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On an Erdős–Straus type problem

Abstract. Motivated by the Erdős–Straus conjecture, we study whether the Euler totient ratio $\frac{\varphi(n)}{n}$ can be expressed as a sum of three unit fractions,

$$\frac{\varphi(n)}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}, \qquad x, y, z \in \mathbb{N}.$$

We prove that such a representation is impossible for prime powers p^k with $p \geq 11$, and for integers of the form $p^{k_1} q^{k_2}$ with $p \geq 5$ and $q \geq 15p$. In contrast, for $2 \leq p \leq 7$ the representation exists for all p^k. Moreover, for every integer $r \geq 3$, there are infinitely many integers n with $\omega(n) = r$ for which no such representation exists. Further, if a representation holds, we obtain the bounds

$$2 \leq x \leq 9 + 3e^{\gamma} \log\log n, \qquad \frac{nx}{\varphi(n)x - n} \leq y \leq \frac{2nx}{\varphi(n)x - n}.$$

In addition, if $x \geq \frac{3(n-1)(e^{\gamma} \log\log n + 3)}{n}$, then $\operatorname{rad}(n) \leq \frac{4^{2^r} - 4^{2^{r-1}}}{3}$, where γ is Euler's constant and $r = \omega(n)$.

1. Introduction

When a rational number is expanded into a sum of unit fractions, the expansion is called an Egyptian fraction. Every fraction $\frac{m}{n}$ has an expansion of at most m terms

$$\Big(\underbrace{\frac{1}{n} + \frac{1}{n} + \frac{1}{n} + \ldots + \frac{1}{n}}_{m \text{ times}}\Big),$$

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so in particular $\frac{2}{n}$ needs at most two terms, $\frac{3}{n}$ needs at most three terms, and $\frac{4}{n}$ needs at most four terms. For $\frac{2}{n}$ two terms are always needed and for $\frac{3}{n}$ three terms are sometimes needed, for instance $\frac{3}{7}$ cannot be expressed as a sum of two unit fractions due to the following lemma

Lemma 1 ([3, 7])
Let m and n be two coprime positive integers. Then

$$\frac{m}{n} = \frac{1}{a} + \frac{1}{b}$$

for some positive integers a and b, if and only if there exist positive integers x and y such that

(i) *xy divides n,*
(ii) *$x + y \equiv 0 \pmod{m}$.*

Thus, for the numerators 2 and 3 the question of how many terms are needed in an Egyptian fraction is known. The numerator 4 is the first case in which the question is still unanswered. That is, is it possible to express all fractions of the form $\frac{4}{n}$ using only two or three unit fractions? For the question on two unit fractions the answer is no, for example $\frac{4}{5}$ cannot be written as a sum of two unit fractions (use Lemma 1). Therefore, the question is whether all fractions of the form $\frac{4}{n}$ can be expressed as sum of at most three unit fractions. The Erdős–Straus conjecture states that for $\frac{4}{n}$ case the maximum number of required terms in a unit fraction sum is three. More precisely, the conjecture states that for all integers $n \geq 2$, there exist positive integers x, y and z such that

$$\frac{4}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}.$$

This conjecture was formulated in 1948 by Paul Erdős and Ernst G. Straus and published by Erdős in 1950 [5]. The problem soon attracted the attention of several prominent mathematicians, including Obláth [10], Rosati [11], Yamamoto [19], and Mordell [8]. Over the years, the problem has been studied from various perspectives, including analytic number theory, algebraic methods, and computational techniques [1, 2, 4, 6, 7, 13, 14, 15, 16, 17, 18, 20]. Recall that $\{\frac{\varphi(n)}{n}\}_{n=1}^{\infty}$ is dense in $[0, 1]$ and $\varphi(n) \mid n$ if and only if $n = 1$ or $n = 2^{k_1} \cdot 3^{k_2}$ where $k_1 \geq 1, k_2 \geq 0$ are integers. Such n is listed in A007694. Therefore, for n of this form we have

$$\frac{\varphi(n)}{n} = \frac{1}{\frac{3n}{\varphi(n)}} + \frac{1}{\frac{3n}{\varphi(n)}} + \frac{1}{\frac{3n}{\varphi(n)}}.$$

