. The technique of developing asymptotic expansions of means is presented in a series of recently published papers and is successfully used in the comparison of classical means and establishing various relations between means.

Jacek Chmieliński *On approximate smoothness*

The concept of *approximate smoothness* in Banach spaces was introduced in the paper [1]. The results obtained there were further developed in [2]. We will report on selected findings on this topic.

The presented results were obtained jointly with co-authors of [1] and [2].

References

[1] J. Chmieliński, D. Khurana, D. Sain, *Approximate smoothness in normed linear spaces*, Banach J. Math. Anal. (2023), 17:41

[2] J. Chmieliński, S. Ghosh, K. Paul, D. Sain, *Smoothness and approximate smoothness in normed linear spaces and operator spaces*, Ann. Funct. Anal. **16** (2025), Art. 23.

Jacek Chudziak *Buying and selling prices under rank-dependent utility*

The *buying price* $B(w, X)$ is a real number that expresses the maximal price at which the investor, with an initial wealth level $w \in \mathbb{R}$, is willing to buy a risky asset X. If the investor has already received X, the selling price $S(w, X)$ expresses the minimal price at which he is willing to sell X. In the expected utility model, where the risks are represented by essentially bounded random variables on a given probability space, *the buying price for X at a wealth level w* is defined implicitly through the equation

$$E\Big[u\big(w + X - B_u(w, X)\big)\Big] = u(w),$$

where $u\colon \mathbb{R} \to \mathbb{R}$ is a continuous and strictly increasing utility function. *The selling price* at a wealth level w is given by

$$E\Big[u(w + X)\Big] = u\big(w + S_u(w, X)\big).$$

Several properties of the buying and selling prices under the expected utility model have been investigated in [1]. In the talk, we introduce the buying and selling prices under rank-dependent utility, the alternative behavioral model of decision making under risk and uncertainty. Moreover, we prove various properties of these prices.

References

[1] Lewandowski M., "Buying and selling price for risky lotteries and expected utility theory with gambling wealth" *J. Risk Uncertainty* 48 (2014): 253–283.

Beata Deręgowska *One-complemented subspaces of ℓ^n_∞*

During this talk, we investigate when a subspace of ℓ_n^∞ is isometrically isomorphic to ℓ_{n-m}^∞. Building on known results for codimension one, we give a concrete and verifiable criterion for codimension two, and extend it to an arbitrary codimension.

Włodzimierz Fechner From the Leibniz rule to a second-order operator identity (joint work with Eszter Gselmann)

This talk surveys recent results concerning operator relations that characterize the differential operator on the space of smooth functions. The foundations for this topic were laid by Hermann König and Vitali Milman in a series of articles, summarized in their monograph [6]. A key result is their complete characterization of the solutions to the Leibniz rule [6, Theorem 3.1]:

$$T(f \cdot g) = f \cdot T(g) + g \cdot T(f), \qquad f, g \in C^k(I).$$

We will discuss further developments and recent results on this topic [1-5]. In particular, we will study the operator equation

$$D(f \cdot g \cdot h) - fD(g \cdot h) - gD(f \cdot h) - hD(f \cdot g) + f \cdot gD(h) + f \cdot hD(g) + g \cdot hD(f) = 0.$$

This equation is motivated by a corresponding identity for the second-order differential operator.

References

[1] Włodzimierz Fechner, Eszter Gselmann and Aleksandra Świątczak-Kolenda. *Characterizations of second-order differential operators.* Bull. Malays. Math. Sci. Soc. (2025) 48:47.

[2] Włodzimierz Fechner, Eszter Gselmann and Aleksandra Świątczak-Kolenda. *Operator relations characterizing higher-order differential operators.* Acta Math. Hungar., 17176 (1) (2025), 264–275.

[3] Włodzimierz Fechner and Eszter Gselmann. *Second-order derivations of functions spaces - a characterization of second-order differential operators.* arXiv:2502.09028.

[4] Włodzimierz Fechner and Eszter Gselmann. *A direct and algebraic characterization of higher-order differential operators.* arXiv:2504.10100.

[5] Włodzimierz Fechner and Aleksandra Światczak. *Characterizations of derivations on spaces of smooth functions. Aequationes Math.*, 97:777–786, 2023.

[6] Hermann König and Vitali Milman. *Operator relations characterizing derivatives.* Birkhäuser / Springer, Cham, 2018.

Gian Luigi Forti *Alternative functional equations: some open problems*

During the last months I was writing a survey on alternative functional equations and a number of open problems came to light. In this talk I intend to present some of them in the context where they arise.

References

[1] Forti, Gian Luigi. *Alternative functional equations. A survey.* Ann. Math. Sil. (submitted)

Souvik Ghosh *Smoothness and its generalizations in Banach spaces and in the space of bounded linear operators*

Let $\mathbb{X}$ be a Banach space and let $x \in \mathbb{X} \setminus \{0\}$ be fixed. The set of all supporting functionals of x is denoted by $J(x)$ and defined by the formula

$$J(x) := \{ f \in \mathbb{X}^* \ : \ \|f\| = 1, \ f(x) = \|x\| \}.$$

The element x is said to be *smooth* if $J(x)$ is singleton. It is a very natural question that: *Whether the set of all smooth points in a Banach space is open or closed?*

In this talk, we discuss a sufficient condition for the set of all smooth points is open in a Banach space through a newly defined property. Next we study a generalized notion of smoothness (approximate smoothness) in the space of bounded linear operators. We obtain a characterization of the same via a numerical range approach. Further, we explore the coincidence of smoothness and approximate smoothness in the space of bounded linear operators.

References

[1] J. Chmieliński, **S. Ghosh**, K. Paul, and D. Sain, *Smoothness and approximate smoothness in normed linear spaces and operator spaces*, Ann. Funct. Anal., **16** (2025), no. 23, 1-24.

Dorota Głazowska Weakly associative functions and means

The associativity functional equation

$$M\big(M(x,y),z\big) = M\big(x, M(y,z)\big), \tag{1}$$

assumed to be satisfied for all three variables x, y, z from a set I, being a principal subject of algebra, under some special natural conditions, is used to construct t-norms and t-conorms, the basic tools in fuzzy sets theory. Assuming that I is a one side open interval, J. Aczél proved, that if a continuous and strictly increasing in both variables function $M\colon I^2 \to I$ satisfies the associativity equation, then there exists a continuous and strictly monotonic function $f\colon I \to \mathbb{R}$ such that

$$M(x,y) = f^{-1}\big(f(x) + f(y)\big), \qquad x, y \in I;$$

in particular, M must be symmetric. It is also well known that, except for some very special cases like

$$M(x,y) = \min(x,y), \qquad M(x,y) = \max(x,y),$$
$$M(x,y) = x, \qquad M(x,y) = y,$$

in general, means and premeans do not enjoy the associativity property. But, it turns out that if we weaken the associativity condition a bit, the situation changes significantly.

