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On a Pascal-type theorem for lines in $\mathbb{P}^3$

Abstract. We prove a three-dimensional Pascal-type theorem for configurations of lines on a smooth quadric surface in $\mathbb{P}^3$. Starting from two triples of skew lines in the two rulings of the quadric, we show that the associated grid of nine intersection points determines natural triples of planes whose residual intersection lines are coplanar. This construction recovers the classical Pascal line after taking a plane section of the quadric. We further study the six diagonal planes associated with permutations of the grid points, showing that they split into two pencils with skew polar axes, and describe a canonical involutive mirror construction on the grid. We work over the field of complex numbers $\mathbb{C}$.

1. Introduction

Pappus theorem, dating back to the fourth century, is one of the first and best understood incidence theorems in geometry; see [5, Chapter 1] for a comprehensive introduction, proofs and variations. We recall its statement. For a subset X in the projective space, we denote by $\langle X\rangle$ the projective subspace generated by X.

Theorem 1.1 (Pappus)
Let A, B, C be three points on a line and let a, b, c be three points on another line in the projective plane. Then the three points

$$\langle A, b\rangle \cap \langle a, B\rangle, \quad \langle B, c\rangle \cap \langle b, C\rangle, \quad \langle C, a\rangle \cap \langle c, A\rangle$$

are collinear.

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It was generalized in 1639 by Blaise Pascal. In his statement the two lines are replaced by a smooth conic.

Theorem 1.2 (Pascal)
Let A, B, C, a, b, c be six distinct points on a smooth conic in the projective plane. Then the three points

$$\langle A, b\rangle \cap \langle a, B\rangle, \quad \langle B, c\rangle \cap \langle b, C\rangle, \quad \langle C, a\rangle \cap \langle c, A\rangle$$

are collinear.

Equivalently, if a hexagon is inscribed in a conic, then the three intersection points of its pairs of opposite sides are collinear. The line obtained in this way is classically called the Pascal line of the hexagon.

There was a lot of interest in generalizations of Pascal's result in recent years. Ciliberto and Miranda [3] study conditions for $d+4$ points in $\mathbb{P}^d$ to lie on a rational normal curve; for $d = 2$ one recovers the classical Pascal theorem. Related higher-dimensional Pascal-type conditions for rational normal curves were also studied by Caminata and Schaffler [1]. Traves builds his article [6] around the Braikenridge–Maclaurin theorem, which may be viewed as a converse to Pascal's theorem.

Theorem 1.3 (Braikenridge–Maclaurin)
Let $\ell_1,\ \ell_2,\ \ell_3$ and $m_1,\ m_2,\ m_3$ be two triples of lines in the projective plane, and assume that they meet in nine distinct points

$$P_{ij} = \ell_i \cap m_j, \qquad 1 \leqslant i, j \leqslant 3.$$

If three of these nine points are collinear, then the remaining six points lie on a conic.

The common feature of these results is residuality. One starts with a complete intersection, imposes that part of it lies on a curve of small degree, and concludes that the residual part lies on another curve of complementary degree. This point of view is closely related to the Cayley–Bacharach theorem and to liaison theory; see [4]. The aim of the present note is to investigate a three-dimensional analogue in which points and lines in the plane are replaced by lines and planes in $\mathbb{P}^3$. The precise configuration is introduced in the next section.

2. The Braikenridge–Maclaurin theorem revisited

We begin with a more general statement of Theorem 1.3. The original version is recovered by setting $k = 3$.

Theorem 2.1 (Generalized 2D Braikenridge–Maclaurin)
Let $\ell_1, \ldots, \ell_k$ and $m_1, \ldots, m_k$ be two k-tuples of lines in the projective plane, and assume that they meet in k^2 distinct points

$$P_{ij} = \ell_i \cap m_j, \qquad 1 \leqslant i, j \leqslant k.$$

If k of these k^2 points are collinear, then the remaining $k^2 - k$ points lie on a curve of degree $k - 1$.

Proof. The following argument was suggested to us by Janusz Gwoździewicz. By a slight abuse of notation we denote equations of the lines with the same symbols as the lines themselves. The products

$$\ell = \ell_1 \cdots \ell_k \quad \text{and} \quad m = m_1 \cdots m_k$$

span a pencil $\mathcal{C}$ of degree k curves whose base locus consists exactly of the k^2 intersection points P_{ij}. Assume that k of these points are collinear, i.e. contained in a line s. Let P be a point in s different from all the points P_{ij}. Then there is a curve $c \in \mathcal{C}$ vanishing at P. This curve vanishes thus at $k+1$ points in s, hence it contains s as a component. Thus $c = s \cup c'$, where c' is a curve of degree $k-1$. Since c contains the whole base locus of the pencil, the residual curve c' contains all points P_{ij} not lying on s.

The statement generalizes with necessary adjustments to dimension 3 (and in fact any dimension but we don't dwell on that).