This motivates us to consider a related question and ask whether there exist positive integers x, y, z such that

$$\frac{\varphi(n)}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z},$$

for given positive integers n. Observe that if we write

$$n = \prod_{p|n} p^{v_p(n)} \quad \text{and} \quad \mathrm{rad}(n) = \prod_{p|n} p,$$

where $v_p(n)$ denotes the highest exponent of a prime number p dividing n, then

$$\frac{\varphi(n)}{n} = \frac{\varphi(\mathrm{rad}(n))}{\mathrm{rad}(n)}.$$

Hence, in order to determine whether n admits such a representation, it is sufficient to verify the condition for $\mathrm{rad}(n)$. In this paper, we establish the following theorems.

Theorem 2
Let p be a prime number and k be a positive integer.

(1) *If $2 \le p \le 7$, then there exist positive integers x, y, z such that*

$$\frac{\varphi(p^k)}{p^k} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}.$$

(2) *If $p \ge 11$, then there do not exist positive integers x, y, z such that*

$$\frac{\varphi(p^k)}{p^k} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}.$$

Note that Theorem 2 shows that there are infinitely many positive integers n with $\omega(n) = 1$ that admit such a representation, also, there are infinitely many positive integers n with $\omega(n) = 1$ that do not admit such a representation. For $n = 2^{k_1} \cdot 3^{k_2}$, where $k_1 \ge 1, k_2 \ge 1$, we have $\varphi(n) \mid n$. Consequently, there are infinitely many positive integers n with $\omega(n) = 2$ that admit such a representation. The following theorem demonstrates that there also exist infinitely many positive integers n with $\omega(n) = 2$ that do not admit such a representation.

Theorem 3
Let p, q be two prime numbers such that $q \ge 15p$ and $p \ge 5$. Then there do not exist positive integers x, y, z such that

$$\frac{\varphi(p^{k_1} q^{k_2})}{p^{k_1} q^{k_2}} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z},$$

where k_1, k_2 are positive integers.

If $n = 2^{k_1} \cdot 3^{k_2} \cdot p^{k_3}$, where $p = 5, 7$ and k_1, k_2, k_3 are positive integers, then from Theorem 2, we have some positive integers x, y, z such that

$$\frac{\varphi(p^{k_3})}{p^{k_3}} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}.$$

It follows that

$$\frac{\varphi(2^{k_1} \cdot 3^{k_2} \cdot p^{k_3})}{2^{k_1} \cdot 3^{k_2} \cdot p^{k_3}} = \frac{1}{x\left(\frac{2^{k_1} \cdot 3^{k_2}}{\varphi(2^{k_1} \cdot 3^{k_2})}\right)} + \frac{1}{y\left(\frac{2^{k_1} \cdot 3^{k_2}}{\varphi(2^{k_1} \cdot 3^{k_2})}\right)} + \frac{1}{z\left(\frac{2^{k_1} \cdot 3^{k_2}}{\varphi(2^{k_1} \cdot 3^{k_2})}\right)}.$$

If $m = 5^{k_3} \cdot p^{k_4}$, where p is a prime and k_3, k_4 are positive integers, then m may admit such a representation when $p \leq 73$ (follows from Theorem 3). Therefore, setting $p = 13$, we get

$$\frac{\varphi(5^{k_3} \cdot 13^{k_4})}{5^{k_3} \cdot 13^{k_4}} = \frac{1}{2} + \frac{1}{5} + \frac{1}{26}.$$

Then for positive integers k_1, k_2, we have

$$\frac{\varphi(2^{k_1} \cdot 3^{k_2} \cdot 5^{k_3} \cdot 13^{k_4})}{2^{k_1} \cdot 3^{k_2} \cdot 5^{k_3} \cdot 13^{k_4}} = \frac{1}{2\left(\frac{2^{k_1} \cdot 3^{k_2}}{\varphi(2^{k_1} \cdot 3^{k_2})}\right)} + \frac{1}{5\left(\frac{2^{k_1} \cdot 3^{k_2}}{\varphi(2^{k_1} \cdot 3^{k_2})}\right)} + \frac{1}{26\left(\frac{2^{k_1} \cdot 3^{k_2}}{\varphi(2^{k_1} \cdot 3^{k_2})}\right)}.$$

Therefore, there are infinitely many positive integers n with $\omega(n) = 3, 4$ that admit such a representation. For positive integers n with $\omega(n) \geq 3$, a corresponding negative result is given below.