During this talk we shall present an overview of the results concerning the so-called weakly associative functions. Namely, a function $M\colon I^2 \to I$ is said to be *weakly associative*, if

$$M\big(M(x,y),x\big) = M\big(x, M(y,x)\big), \qquad x, y \in I. \tag{2}$$

Thus M is weakly associative if it satisfies the original associativity equation (1) for all $x, y, z \in I$ such that $z = x$.

One can easily check that every weighted quasi-arithmetic mean satisfies the above condition (2), so it is weakly associative. We give the characterization of weakly associative functions in the class of some generalized weighted quasi-arithmetic means. Moreover, premeans which are rational functions of degree at most 2 and weakly associative will be presented. We will also discuss, some new broader class of weakly associative non-symmetric means, its characterization and some open problems in this area.

Angshuman Robin Goswami *Some stability results in sequences*

The primary goal of this talk is to introduce stability results for several well-known sequence classes that can be represented via discrete functional equations or inequalities. We will focus on four fundamental types of sequences: monotone, convex, subadditive and periodic.

A special attention will be given to convex sequences, including their characterizations and applications. We will also explore the deep connection between Fekete's Subadditive Lemma and Hille's Theorem, illustrating how these classical results are interrelated. Furthermore, we propose a Hermite-Hadamard-type inequality in the discrete setting, along with a corresponding sandwich-type result.

Finally, we will present decomposition and equivalence theorems for approximately monotone and approximately periodic sequences, offering insight into their structure and behavior.

Eszter Gselmann *Characterizations of higher-order differential operators* (joint work with Włodzimierz Fechner)

This talk presents an approach to characterizing higher-order differential operators using direct and algebraic methods. Building on the foundational Leibniz rule for first-order derivatives, we extend the characterization to higher-order operators. Instead of exploring the higher-order Leibniz rule equation involving several differential operators (as it was done e.g. in [2, 3, 4]), we propose a new identity that characterizes the nth order differential operator with an operator equation involving a single operator. Our results show that under certain, mild conditions, these operators can be characterized as solutions of single operator equation. Our findings provide a comprehensive framework for understanding the algebraic structure of higher-order differential operators in function spaces. The background of this problem area can be found in [1], where we dealt with second-order differential operators.

References

[1] Włodzimierz Fechner and Eszter Gselmann. Second-order derivations of functions spaces—a characterization of second-order differential operators. *arXiv*, 2025.

[2] Helmut Goldmann and Peter Šemrl. Multiplicative derivations on $C(X)$. *Monatsh. Math.*, 121(3):189–197, 1996.

[3] Hermann König and Vitali Milman. Characterizing the derivative and the entropy function by the Leibniz rule. *J. Funct. Anal.*, 261(5):1325–1344, 2011.

[4] Hermann König and Vitali Milman. *Operator relations characterizing derivatives.* Cham: Birkhäuser, 2018.

László Horváth *Characterization of dual Steffensen-Popoviciu measures on compact intervals*

The Steffensen-Popoviciu measures have fundamental importance, e.g. in the extension of Jensen's and Hermite-Hadamard inequalities to signed measures and in the theory of majorization. The dual Steffensen-Popoviciu measures appear in few applications, although its applicability is demonstrated by, for example, the results on the extension of the Bennett's Inequality. One obvious reason for this is that, apart from a few specific such measures, little was known about these measures.

Another important and interesting area where the dual Steffensen-Popoviciu measure comes up is the right-hand side of the Hermite-Hadamard inequality on compact intervals. The characterization of the dual Steffensen-Popoviciu measures has so far been an open problem. In this talk, we give a complete characterization of the dual Steffensen-Popoviciu measures on compact intervals. We give two approaches to the problem. Using the general case, we provide a more easily verifiable necessary and sufficient condition for discrete signed measures.

Szymon Ignaciuk *Halpern's algorithm for multi-agent nonexpansive operators* (joint work with Paweł Kluza, Paweł Kurasiński, Zdzisław Otachel, Andrzej Wiśnicki)

The presented results concern the problem of computing fixed points of nonexpansive operators in a distributed setting, where agents collaborate over time-varying networks in real Hilbert spaces using local data. Each agent updates its state using a local nonexpansive operator and acombination of the states of its neighbours.

Inspired by the centralized Halpern algorithm and prior works on multi-agent Krasnosel'skiĭ–Mann algorithms (D-KM and B-DKM), the distributed Halpern algorithm with static weights (D-HI-SW), and its more general version, the distributed Halpern algorithm (D-HI) with time-varying communication graphs, are introduced. It is shown that the proposed iterations converge strongly to a fixed point of the global operator.

Requiring minimal assumptions, these algorithms offer versatile tools for distributed optimization and convex feasibility. The theoretical results are illustrated with several numerical examples.

Wojciech Jabłoński *On the set of differences for some manifolds*

The famous Steinhaus theorem states that $0 \in \mathbb{R}^n$ is an inner point of the set of differences $A - A$ for a set A of positive Lebesgue measure in $\mathbb{R}^n$. This property has numerous generalizations in various directions. We will consider two classes $\mathcal{M}_1$ and $\mathcal{M}_2$ of manifolds in some vector spaces. Our aim is to show that even if $\operatorname{int}(M - M) \neq \emptyset$ for $M \in \mathcal{M}_j$, we cannot expect that $0 \in \operatorname{int}(M - M)$. However, in some cases we get $0 \in \operatorname{cl} \operatorname{int}(M - M)$.

Justyna Jarczyk *On a multiplicative iterative equation and its relationships to probability theory*

This is a joint work with Witold Jarczyk. The aim of the talk is briefly present new directions and progress in our study of the functional equation

$$\psi(x) = \prod_{j=1}^{\infty} \psi\big(f_j(x)\big)^{p_j(x)}.$$

I also intend to draw your attention to connections of this equation with certain problems in probability theory.

Witold Jarczyk *On iterative roots of some multifunctions*

This is a joint work with Chaitanya Gopalakrishna. Given an arbitrary nonvoid set X, without any structure, and a multifunction mapping X into 2^X, we are interested in the existence of their multivalued iterative roots. More precisely, given a set-valued $f\colon X \to 2^X$ and an integer $n \geqslant 2$, we are asking about multifunctions $g\colon X \to 2^X$ such that $g^n = f$. Here

$$g^n := \underbrace{g \circ \ldots \circ g}_{n \text{ times}},$$

where the composition $h \circ g\colon X \to 2^X$ of multifunctions $g, h\colon X \to 2^X$ is defined by

$$(h \circ g)(x) := h\big(g(x)\big)$$

and

$$h(A) := \bigcup_{x \in A} h(x)$$

for each $A \subseteq X$.

In the talk I am going to present our first results providing some necessary conditions of the existence of iterative roots in a special class of multifunctions. This refers, in a sense, to some earlier results obtained, among others, by Grażyna Łydzińska [1].

References

[1] Grażyna Łydzińska, "On iterative roots of order n of some multifunctions with a unique set-value point", *Publ. Math. Debrecen* 93 (2018): 1-8.