Theorem 2.2 (3D Braikenridge–Maclaurin)
Let $F_1, \dots, F_k$ and $G_1, \dots, G_k$ be two k-tuples of planes in the projective space $\mathbb{P}^3$, and assume that they meet in k^2 distinct lines

$$\ell_{ij} = F_i \cap G_j, \qquad 1 \leqslant i, j \leqslant k.$$

If k of these k^2 lines are coplanar, i.e. they are contained in a plane $H \subset \mathbb{P}^3$, then the remaining $k^2 - k$ lines lie on a surface of degree $k-1$.

Proof. The proof follows exactly the same lines as the proof of Theorem 2.1. Indeed, the products

$$F = F_1 \cdots F_k \quad \text{and} \quad G = G_1 \cdots G_k$$

span a pencil of surfaces of degree k. The base locus of this pencil consists of the k^2 lines ℓ_{ij} for $1 \leqslant i, j \leqslant k$. Let P be a point in the plane H not contained in any line ℓ_{ij}. Let S be the element of the pencil vanishing at P. Then S must contain H, since it vanishes along k lines in H and in an additional point P. The residual surface $S' = S - H$ contains then the remaining $k^2 - k$ lines.

Remark 2.3
Theorem 2.2 asserts for $k = 2$, that if two pairs of planes intersect in 4 distinct lines and two of these lines are contained in a plane, then the remaining two are coplanar as well.

There is a simple direct argument showing this case of the theorem.

Proof. Let F_1, F_2 and G_1, G_2 be two pairs of planes in $\mathbb{P}^3$, and set

$$\ell_{ij} = F_i \cap G_j.$$

If two of the four lines ℓ_{ij} sharing one of the planes are coplanar, then the assertion is immediate. For instance, ℓ_{11} and ℓ_{12} both lie in F_1, while the remaining two lines ℓ_{21} and ℓ_{22} both lie in F_2.

It remains to consider the diagonal case. Suppose that ℓ_{11} and ℓ_{22} are contained in a plane. Write

$$F_i = V(f_i), \quad G_j = V(g_j).$$

If a plane $V(h)$ contains both ℓ_{11} and ℓ_{22}, then

$$h \in \langle f_1, g_1\rangle \cap \langle f_2, g_2\rangle.$$

Thus

$$h = \alpha f_1 + \beta g_1 = \gamma f_2 + \delta g_2.$$

Rearranging, we obtain

$$\alpha f_1 - \delta g_2 = \gamma f_2 - \beta g_1.$$

The linear form on the left belongs to $\langle f_1, g_2\rangle$, while the same linear form on the right belongs to $\langle f_2, g_1\rangle$. Hence it defines a plane containing both

$$\ell_{12} = V(f_1, g_2) \quad \text{and} \quad \ell_{21} = V(f_2, g_1).$$

Therefore the remaining two lines are coplanar.

For $k = 3$, we want to study the geometry of the six lines away from the plane H. To this end it is convenient to assume that the three coplanar lines are the diagonal lines

$$\ell_{11},\ \ell_{22},\ \ell_{33}.$$

Let Q be the quadric containing the six remaining lines ℓ_{ij} with $i \neq j$. For a sufficiently general choice of the planes F_i and G_j, this quadric is smooth.

Lemma 2.4
Assume that the quadric Q containing the six lines ℓ_{ij}, $i \neq j$, is smooth. Then the triples

$$\ell_{12},\ \ell_{23},\ \ell_{31} \quad \textit{and} \quad \ell_{13},\ \ell_{21},\ \ell_{32}$$

belong to distinct rulings on Q. In particular, the lines in each of these triples are pairwise skew, while every line from one triple meets every line from the other triple.

Proof. Let $f_i = 0$ and $g_i = 0$ be equations of the planes F_i and G_i, respectively, and let $h = 0$ be an equation of the plane H. Since

$$\ell_{ii} = F_i \cap G_i \subset H,$$

the linear form h belongs to the pencil of linear forms generated by f_i and g_i. Note that $h \neq f_i$ and $h \neq g_i$ holds. Otherwise, if there were say $h = f_1$, then $H = F_1$ would contain ℓ_{12}, which were forced to be equal ℓ_{22} contradicting the assumption that there are k^2 distinct lines. Thus, after rescaling f_i and g_i, we may write

$$h = f_i + g_i$$

for every $i = 1, 2, 3$.

We first show that, for $i \neq j$, the two lines ℓ_{ij} and ℓ_{ji} meet. Indeed, they are defined by

$$\ell_{ij} = \{f_i = g_j = 0\}, \quad \ell_{ji} = \{f_j = g_i = 0\}.$$

Using $g_i = h - f_i$ and $g_j = h - f_j$, the common solutions of these equations satisfy

$$f_i = f_j = h = 0.$$

Thus ℓ_{ij} and ℓ_{ji} meet at the point

$$F_i \cap F_j \cap H.$$

Indeed, the intersection cannot be a line, because then two lines ℓ_{ij} and ℓ_{ji} would coincide, contrary to the assumed distinctness of the k^2 intersection lines. Provided the configuration is general enough for this intersection to be a single point. Equivalently, this is the same as $G_i \cap G_j \cap H$.