Theorem 4
For any integer $r \geq 3$, there are infinitely many positive integers n with $\omega(n) = r$ that cannot be written as

$$\frac{\varphi(n)}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z},$$

where x, y, z are positive integers.

A natural question is whether, for a positive integer n with $\omega(n) \geq 2$ which admits such a representation, one can obtain an upper bound for the prime divisors of n. The following theorem partially answers this question.

Theorem 5
Let n be a positive integer with $\omega(n) = r \geq 2$ such that

$$\frac{\varphi(n)}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z},$$

where $x \leq y \leq z$ are positive integers.

(1) *Then*

$$2 \leq x \leq 9 + 3e^{\gamma} \log\log n, \quad \frac{nx}{\varphi(n)x - n} \leq y \leq \frac{2nx}{\varphi(n)x - n},$$

where γ is Euler's constant.

(2) *If*

$$x \geq \frac{3(n-1)(e^{\gamma} \log\log n + 3)}{n},$$

where γ is Euler's constant. Then

$$\mathrm{rad}(n) \leq \frac{4^{2^r} - 4^{2^{r-1}}}{3}.$$

Corollary 6
Let n be a positive integer with $\omega(n) = r \geq 2$ such that

$$\frac{\varphi(n)}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z},$$

where $x \leq y \leq z$ are positive integers. If p is a prime divisor of n such that

$$e^{e^{\frac{p}{9}-1}} \geq n,$$

then $p \nmid x$.

2. Proof of main results

Proof of Theorem 2. Since we have $\frac{\varphi(p^k)}{p^k} = \frac{\varphi(p)}{p}$, we only consider $\frac{\varphi(p)}{p}$. Observe that

$$\frac{\varphi(2)}{2} = \frac{1}{2} = \frac{1}{4} + \frac{1}{6} + \frac{1}{12}, \quad \frac{\varphi(3)}{3} = \frac{2}{3} = \frac{1}{3} + \frac{1}{4} + \frac{1}{12},$$
$$\frac{\varphi(5)}{5} = \frac{4}{5} = \frac{1}{2} + \frac{1}{5} + \frac{1}{10}, \quad \frac{\varphi(7)}{7} = \frac{6}{7} = \frac{1}{2} + \frac{1}{3} + \frac{1}{42}.$$

Let us assume that there exist positive integers x, y, z such that

$$\frac{\varphi(p)}{p} = \frac{p-1}{p} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}, \tag{1}$$

for a prime $p \geq 11$. Without loss of generality, assume that $x \leq y \leq z$. Since $p \geq 11$, from (1) we get

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \geq \frac{10}{11}. \tag{2}$$

Since $x \leq y \leq z$, we have $\frac{1}{x} \geq \frac{1}{y} \geq \frac{1}{z}$. Thus, we can write

$$\frac{3}{x} \geq \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \geq \frac{10}{11}.$$

Therefore, $x \leq 33/10$, i.e. x can be $1, 2$ or 3. Note that $\frac{\varphi(p)}{p} < 1$, which implies that $x \neq 1$. If $x = 2$ then from (2) we obtain

$$\frac{2}{y} \geq \frac{1}{y} + \frac{1}{z} \geq \frac{9}{22}.$$

Therefore, $y \leq 44/9$, i.e. y can be $2, 3$ or 4. $y \neq 2$ as if $y = 2$ then $\frac{\varphi(p)}{p} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} > 1$. If $y = 3$, then from (1) we get

$$\frac{\varphi(p)}{p} = \frac{5}{6} + \frac{1}{z},$$

that is $(p-6)z = 6p$. Since $p \nmid (p-6)$, we must have $p \mid z$. Let $z = pk$ for some positive integer k. Then $(p-6)z = (p-6)pk = 6p$ implies $k = \frac{6}{p-6} \leq 6/5$. Therefore, $k = 1$ and $z = p$ but for any prime $p \geq 11$ we have $(p-6)p \neq 6p$. Thus, the only choice for y is 4.