Divya Khurana *Some results on operators preserving and reversing orthogonality*

In this talk, we will discuss operators approximately preserving and reversing orthogonality (in the sense of Dragomir). We will discuss a complete characterization of operators approximately preserving and reversing orthogonality. Using this characterization, we will show that for some orthogonality notations, an operator defined from a finite-dimensional Banach space to a normed linear space is approximately preserving/reversing orthogonality if and only if it is an injective operator.

Tibor Kiss *Conditional t-convexity of sets*

Let X be a linear space. For a given $t \in [0,1]$, we say that a subset $A \subseteq X$ is *conditionally t-convex with respect to $R \subseteq A \times A$*, if for $x, y \in A$ we have $tx + (1-t)y \in A$ provided that $(x, y) \in R$. In the talk, we formulate connections between the above concept and the classical t-convexity of sets.

Paweł A. Kluza *On the h-additive functions and their symmetry properties* (joint work with Mahmood Kamil Shihab)

Our goal in this paper is to generalize the class of additive functions to the more general class of functions, h–additive functions, and give several characterizations of this class by using its symmetry. Also, we provide a definition of this class h_n–additive functions and give several characteristics for it. The density of this class of functions in the plane is given as well. The characterization of the continuity of this class also is presented.

References

[1] Shihab, M.K.; Kluza, P.A. On the h-Additive Functions and Their Symmetry Properties. Symmetry 2025, 17, 158. https://doi.org/10.3390/sym17020158

[2] Páles, Z., Shihab, M.K. Cauchy–Schwarz-type inequalities for solutions of Levi-Civita-type functional equations. Period Math Hung 90, 228–238 (2025). https://doi.org/10.1007/s10998-024-00608-y

[3] Páles, Z., Shihab, M.K. Decomposition of Higher-Order Wright Convex Functions Revisited. Results Math 77, 73 (2022). https://doi.org/10.1007/s00025-022-01606-7

[4] Páles, Z.; Shihab, M.K. On convexity properties with respect to a Chebyshev system. J. Math. Anal. Appl. 2024, 530, 127728.

Mario Krnić *More accurate Jensen-type inequalities based on the sum of the Lidstone polynomials*

The main aim of this talk is to establish refinements of the Jensen inequality for the classes of completely convex and absolutely convex functions. In the first case the refinement is expressed in terms of the alternating sum of Lidstone polynomials, while the second case deals with the sum of the Lidstone polynomials. As an application, more accurate power mean inequalities are derived. In particular, we obtain strengthened versions of arithmetic-geometric mean inequality in difference and quotient form.

References

[1] M. Bošnjak, M. Krnić, N. Lovričević, J. Pečarić, *More accurate Jensen-type inequalities based on the Lidstone interpolation formula*, Results Math. (2023) 78:247

[2] R.P. Agarwall, P.J.Y. Wong, *Error Inequalities in Polynomial Interpolation and Their Applications*, Kluwer Academic Publishers, Dordrecht, Boston, London, 1993.

[3] S.S. Dragomir, J.E. Pečarić, L.E. Persson, *Properties of some functionals related to Jensen's inequality*, Acta Math. Hungar. (70) **1-2** (1996), 129–143.

[4] M. Krnić, *Improving Jensen-type inequalities via the sum of the Lidstone polynomials*, Bull. Malays. Math. Sci. Soc. (2024) 47:71

[5] M. Krnić, N. Lovričević, J. Pečarić, *On the properties of McShane's functional and their applications*, Period. Math. Hung. **66** (2013), 159–180.

[6] D.S. Mitrinović, J.E. Pečarić, A.M. Fink, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Dordrecht/Boston/London, 1993.

[7] D.V. Widder, *Completly convex functions and Lidstone series*, Trans. Amer. Math. Soc. **51** (1942), 387–398.

Abdellatif Lalmi *A nonlinear eigenvalue problem including a logarithmic source term*

This talk focuses on the analysis of a nonlinear elliptic problem involving a logarithmic nonlinearity. We express the problem under consideration as follows:

$$\begin{cases} -\Delta_p u = \lambda \left|u\right|^{q-2} u \log\left|u\right|, & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases}$$

where Ω represents the bounded and regular domain of $\mathbb{R}^n$, while $\lambda > 0$ is a spectral parameter. The case where this problem lacks a logarithmic nonlinear term has been considered in the book [1] and the articles [2, 3, 4]. The authors examined several existence and uniqueness results, along with certain features of the solutions. Moreover, the authors provided guidance on how to extend certain proofs established in the classical context, where $p = q$.

In this work, we use the variational method to demonstrate the existence of weak solutions to our problem. The theoretical framework relies on Lebesgue and Sobolev spaces, and the energy functional has a special form because of the logarithmic term. We prove that, under appropriate assumptions on λ, the energy functional is coercive and has a nontrivial minimum. We obtain an energy critical point representing this minimum as a weak solution to the problem.

We then discuss the important features of these solutions, like elliptic regularity, the way the nonlinear term affects the spectrum, and the conditions for uniqueness. These results demonstrate how to adapt standard methods of studying partial differential equations to deal with unusual nonlinearities.

References

[1] P. Drábek, A. Kufner and F. Nicolosi *Quasilinear elliptic equations with degenerations and singularities.* Walter de Gruyter. Berlin. New York, 1997.

[2] J.P. García Azorero and I. Peral Alonso. "Existence and nonuniqueness for the p-Laplacian" *Commun in Partial Differential Equations* 12, no. 12 (1987): 126-202.

[3] P. Lindqvist. "A nonlinear eigenvalue problem" *Topics in mathematical analysis, Ser. Anal. Appl. Comput., World Sci. Publ., Hackensack, NJ* 3, (2008): 175-203.

[4] M. Ôtani. "Existence and nonexistence of nontrivial solutions of some nonlinear degenerate elliptic equations" *J of functional analysis* 76, no. 1 (1988): 140-159.

Zbigniew Leśniak *On irregular points and iterative roots of Brouwer homeomorphisms*

We describe properties of the set of irregular points of a Brouwer homeomorphism f, under the assumption that there exists a foliation of the plane in which each leaf is an invariant line of f. In particular, we focus on the asymptotic behavior of points that are strongly irregular. We explicitly determine the form of the limit set for any such point, showing that it is analogous to the first prolongational limit set that would arise if f were embeddable in a continuous flow on the plane, thereby underlining a structural analogy with time-one maps of continuous planar flows.

Based on this result, we present a method for constructing continuous iterative roots g of the homeomorphism f. These iterative roots are obtained in a recursive manner, defined step-by-step on a collection of maximal parallelizable regions of f. To guarantee the global continuity of g, we use a matching property applied along the boundaries of these parallelizable regions.

References

[1] Frédéric Le Roux, Anthony G. O'Farrell, Maria Roginskaya, and Ian Short. "Flowability of plane homeomorphisms", *Ann. Inst. Fourier Grenoble* 62 (2012): 619–639.

[2] Zbigniew Leśniak. "On families of invariant lines of a Brouwer homeomorphism", *J. Difference Equ. Appl.* 25 (2019): 1363–1371.

[3] Zbigniew Leśniak. "On limit sets of a Brouwer homeomorphism generating foliations of the plane", *J. Difference Equ. Appl.* (2025): 1–13, https://doi.org/10.1080/10236198.2025.2509779.

Tomasz Małolepszy *On some functional equations related to Cauchy differences* (joint work with Janusz Matkowski)

The additive type Cauchy differences, i.e. the bivariable functions of the form

$$\Phi_f(x,y) := f(x) + f(y) - f(x+y),$$

generated by a single variable function f, as well as other types of Cauchy differences (exponential, logarithmic and multiplicative) were the subject of research by several mathematicians, for instance Baron, Kannappan, Kominek and Volkmann [1], [2], [3]. In this talk we examine the existence of functions $f\colon \mathbb{R} \to \mathbb{R}$, for which the Cauchy difference of a certain type, defined on $\mathbb{R}^2$, is just a biadditive function of the form $\alpha(x)\alpha(y)$, where $\alpha\colon \mathbb{R} \to \mathbb{R}$ is a prescribed additive function. We also present counterparts of our results for Cauchy differences of several variables.

References

[1] Baron, K., Kannappan, Pl., "On the Cauchy difference", *Aequationes Math.* 46, no. 1-2 (1993): 112-118.

[2] Baron, K., Kominek, Z., "On functionals with the Cauchy difference bounded by a homogeneous functional", *Bull. Polish Acad. Sci. Math.* 51, no. 3 (2003): 301-307.

[3] Baron, K., Volkmann, P., "On a theorem of van der Corput", *Abh. Math. Sem. Univ. Hamburg* 61, (1991): 189-195.

Lenka Mihoković *Asymptotic expansions techniques in functional equations and inequalities related to means*

As part of the analysis of the properties and relations between means, i.e. functions that assume values between minimum and maximum of their variables, using the asymptotic expansions has proven useful, especially when the underlying problem relies on functional equations. The purpose of this talk is to present some of the results obtained as a consequence of using asymptotic expansions techniques.

On a set of bivariate means, the symmetric mean M_2 of the mean M_1 with respect to the mean M_0 can be defined in several ways. The first one is related to the group structure on the set of bivariate means: $S_{M_0}(M_1) = M_2 \Leftrightarrow M_1 * M_2 = M_0 * M_0$. The second one is defined through Gauss functional equation: $\sigma_{M_0}(M_1) = M_2 \Leftrightarrow M_0(M_1, M_2) = M_0$. Motivated by the question of the matching of these two different mappings, as a result of the comparison of the coefficients, a new class of means was discovered, which interpolates between harmonic, geometric, and arithmetic means [1].

Furthermore, for three homogeneous symmetric bivariate means, define the resultant mean-map of K, M and N as

$$\mathcal{R}(K, N, M)(s, t) := K\big(N\left(s, M(s, t)\right), N\left(M(s, t), t\right)\big).$$

A symmetric mean M is said to be *stable* or *balanced* if $\mathcal{R}(M, M, M) = M$. By considering various other functional equations involving the resultant mean-map, we may arrive to the notions of (K, N)-stabilizable and (K, N)-stabilized mean M. Also the (strict) sub-stabilizability and super-stabilizability concepts can be introduced with the appropriate inequality sign in stabilizability relation. Based on the complete asymptotic expansion of the resultant mean–map and also the asymptotic expansions of stable, stabilizable and stabilized means, and using a notion of asymptotic inequality, we analyse some relations in this setting and show how to obtain the best possible parameters when the parametric means are involved [2].

References

[1] Mihoković, Lenka. Coinciding Mean of the Two Symmetries on the Set of Mean Functions. *Axioms.* 2023; 12(3):238.

[2] Mihoković, Lenka. Asymptotic expansions of stable, stabilizable and stabilized means with applications. *Appl. Anal. Discrete Math.* 18 (2024): 116–147.

Kazimierz Nikodem *Integral means of set-valued maps*

Let $a, b \in \mathbb{R}$, $a < b$, and let Y be a Banach space. Denote by $S(Y)$ the family of all nonempty subsets of Y. Given a set-valued map $F\colon [a,b] \to S(Y)$ we define the *integral mean of F on the interval* $[a,b]$ by

$$M_F(a,b) := \frac{1}{b-a}\int_a^b F(x)dx.$$

We present some basic properties of the integral means of set-valued maps. In particular, a counterpart of the classical integral mean value theorem for set-valued maps is presented and corresponding examples are given. We prove also (under appropriate assumptions) that if a set-valued map F is convex on an interval $I \subseteq \mathbb{R}$, then its integral mean M_F is Schur-convex on I^2. This is a set-valued generalization of the Elezović - Pečarić Result obtained for real valued functions.

Kazuki Okamura *Construction of graph-directed invariant sets of weak contractions on semi-metric spaces*

This talk will be based on [5]. We present a construction of graph-directed invariant sets of weak contractions in the sense of Matkowski-Rus [3, 6] on semi-metric spaces. The notion of graph-directed invariant sets was introduced by [4]. We follow the approach by Bessenyei and Pénzes [2], which applies the Kuratowski noncompactness measure without relying on Blascke's completeness theorem. We adopt the framework of regular semi-metric spaces introduced by [1], where the basic triangle function is continuous at the origin. We also establish a relationship between this approach and a generalized de Rham's functional equation indexed by a finite directed graph. One example is given by the formula

$$\begin{cases} \varphi_1(x) = \begin{cases} \frac{1}{3}\varphi_1(2x) & x \in \left[0, \frac{1}{2}\right], \\ \frac{2}{3}\varphi_2(2x-1) + \frac{1}{3} & x \in \left[\frac{1}{2}, 1\right], \end{cases} \\ \\ \varphi_2(x) = \begin{cases} \frac{1}{4}\varphi_1(2x) & x \in \left[0, \frac{1}{2}\right], \\ \frac{3}{4}\varphi_2(2x-1) + \frac{1}{4} & x \in \left[\frac{1}{2}, 1\right], \end{cases} \end{cases}$$

where φ_1 and φ_2 are strictly increasing and continuous functions on $[0,1]$.

References

[1] Mihály Bessenyei and Zsolt Páles. A contraction principle in semimetric spaces. *J. Nonlinear Convex Anal.*, 18(3):515–524, 2017.

[2] Mihály Bessenyei and Evelin Pénzes. Generalized fractals in semimetric spaces. *Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods*, 220:15, 2022. Id/No 112853.

[3] Janusz Matkowski. Integrable solutions of functional equations. *Diss. Math.*, 127, 1975.

[4] R. Daniel Mauldin and S. C. Williams. Hausdorff dimension in graph directed constructions. *Trans. Am. Math. Soc.*, 309(2):811–829, 1988.

[5] Kazuki Okamura. Construction of graph-directed invariant sets of weak contractions on semi-metric spaces. *Aequationes Mathematicae* (2025).