For $y = 4$ we have

$$\frac{\varphi(p)}{p} = \frac{1}{2} + \frac{1}{4} + \frac{1}{z} = \frac{3}{4} + \frac{1}{z},$$

that is $(p-4)z = 4p$. Since $p \nmid (p-4)$, we must have $p \mid z$. Let $z = pk$ for some positive integer k. Then $(p-4)z = (p-4)pk = 4p$ implies $k = \frac{4}{p-4} < 1$, a contradiction. Hence $x \neq 2$.

If $x = 3$ then from (2) we obtain

$$\frac{2}{y} \geq \frac{1}{y} + \frac{1}{z} \geq \frac{19}{33}.$$

Therefore $y \leq 66/19$, i.e. $y = 3$. For $y = 3$ we have

$$\frac{\varphi(p)}{p} = \frac{2}{3} + \frac{1}{z},$$

that is $(p-3)z = 3p$. Since $p \nmid (p-3)$, we must have $p \mid z$. Let $z = pk$ for some positive integer k. Then $(p-3)z = (p-3)pk = 3p$ implies $k = \frac{3}{p-3} < 1$, a contradiction.

Therefore, for any choice of x, we have no y, z such that

$$\frac{\varphi(p)}{p} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}.$$

This completes the proof.

To prove Theorem 3, we make use of the following proposition.

Proposition 7
Let $f_\alpha \colon [7, \infty) \to \mathbb{R}$ be a function defined by

$$f_\alpha(x) = \frac{x}{1 - 16x + \alpha x^2}.$$

If $\alpha \geq 2.5$ then $f_\alpha(x)$ is a decreasing function of x in $[7, \infty)$ and $f_\alpha(x) < 1$ for all $x \in [7, \infty)$.

Proof. For $x \geq 7$ and $\alpha \geq 2.5$, we have $\alpha x^2 - 16x + 1 = x(\alpha x - 16) + 1 > 0$. Also, we have

$$f'_\alpha(x) = \frac{1 - \alpha x^2}{(1 - 16x + \alpha x^2)^2}.$$

Since $\alpha \geq 2.5$ and $x \geq 7$ we have $f'_\alpha(x) < 0$. It follows that $f_\alpha(x)$ is a strictly decreasing function of x in $[7, \infty)$. Therefore, $f_\alpha(x) \leq f_\alpha(7) = \frac{7}{49\alpha - 111} < 1$ for all $x \geq 7$ and $\alpha \geq 2.5$. This completes the proof.

Proof of Theorem 3. Since we have

$$\frac{\varphi(p^{k_1}q^{k_2})}{p^{k_1}q^{k_2}} = \frac{\varphi(pq)}{pq},$$

we only consider $\frac{\varphi(pq)}{pq}$. Let us assume that there exist positive integers x, y, z such that

$$\frac{\varphi(pq)}{pq} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$$

for some $q \geq 15p$ and $p \geq 5$. Without loss of generality, assume that $x \leq y \leq z$. Then

$$\frac{3}{x} \geq \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{\varphi(pq)}{pq} = \Big(1 - \frac{1}{p}\Big)\Big(1 - \frac{1}{q}\Big) \geq \Big(1 - \frac{1}{p}\Big)\Big(1 - \frac{1}{15p}\Big) \geq \frac{296}{375}. \quad (3)$$

Therefore, $x \leq 1125/296$, that is, $1 \leq x \leq 3$. Note that $x \neq 1$ as $\frac{\varphi(pq)}{pq} < 1$.