[6] Ioan A. Rus. *Generalized contractions and applications.* Cluj University Press, Cluj-Napoca, 2001.

Zdzisław Otachel *Jensen-Jessen inequality for convex maps*

Given a set $E \neq \emptyset$, let L be a real linear space of some real-valued functions x defined on E between others; the function $\mathbb{I} \equiv 1$ belongs to L, and let A be a *linear mean* on L, i.e. any linear positive functional on L with $A(\mathbb{I}) = 1$.

For any convex functions ϕ defined on an interval I, inequalities of the form

$$\phi\big(A(x)\big) \leqslant A\big(\phi \circ x\big), \tag{1}$$

where $x, \phi \circ x \in L$, are classical. The fundamental examples of linear means are

$$A(x) = \sum_{k=1}^{n} w_k x_k \quad \text{or} \quad A(x) = \int_0^1 w(t)x(t)dt,$$

where $w_1, \ldots, w_n$ are positive weights summable to 1 and $w(t), t \in [0,1]$, is a fixed non-negative function integrable to 1, while $x(k) = x_k \in I$ or $x(t) \in I$ is integrable. For these functionals, inequalities (1) were established in 1906 by Jensen. Presently, they are just called Jensen's inequalities; moreover, a convex function itself is defined by (1) in case of the mentioned discrete functional.

Jessen (1931) proved (1) in a general case for abstract linear means. The case of multi-variable convex function ϕ was developed by McShane (1937). The inequality remains true for the wider class of functionals compared to linear means, namely sublinear isotonic functionals A preserving constants. Results of that type come from Pečarić and Raşa (1992) and Dragomir et al. (1995). The latest results on this topic can be found in Cheung et al. (2006), Guessab and Schmeisser (2013), Cŭljak et al. (2019), Aslam et al. (2022), Khan et al. (2020) and others. Between others, Otachel (2024) obtained (1) with weaker assumptions about functionals than those made in Pečarić and Raşa or in Dragomir et al. and generalized Jessen's and McShane's inequalities.

Now, we shall present a vectorial version of inequality (1) and a counterpart of McShane's result, where convex functions are replaced by convex maps Φ with values in ordered linear spaces and functionals A being linear means are altered to *vectorial means.*

The key tool for proving inequalities of type (1) includes the supports of convex functions. We will show that certain convex maps admit analogous characterizations. It follows from versions of Hahn–Banach theorems on extensions of linear or affine maps dominated by sublinear or convex maps taking values in ordered linear spaces, which also will be discussed.

Diana Otrocol Some results on Ulam stability for differential, partial differential and difference operators

The Ulam stability problem, initially formulated for functional equations, was first developed in this direction with results obtained for classical functional equations. In recent years, many works have been dedicated to the study of Ulam stability for some classes of differential equations, differential operators, and difference equations.

In this regard, we will present some recent results obtained for the previously mentioned equations. Specifically, we present characterizations of Ulam stability for differential operators with constant and variable coefficients, differential equations with modified argument, partial differential operators of the first order, and difference equations. We also present some results on the best Ulam constant.

References

[1] J. Brzdęk, D. Popa, I. Raşa, B. Xu, *Ulam Stability of Operators*, Academic Press: Cambridge, MA, USA, 2018.

[2] A. Novac, D. Otrocol, D. Popa, Ulam stability of a linear difference equation in locally convex spaces, *Results Math.*, 76, 33 (2021).

[3] A. Novac, D. Otrocol, D. Popa, Best Ulam constant of a partial differential operator, submitted for publication.

[4] D. Otrocol, V.A. Ilea, Ulam stability for a delay differential equation, *Cent. Eur. J. Math.*, 11 no. 7 (2013): 1296-1303.

Paweł Pasteczka *Iterations of mean-type mappings and their convergence*

We define so-called residual means, which have a Taylor expansion of the form

$$M(x) = \bar{x} + \tfrac{1}{2}\xi_M(\bar{x})\mathrm{Var}(x) + o\big(\|x - \bar{x}\|^\alpha\big)$$

for some $\alpha > 2$ and a single-variable function ξ_M ($\bar{x}$ stands for the arithmetic mean of the vector x), and show that all symmetric means which are three times continuously differentiable are residual. We also calculate the value of residuum for quasideviation means and a few subclasses of this family.

Later, we apply it to establish the limit of the sequence $\Big(\frac{\mathrm{Var}\ \mathbf{M}^{n+1}(x)}{(\mathrm{Var}\ \mathbf{M}^{n}(x))^2}\Big)_{n=1}^{\infty}$, where $\mathbf{M}\colon I^p \to I^p$ is a mean-type mapping consisting of p-variable residual means on an interval I, and $x \in I^p$ is a nonconstant vector.

References

[1] Pasteczka, Paweł. "Iterations of mean-type mappings and their convergence." *Arxiv* (2024): arxiv.org/pdf/2406.12491.pdf

Dorian Popa *On the inhomogeneous Sincov's equation*

In this talk we present some results on the inhomogeneous Sincov's equation

$$f(x,z) - f(x,y) - f(y,z) = I(x,y,z),$$

where $f\colon S^2 \to G$ is the unknown function and $I\colon S^3 \to G$ is the given function (S is a nonempty set and $(G,+)$ is an abelian group). We give necessary and sufficient conditions on I for the existence of a solution for the inhomogeneous Sincov's equation. Moreover we give a result on Ulam stability for the Sincov's equation and obtain the best Ulam constant of it. In this way we give an answer to a problem formulated by L. Reich at the 61st ISFE.

Patryk Rela *The Orlicz premium principle under uncertainty* (joint work with Jacek Chudziak)

Assume that $(\Omega, \mathcal{F})$ is a measurable space and $\mu\colon \mathcal{F} \to [0,1]$ is a *capacity*, that is a monotone set function satisfying $\mu(\emptyset) = 0$ and $\mu(\Omega) = 1$. Let $\mathcal{X}_0$ be a family of all $\mathcal{F}$-measurable functions $X\colon \Omega \to [0,\infty)$ such that $\mu\big(\{X > t\}\big) = 0$ for some $t \in \mathbb{R}$. The *Orlicz premium principle under uncertainty for* $X \in \mathcal{X}_0$ is defined through the equation

$$E_\mu \left[\Phi\left(\frac{X}{H_{(\mu,\alpha,\Phi)}(X)}\right)\right] = 1 - \alpha, \tag{1}$$

where $\alpha \in [0,1)$ is a given parameter and $\Phi\colon [0,\infty) \to [0,\infty)$ is a *normalized Young function*, that is a strictly increasing, convex function $\Phi\colon [0,\infty) \to [0,\infty)$ satisfying $\Phi(0) = 0$, $\Phi(1) = 1$ and $\lim\limits_{x\to\infty} \Phi(x) = \infty$ and

$$E_\mu\left[X\right] = \int\limits_0^\infty \mu(\{X > x\})dx \quad \text{for} \quad X \in \mathcal{X}_0,$$

is the Choquet integral with respect to the capacity μ.

The aim of this talk is to prove the existence and uniquenes of the Orlicz premium defined by (1) and to characterize its several important properties.

References

[1] Bellini, F., Gianin, E. R. "On Haezendonck risk measures" *Journal of Banking and Finance*, no. 32 (2007): 986–994.

[2] Haezendonck, J., Goovaerts, M. "A new premium calculation principle based on Orlicz norms" *Insurance: Mathematics and Economics*, no. 1 (1982): 41-53.

Wutiphol Sintunavarat *On stability of two new generalized set-valued functional equations*

Denote by $C_c(Y)$ the set of all closed convex subsets of a Banach space Y. For every $A, B \in C_c(Y)$, we define an operation $\oplus$ by $A \oplus B := \overline{A+B}$, which is a closure of the set $A + B := \big\{a + b : a \in A,\, b \in B\big\}$. This work aims to establish the Hyers-Ulam-Rassias stability of the following *generalized set-valued functional equations*

$$\begin{aligned} &F\left(\frac{x+y}{2} + (\alpha-1)z\right) \oplus F\left(\frac{x+z}{2} + (\alpha-1)y\right) \oplus F\left(\frac{y+z}{2} + (\alpha-1)x\right) \\ &\quad = \alpha\left(F(x) \oplus F(y) \oplus F(z)\right) \end{aligned}$$

and

$$F(\beta x + y) \oplus F(\beta x - y) \ = \ F(x + y) \oplus F(x - y) \oplus 2(\beta^2 - 1)F(x)$$

for all $x, y, z \in X$, where $F\colon X \to C_c(Y)$ is an unknown set-valued function while X is a real vector space, $\alpha \geq 2$ and $\beta \notin \{-1, 0, 1\}$ are fixed integers. These two equations are respectively related to Cauchy-Jensen type and quadratic type set-valued functional equations.

Rafał Stypka *Linearity of additive operators approximately preserving Birkhoff-James orthogonality*

We consider additive operators between real normed spaces that approximately preserve Birkhoff-James orthogonality. In a normed space X over the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and of dimension at least 2, $\varepsilon \in [0, 1)$, one can consider the:

- *Birkhoff–James orthogonality*:

$$x \perp_B y \quad \iff \quad \|x + \lambda y\| \geqslant \|x\| \quad \text{for all} \quad \lambda \in \mathbb{K}.$$

- *Approximate Birkhoff–James orthogonality*:

$$x \perp_B^\varepsilon y \quad \iff \quad \|x + \lambda y\|^2 \geqslant \|x\|^2 - 2\varepsilon\|x\|\|\lambda y\| \quad \text{for all} \quad \lambda \in \mathbb{K},$$

  or

$$x \perp_B^\varepsilon y \quad \iff \quad \|x + \lambda y\| \geqslant \|x\| - \varepsilon\|\lambda y\| \quad \text{for all} \quad \lambda \in \mathbb{K}.$$

An operator $T\colon X \to Y$ between normed spaces is said to be *approximately preserve Birkhoff–James orthogonality* if

$$x \perp_B y \quad \Rightarrow \quad T(x) \perp_B^\varepsilon T(y) \quad \text{for all} \quad x, y \in X.$$

The main result for real normed spaces is to prove that every additive operator approximately preserving Birkhoff-James orthogonality must be linear. This is achieved without any additional assumptions such as continuity or homogeneity. Our approach is based on the geometric structure of normed spaces and on inequalities obtained from the definition of approximate Birkhoff–James orthogonality. This result extends the theorem obtained by Wójcik [2], who proved the linearity of additive mappings that exactly preserve Birkhoff-James orthogonality. As an easy consequence of this result, we observe that any additive operator that preserves (or approximately preserves) the inner product must also be linear.

References

[1] J. Chmieliński, *On an ε-Birkhoff orthogonality*, J. Inequal. Pure and Appl. Math., **6** (2005), Art. 79.

[2] P. Wójcik, *Mappings preserving B-orthogonality*, Indag. Math. (N.S.) **30** (2019), 197–200.

László Székelyhidi *Remarks on the localization of ideals in the Fourier algebra*

In our former paper [1] we introduced the concept of localization of ideals in the Fourier algebra of a locally compact abelian group. Roughly speaking, an ideal is localizable, if it is completely determined by the set of differential operators, which annihilate it at the zeros of the ideal. It turns out that localizability is equivalent to synthesizability.

References

[1] L. Székelyhidi, *Characterization of Locally Compact Abelian Groups Having Spec tral Synthesis*, to appear in Forum of Math. Sigma.

Patricia Szokol *Weighted Lagrange interpolation with exponential weight*

The famous Bernstein conjecture about optimal node systems of classical polynomial Lagrange interpolation, standing unresolved for over half a century, was solved by T. Kilgore in 1978 [2]. Immediately following him, also the additional conjecture of Erdős was solved by deBoor and Pinkus [1]. These breakthrough achievements were built on a fundamental auxiliary result on nonsingularity of derivative (Jacobian) matrices of certain interval maxima in function of the nodes.

After the above breakthrough, a considerable effort was made to extend the results to the case of at least certain restricted classes of functions and Chebyshev-Haar subspaces. The Chebyshev–Haar system of weighted polynomials with exponential weight adjoined with constant functions and the corresponding interpolation were previously studied, as well. Some hints were also mentioned for he proof of Bernstein and Erdős conjectures. Our aim is to present in detail the proof of all the auxiliary results needed in the new setting.

References

[1] C. deBoor, A. Pinkus, *Proof of the conjectures of Bernstein and Erdős concerning the optimal nodes for polynomial interpolation*, J. Approx. Th. **24** (1978), 289–303.

[2] T. Kilgore, *A characterization of the Lagrange interpolating projection with minimal Tchebycheff norm*, J. Approx. Th. **24** (1978), 273–288.

Andrzej Wiśnicki *Invariant measures for Markov operators on metric spaces*

The theory of Markov operators unites stochastic analysis with methods of fixed-point theory and ergodic theory and provides insight into the dynamics of random processes. The present talk deals with the properties and the existence of invariant measures of Markov operators on metric spaces with particular emphasis on Polish spaces.

A linear operator $P\colon B_b(X) \to B_b(X)$ on the space of bounded and Borel-measurable functions $B_b(X)$ is said to be a *Markov operator* if

(i) $P\mathbf{1}_X = \mathbf{1}_X$, and

(ii) $P\varphi \geq 0$ whenever $\varphi \geq 0$, and

(iii) if $\varphi_n \to \varphi$ pointwise, then $P\varphi_n \to P\varphi$ pointwise.

The *dual operator* $P^*\colon \mathcal{M}_1(X) \to \mathcal{M}_1(X)$ acts on the space of Borel probability measures $\mathcal{M}_1(X)$ and is defined by the transition kernel

$$P^*\mu(A) = \int_X (P\mathbf{1}_A)(x)\,\mu(dx).$$

A measure $\mu_0 \in \mathcal{M}_1(X)$ is *invariant* for P^* if $P^*\mu_0 = \mu_0$.