If $x = 2$, then from (3) we get

$$\frac{2}{y} \geq \frac{1}{y} + \frac{1}{z} \geq \frac{217}{750}.$$

Then $y \leq 1500/217$, i.e. $2 \leq y \leq 6$. Since $x = 2$, we must have $y \neq 2$ as $\frac{\varphi(pq)}{pq} < 1$.

If $x = 3$, then from (3) we get

$$\frac{2}{y} \geq \frac{1}{y} + \frac{1}{z} \geq \frac{57}{125}.$$

Then $y \leq 250/57$, i.e. $3 \leq y \leq 4$. Note that $q \geq 75$. Since $q \mid (p-1)(q-1)xyz$ and $q \nmid (p-1)(q-1)$ we have $q \mid xyz$ but $2 \leq x \leq 3$ which implies that $q \mid y$ or $q \mid z$. But for $x = 2, 3$ we have $y < 7$ which implies that $q \mid z$.

Therefore, let $z = qk$ for some positive integer k. Then

$$\frac{\varphi(pq)}{pq} = \frac{1}{x} + \frac{1}{y} + \frac{1}{qk}.$$

It follows that

$$k = \frac{pxy}{q(xy(p-1) - p(x+y)) - xy(p-1)}. \quad (4)$$

Since $q \geq 15p$ and $xy(p-1) - p(x+y) > 0$ for $p \geq 7$ and $x \geq 2, y \geq 3$ we get from (4) that

$$\begin{aligned} k &\leq \frac{pxy}{15p(xy(p-1) - p(x+y)) - xy(p-1)} \\ &= \frac{p}{1 - 16p + 15(1 - \frac{x+y}{xy})p^2} = f_{15(1-\frac{x+y}{xy})}(p). \end{aligned} \quad (5)$$

Since $2.5 \leq 15(1 - \frac{x+y}{xy})$ for the pairs $(x, y) = (2, 3), (2, 4), (2, 5), (2, 6), (3, 3), (3, 4)$, we get from Proposition 7 and (5) that $k < 1$, a contradiction. Therefore, for $q \geq 15p$ and $p \geq 7$, there do not exist positive integers x, y, z such that

$$\frac{\varphi(pq)}{pq} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}.$$

It remains to show that such a representation is impossible for $p = 5$, $q \geq 15p$. From (4) we obtain

$$k = \frac{5xy}{q(4xy - 5(x+y)) - 4xy}. \tag{6}$$

Observe that for the pairs $(x, y) = (2, 4), (2, 5), (2, 6), (3, 3), (3, 4)$ we have $4xy - 5(x + y) > 0$. Therefore, since $q \geq 75$, we get

$$k \leq \frac{5xy}{75\{4xy - 5(x+y)\} - 4xy} = \frac{5}{296 - 375(\frac{x+y}{xy})}.$$

But $\frac{5}{296-375(\frac{x+y}{xy})} < 1$ for the pairs $(x, y) = (2, 4), (2, 5), (2, 6), (3, 3), (3, 4)$, which implies that $k < 1$, a contradiction. Now for the pair $(x, y) = (2, 3)$ we have from (6) that

$$k = \frac{30}{-q - 24} < 1,$$

again a contradiction. This completes the proof.

Proof of Theorem 4. Let $r \geq 3$ and $k_1, \ldots, k_r$ be positive integers. Put $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$, where $\frac{1}{1-\sqrt[r]{4/5}} \leq p_1 < p_2 < \ldots < p_r$ are prime numbers such that for $2 \leq j \leq r$ we have

$$p_j \equiv 2 \pmod{p_1 p_2 \cdots p_{j-1}}. \tag{7}$$

The existence of each p_j is guaranteed by the Dirichlet prime number theorem. Assume that there exist positive integers x, y, z such that $\frac{\varphi(n)}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$. Since

$$\frac{\varphi(n)}{n} = \prod_{i=1}^{r} \frac{(p_i - 1)}{p_i},$$

we have

$$\prod_{i=1}^{r} \frac{(p_i - 1)}{p_i} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}.$$

Without loss of generality, assume that $x \leq y \leq z$. Since $p_1 \geq \frac{1}{1-\sqrt[r]{4/5}}$, we have

$$\frac{3}{x} \geq \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \prod_{i=1}^{r} \frac{(p_i - 1)}{p_i} \geq \left(1 - \frac{1}{p_1}\right)^r \geq \frac{4}{5}. \tag{8}$$

It follows that $1 \leq x \leq 3$. Again, $x \neq 1$ as $\frac{\varphi(n)}{n} < 1$. Using (8), we get if $x = 2$ then $y \in \{3, 4, 5, 6\}$ and if $x = 3$ then $y \in \{3, 4\}$.