A classical result on the existence of invariant measures for Markov operators is the Lasota–Yorke theorem (1994) in proper metric spaces, extended by Szarek (2000) to Polish spaces. It states that if a Markov operator P^* is nonexpansive in the Fortet–Mourier metric (which metrizes weak convergence of measures) and for every $\varepsilon > 0$ there exists a Borel set $C \subseteq X$ with diameter $\operatorname{diam} C < \varepsilon$ such that

$$\inf_{\mu\in\mathcal{M}_1} \liminf_{n\to\infty} (P^*)^n\mu(C) > 0,$$

then P^* is asymptotically stable, i.e. the sequence of iterates $\{(P^*)^n\}$ converges weakly to a unique invariant measure μ_0.

The above results have been generalized in various directions. In this talk I review some of them and, motivated by the Lasota–Yorke "lower bound" technique, present some new ones that considerably relax the assumptions on both the underlying space and operators.

Paweł Wójcik A case of orthogonality equation

Let X, Y be real normed spaces and let ρ'_+, and $[\cdot|\diamond]$, be norm derivative and semi-inner product, respectively. In this talk we consider a functional equation

$$\forall_{x,y\in X} \quad \rho'_+\big(f(x), f(y)\big) = \rho'_+(x, y)$$

with an unknown function $f\colon X \to Y$. The similar function equation

$$\forall_{x,y\in X} \quad \big[f(x)|f(y)\big] = [x|y]$$

is investigated, too. As a consequence, we present partial answer to open problem posed in [1].

References

[1] C. Alsina, J. Sikorska, M.S. Tomás, *Norm Derivatives and Characterizations of Inner Product Spaces*, World Scientific, Hackensack, NJ, 2010.

Sebastian Wójcik *Zero utility principle as a generalized shortfall risk measure* (joint work with Jacek Chudziak)

In a process of insurance contracts pricing, the insurance company assigns to any risk a non-negative real number, being a premium for the risk. There are various methods of insurance contracts pricing. The zero utility principle was introduced by H. Bühlmann in 1970. This method presents the problem from the point of view of an insurance company, assuming that the premium for a given risk is determined in such a way that the company is indifferent between entering into contract and rejecting it.

In 2012 Kałuszka and Krzeszowiec introduced and studied the zero utility principle under the Cumulative Prospect Theory (cf. the results obtained by Tversky and Kahneman in 1992). In this setting, the premium for a risk X is defined as a unique solution $H_{(u,g,h)}(X)$ of the equation

$$E_{gh}\Big[u(H_{(u,g,h)}(X)-X)\Big]=0, \tag{1}$$

where $u \to \mathbb{R} \to \mathbb{R}$ is a continuous strictly increasing function and E_{gh} is the Choquet integral with respect to the probability weighting functions $g,h\colon [0,1]\to[0,1]$ for gains and losses, respectively.

In 2022 Zhang and Xu, weakening the assumptions about the generators, proposed the following risk measure

$$H_{(u,g,h)}(X)=\inf\Big\{x\in\mathbb{R}\ :\ E_{gh}\big[u(x-X)\big]\geqslant 0\Big\}, \tag{2}$$

for random losses X restricted to nonnegative. Such formulations stem for approach of generalized shortfall introduced by Föllmer and Schied (2002).

In this talk we consider the zero utility principle in the Cumulative Prospect Theory under uncertainty and investigate some its properties.

References

[1] Bühlmann, H. *Mathematical Models in Risk Theory* Berlin: Springer-Verlag, 1970.

[2] Tversky, A.; Kahneman, D. "Advances in prospect theory: Cumulative representation of uncertainty", *J. Risk Uncertain.*, no. 5 (1992): 297–323.

[3] Zhang, S., Xu., H., 2022. "Insurance premium-based shortfall risk measure induced by cumulative prospect theory" *Comput. Manag, Sci.* 19, 703–738.

[4] Föllmer, H.;, Schied, A. (2002). "Convex measures of risk and trading constraints". *Finance Stochast..* 6, 429–447.

Marek Cezary Zdun *On arc-valued iteration groups on the circle*

Let S^1 be the unite circle and $cc(S^1)$ be the family of all closed arcs of S^1. We will present the properties of set-valued iteration groups on the circle of a special form that is the families

$$\big\{F^t\colon S^1\to cc(S^1)\ \mid\ t\in\mathbb{R}\big\},$$

where $F^t(z)=\Phi^{-1}\Big[\big\{e^{2\pi it}\Phi(z)\big\}\Big]$ and $\Phi\colon S^1\to S^1$ is a continuous, non-injective and increasing mapping with respect to the cyclic order. We give a characterization of iteration groups of the above form. Let $\{I_\alpha \mid \alpha\in A\}$ be the family of arcs of constancy of Φ. The set of the iterative indexes

$$T:=\Big\{t\in\mathbb{R}\ :\ F^t\big[S^1\setminus \textstyle\bigcup_{\alpha\in A} I_\alpha\big]=F^t\big[S^1\setminus \textstyle\bigcup_{\alpha\in A} I_\alpha\big]\Big\}$$

is an additive group such that $\mathbb{Z} \subseteq T$. The group $T/\mathbb{Z}$ is non trivial and finite if and only if the set of arcs of constancy of Φ is finite. If the group $T/\mathbb{Z}$ is infinite, then no arc I_α is isolated and Φ is a Lebesgue singular function constant on arcs I_α.

Marek Żołdak *Quasiconvexity of deviation means* (joint work with Jacek Chudziak)

A function $E\colon I^2 \to \mathbb{R}$ defined on an open interval $I \subseteq \mathbb{R}$ is called a *deviation* (in the sense of Daróczy [2]) provided that

(E1) $E(x,x) = 0$ for $x \in I$;

(E2) for all $x \in I$, the function $I \ni t \mapsto E(x,t)$ is strictly decreasing and continuous.

If $E\colon I^2 \to \mathbb{R}$ is a deviation, then for all $n \in \mathbb{N}$ and $\bar{x} = (x_1, ..., x_n) \in I^n$, equation

$$\sum_{i=1}^{n} E(x_i, t) = 0 \tag{1}$$

has a unique solution $t := D_E(\bar{x})$ (cf. [2, Theorem 1]). Moreover, we have

$$\min\big\{x_i : i \in \{1, ..., n\}\big\} \leqslant D_E(\bar{x}) \leqslant \max\big\{x_i : i \in \{1, ..., n\}\big\}.$$

Hence, equation (1) defines a mean, called a *deviation mean generated by E*. Some properties of this mean were proved in [2]. An important particular case of a deviation mean is the quasi-arithmetic mean, generated by deviation $E\colon I^2 \to \mathbb{R}$ of the form $E(x,t) = \phi(x) - \phi(t)$ for $(x,t) \in I^2$, where $\phi\colon I \to \mathbb{R}$ is a continuous and strictly increasing function. Recently (cf. [1, 3]), convexity of the quasi-arithmetic mean was characterized. Furthermore, in [4] the Jensen convexity of the quasideviation means, which are a generalization of deviation means, was established. Inspired by these results, we present a characterization of quasiconvexity of the mean D_E. Applying that result we prove a characterization of quasiconvexity for quasi-arithmetic means and Bajraktarević means.