For $x = 2$ and $y = 3$, we get

$$\frac{1}{z} = \prod_{i=1}^{r} \frac{(p_i - 1)}{p_i} - \frac{5}{6} = \frac{6(p_1 - 1)(p_2 - 1) \cdots (p_r - 1) - 5p_1 p_2 \cdots p_r}{6p_1 p_2 \cdots p_r},$$

then

$$6p_1 p_2 \cdots p_r = z(6(p_1 - 1)(p_2 - 1) \cdots (p_r - 1) - 5p_1 p_2 \cdots p_r). \tag{9}$$

If possible, assume that

$$\gcd(6p_1p_2\cdots p_r, 6(p_1-1)(p_2-1)\cdots(p_r-1)-5p_1p_2\cdots p_r) > 1.$$

Then there is a prime p such that

$$6p_1p_2\cdots p_r \equiv 0 \pmod{p} \tag{10}$$

and

$$6(p_1-1)(p_2-1)\cdots(p_r-1)-5p_1p_2\cdots p_r \equiv 0 \pmod{p}. \tag{11}$$

Therefore, from (10), we have $p = 2, 3$ or $p = p_k$, where $1 \le k \le r$. Since each $p_i > 13$, from (11) we get $p \ne 2, 3$. Without loss of generality, set $p = p_t$, where $1 \le t \le r-1$, as clearly from (11) we have $p \ne p_r$. Then from (11), we must have some $t+1 \le l \le r$ such that

$$p_l \equiv 1 \pmod{p_t}. \tag{12}$$

Setting $j = l$ in (7), we get

$$p_l \equiv 2 \pmod{p_1p_2\cdots p_t\cdots p_{l-1}},$$

in particular

$$p_l \equiv 2 \pmod{p_t},$$

which contradicts (12). Therefore, we have

$$\gcd(6p_1p_2\cdots p_r, 6(p_1-1)(p_2-1)\cdots(p_r-1)-5p_1p_2\cdots p_r) = 1.$$

It follows from (9) that $z = 6z'p_1p_2\cdots p_r$ for some positive integer z'. Then from (9), we can write

$$z'(6(p_1-1)(p_2-1)\cdots(p_r-1)-5p_1p_2\cdots p_r) = 1,$$

which forces

$$6(p_1-1)(p_2-1)\cdots(p_r-1)-5p_1p_2\cdots p_r = 1.$$

In particular, we can write

$$6(p_1-1)(p_2-1)\cdots(p_r-1) \equiv 1 \pmod{p_1}. \tag{13}$$

From (7) we have

$$6(p_1-1)(p_2-1)\cdots(p_r-1) \equiv -6 \pmod{p_1},$$

which contradicts (13) as $p_1 > 13$. Hence, $p \ne p_t$. Consequently, we must have $x = 2$ and $y \in \{4, 5, 6\}$. The cases $x = 2, y \in \{4, 5, 6\}$ and $x = 3, y \in \{3, 4\}$ are similar. This completes the proof.

In order to prove Theorem 5, we need the following auxiliary lemmas.

Lemma 8
For $n \geq 3$, we have

$$\frac{n}{\varphi(n)} \leq e^{\gamma} \log\log n + 3.$$

Proof. See (3.41) of [12].