References

[1] Chudziak, J., Głazowska, D., Jarczyk, J., Jarczyk, W.: *On weighted quasi-arithmetic means which are convex*, Math. Inequal. Appl. **22**(4) (2019), 1123-1136.

[2] Daróczy, Z.: *Über eine eine Klasse von Mittelwerten*, Publ. Math. Debrecen **19** (1972), 211-217.

[3] Páles, Zs., Pasteczka, P.: *On the Jensen convex and Jensen concave envelopes of means*, Arch. Math. **116**(4) (2021), 423-432.

[4] Páles, Zs., Pasteczka, P.: *On the Jensen convexity of quasideviation and Bajraktarević means*, Aequat. Math. **97**(5-6) (2023), 1201-1219.

2. Problems and Remarks

Problem and Remark posed by Gian Luigi Forti

Consider the following alternative equation

$$f(x+y)-f(x)-f(y)\in\big\{0,\mathbf{e^1},\mathbf{e^2},\mathbf{e^1}+\mathbf{e^2},\mathbf{e^1}+\mathbf{e^3},\mathbf{e^2}+\mathbf{e^3}\big\},$$

where $f=(f_1,f_2,f_3)\colon G\to\mathbb{R}^3$, G is a group where the additive Cauchy equation is stable and $\mathbf{e^1},\mathbf{e^2},\mathbf{e^3}$ are the vectors of the standard basis of $\mathbb{R}^3$. This equation was investigated by Costanza Borelli and GLF and after reducing via stability to the case of bounded functions, that is $f_i(x)\in[-1,0]$, $i=1,2,3$, three types of nontrivial solutions are described. One of these types has the form $f=(f_1,0,f_3)$ where the sets

$$\begin{aligned}A&=\big\{x\in G: f_1(x)=f_3(x)=0\big\};\\ B&=\big\{x\in G: f_1(x)\neq 0, f_3(x)=0\big\};\\ C&=\big\{x\in G: f_1(x)=f_3(x)=-1\big\};\\ D&=\big\{x\in G: f_1(x)=f_3(x)\notin\{-1,0\}\big\};\end{aligned}$$

are non empty and satisfy the following conditions:

$$\begin{aligned}&A,B,C,A\cup B \quad \text{are subsemigroups of} \quad G;\\ &x\in A, y\in B \quad\Rightarrow\quad x+y, y+x\in A;\\ &x\in B, y\in C \quad\Rightarrow\quad x+y, y+x\in B;\\ &x\in A, y\in C \quad\Rightarrow\quad x+y, y+x\in A\cup C;\\ &x\in A\cup B\cup C, y\in D \quad\Rightarrow\quad x+y, y+x\in D.\end{aligned}$$

The problem is to identify the groups for which it is possible to have subsets A,B,C,D with the properties above. It is easy to see that for instance in $\mathbb{Z}$ this is not possible.

REFERENCES

C. Borelli, G.-L. Forti: *On an alternative functional equation in* $\mathbb{R}^n$. In "Functional Analysis, Approximation Theory and Numerical Analysis", Ed. John M. Rassias, World Sci. Publishing Co., Singapore, 1994, 33-44.

Problem and Remark posed by László Székelyhidi

This remark and problem is related to the talk "Remarks on the localization of ideals in the Fourier algebra". Let G be a locally compact abelian group, let $\mathcal{C}(G)$ (resp. $\mathcal{M}_c(G)$) denote the space of continuous complex valued functions on G (resp. the measure algebra on G), and let $\mathcal{A}(G)$ denote the Fourier algebra of G. It is known that $\mathcal{C}(G)$ is a topological module over the measure algebra and there is a one-to-one correspondence between the closed submodules (called *varieties*) of this module and the closed ideals of the measure algebra via the annihilator correspondence, and also there is a one-to-one correspondence between the varieties in $\mathcal{C}(G)$ and the closed ideals of the Fourier algebra via the Fourier transformation. In the talk we introduced the concept of polynomial differential operators on the Fourier algebra and also the concept of localizability of ideals in $\mathcal{A}(G)$. Roughly speaking, an ideal $\widehat{I}$ is *localizable* in $\mathcal{A}(G)$ if it has the following property: if a Fourier transform $\widehat{\mu}$ is annihilated by all polynomial differential operators, which annihilate the ideal $\widehat{I}$ at every root of this ideal, then $\widehat{\mu}$ is in $\widehat{I}$. The problem is the following: find a non-localizable ideal. In fact, we have proved that an ideal $\widehat{I}$ is localizable if and only if the corresponding variety $\operatorname{Ann} I$ is synthesizable. Based on this result, and on the characterization theorem of M. Laczkovich and myself (see Spectral synthesis on discrete abelian groups, *Math. Proc. Cambridge Philos. Soc.*, **143(1)**, 103–120, 2007.) there are non-localizable ideals in the Fourier algebra of a discrete ableian group, if its torsion-free rank is infinite. One example is the following: let G be the non-complete direct product (also called direct sum) of ω copies of $\mathbb{Z}$. Namely, $G = \mathbb{Z}^{(\omega)}$ and we denote by p_k the projection of G onto the k-th copy of $\mathbb{Z}$. Then p_k is an additive function. We define the function

$$B(x, y) := \sum_k p_k(x)\, p_k(y)$$

for $x, y \in G$. The sum is finite for each x, y and B is a symmetric bi-additive function. We let $f(x) = B(x, x)$, and let V denote the variety of f. It is the linear space spanned by $\{1, f\}$, and the pojections p_k for $k = 0, 1, 2, \ldots$. Let $I = \operatorname{Ann} V$. we show that $\widehat{I}$ is non-localizable. Indeed, the exponential monomials in V are exactly the linear combinations of the functions $x \mapsto p_k(x) + d$ for $k = 0, 1, \ldots$, where d is any complex number. The closure of this set consists of linear functions, hence f is not included in the closure, as f is quadratic. We infer that the exponential monomials do not span a dense subset in V, hence V is not synthesizable, which implies that $\widehat{I}$ is not localizable.

On the other hand, if we ask for a non-localizable ideal in the Fourier algebra of a discrete abelian group with **finite** torsion-free rank, then this example does not work anymore. We can modify our problem in the following way: find a non-localizable ideal in the Fourier algebra of a **compactly generated** locally compact abelian group. This formulation is related to the counter-examples of D. I. Gurevič, Counterexamples to a problem of L. Schwartz. *Funkcional. Anal. i Priložen.*, **9(2)**, 29–35, 1975. Unfortunately, those counter-examples are not really transparent.

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