Lemma 9 ([9])
Let $r, a, b \in \mathbb{N}$ and let $x_1, \ldots, x_r$ be integers such that $1 < x_1 < x_2 < \cdots < x_r$ and

$$\prod_{j=1}^{r}\Big(1 - \frac{1}{x_j}\Big) \leq \frac{a}{b} < \prod_{j=1}^{r-1}\Big(1 - \frac{1}{x_j}\Big). \tag{14}$$

Then

$$a\prod_{j=1}^{r} x_j \leq (a+1)^{2^r} - (a+1)^{2^{r-1}}. \tag{15}$$

Proof of Theorem 5. Let $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$, where $p_1 < p_2 < \ldots < p_r$ are prime numbers, where k_i are positive integers. Then we have some positive integers $x \leq y \leq z$ such that

$$\frac{\varphi(n)}{n} = \prod_{i=1}^{r} \frac{(p_i - 1)}{p_i} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}.$$

(1) Since $\frac{\varphi(n)}{n} \leq \frac{3}{x}$, using Lemma 8, we get

$$x \leq 9 + 3e^{\gamma} \log\log n.$$

If possible assume that $y > \frac{2xn}{\varphi(n)x - n}$. Then we have

$$\varphi(n)xy - n(x+y) > nx.$$

From the above we can write

$$y > \frac{nxy}{\varphi(n)xy - n(x+y)} = z,$$

which is a contradiction. Therefore, we must have $y \leq \frac{2xn}{\varphi(n)x-n}$. Next, assume that $y < \frac{nx}{\varphi(n)x-n}$. Then we have

$$\frac{\varphi(n)xy - ny}{nx} < 1,$$

which implies that

$$\frac{1}{z} = \frac{\varphi(n)xy - n(x+y)}{nxy} < 0,$$

which is absurd. Therefore, our assumption was wrong and thus we must have $y \geq \frac{nx}{\varphi(n)x-n}$.

(2) Since $x \geq \frac{3(n-1)(e^\gamma \log\log n+3)}{n}$, using Lemma 8, we have

$$\prod_{i=1}^{r}\left(1-\frac{1}{p_i}\right)=\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\leq\frac{3}{x}\leq\frac{\varphi(n)}{n-1}.$$

Since

$$\frac{\dfrac{\varphi(n)}{n-1}}{\displaystyle\prod_{i=1}^{r-1}\left(1-\frac{1}{p_i}\right)}=\frac{n\displaystyle\prod_{i=1}^{r}\left(1-\frac{1}{p_i}\right)}{(n-1)\displaystyle\prod_{i=1}^{r-1}\left(1-\frac{1}{p_i}\right)}=\frac{1-\frac{1}{p_r}}{1-\frac{1}{n}}<1,$$

we have

$$\prod_{i=1}^{r}\left(1-\frac{1}{p_i}\right)\leq\frac{3}{x}\leq\frac{\varphi(n)}{n-1}<\prod_{i=1}^{r-1}\left(1-\frac{1}{p_i}\right).$$

Hence, the inequality (14) is satisfied for $x_j=p_j$, $a=3$, $b=x$. Therefore, from (15) we get

$$\operatorname{rad}(n)\leq\frac{4^{2^r}-4^{2^{r-1}}}{3}.$$

Proof of Corollary 6. Assume for the sake of contradiction that $p \mid x$. Then

$$x\geq p\geq 9+9\log\log n>9+3e^\gamma\log\log n,$$

which contradicts Theorem 5.

3. Future research

A natural direction for future research is to investigate the natural density of the sets

$$\left\{n\in\mathbb{N}:\ \frac{\varphi(n)}{n}=\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\text{ for some }x,y,z\in\mathbb{N}\right\}$$

and

$$\left\{n\in\mathbb{N}:\ \frac{\varphi(n)}{n}\neq\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\text{ for all }x,y,z\in\mathbb{N}\right\}.$$

Note that in the proof of Theorem 4 we actually showed that if $\frac{\varphi(n)}{n}\geq 4/5$, then $\frac{\varphi(n)}{n}$ cannot be written as a sum of three unit fractions. Hence, another interesting problem is to determine, the infimum of the set of the values c with the property that if $\frac{\varphi(n)}{n}\geq c$, then $\frac{\varphi(n)}{n}$ cannot be written as a sum of three unit fractions.

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