On the other hand, some intersections are immediate from the definitions: two lines ℓ_{ij} and ℓ_{pq} meet whenever $i = p$ or $j = q$, because then they are contained in a common plane. Therefore each line in the first triple

$$\ell_{12},\ \ell_{23},\ \ell_{31}$$

meets each line in the second triple

$$\ell_{13},\ \ell_{21},\ \ell_{32}.$$

For example, ℓ_{12} meets ℓ_{13} in the plane F_1, it meets ℓ_{32} in the plane G_2, and it meets ℓ_{21} by the argument above.

Now we use the elementary geometry of a smooth quadric surface. A smooth quadric $Q \subset \mathbb{P}^3$ has two rulings by lines. Two distinct lines on Q meet if and only if they belong to different rulings; lines in the same ruling are skew.

Since ℓ_{12} meets all three lines ℓ_{13}, ℓ_{21}, ℓ_{32}, these three lines belong to the ruling opposite to the ruling of ℓ_{12}. Similarly, ℓ_{23} and ℓ_{31} meet, for instance, the line ℓ_{13}, and hence they cannot belong to the same ruling as ℓ_{13}. Therefore they belong to the same ruling as ℓ_{12}.

Thus

$$\ell_{12},\ \ell_{23},\ \ell_{31}$$

belong to one ruling of Q, while

$$\ell_{13},\ \ell_{21},\ \ell_{32}$$

belong to the other ruling. The final assertions follow immediately.

3. A Pascal-type theorem for lines in the projective space

In this section we want to prove a converse to Lemma 2.4 and study additional properties of the resulting point-line-plane configurations. To this end we assume that we are given skew triples of lines: $k_1,\ k_2,\ k_3$ and $m_1,\ m_2,\ m_3$ which form a *grid*, i.e. each pair of lines k_i and m_j intersect in a point $P_{ij} = k_i \cap m_j$ for

$i, j = 1, 2, 3$. It follows from [2, Remark 3.4 (5)] that there exists a (unique) smooth quadric Q which contains the lines in opposite rulings.

The following result may be viewed as a converse to Lemma 2.4. Starting with two skew triples of lines in opposite rulings of a smooth quadric, we construct two triples of planes. The six off-diagonal intersections of these planes are the original lines, while the three diagonal intersections are new lines. The Pascal-type phenomenon is that these three residual lines are coplanar.

Theorem 3.1 (3D Pascal)
Let $k_1,\ k_2,\ k_3$ and $m_1,\ m_2,\ m_3$ be two triples of lines in $\mathbb{P}^3$ such that the lines in each triple are pairwise skew and such that every line k_i meets every line m_j. Denote their intersection points by

$$P_{ij} = k_i \cap m_j$$

and assume that the nine points P_{ij} are distinct. Then the six lines are contained in a unique smooth quadric surface Q, the lines $k_1,\ k_2,\ k_3$ belonging to one ruling and the lines $m_1,\ m_2,\ m_3$ to the other.

Let the indices be read modulo 3, and set

$$P_{i,i+1} = k_i \cap m_{i+1}, \qquad i = 1, 2, 3.$$

The points $P_{12},\ P_{23},\ P_{31}$ are not collinear (because they are not contained in any ruling line). Let

$$H = \langle P_{12}, P_{23}, P_{31} \rangle.$$

Define six planes

$$F_i = \langle k_i, m_i \rangle, \quad G_i = \langle k_{i-1}, m_{i+1} \rangle, \qquad i = 1, 2, 3.$$

Then the three lines

$$\ell_{ii} = F_i \cap G_i, \qquad i = 1, 2, 3,$$

are contained in the plane H. More precisely,

$$\ell_{11} = \langle P_{12}, P_{31} \rangle, \quad \ell_{22} = \langle P_{12}, P_{23} \rangle, \quad \ell_{33} = \langle P_{23}, P_{31} \rangle.$$

Moreover, the remaining intersections of the two triples of planes recover the original six lines:

$$F_i \cap G_{i+1} = k_i, \quad F_i \cap G_{i-1} = m_i,$$

where indices are read modulo 3.

Proof. The existence and uniqueness of the smooth quadric Q containing the six lines, with the two triples in opposite rulings, follows from [2, Remark 3.4 (5)]. This also can be seen directly using a classical construction of a *regulus*. For a point $P \in k_1$ consider the plane $H_P = \langle P, k_2 \rangle$ and its intersection point Q_P with the line k_3. Moving P along k_1, we obtain skew lines $\langle P, Q_P \rangle$ intersecting the given lines $k_1,\ k_2$ and k_3. The lines are skew because the planes H_P intersect pairwise along the line k_2. If two lines $\langle P, Q_P \rangle$ and $\langle P', Q_{P'} \rangle$ would meet k_2 in the same point, then all points $P,\ P',\ Q_P,\ Q_{P'}$ would be coplanar contradicting the skewness assumption. Clearly, the lines $m_1,\ m_2\ ,m_3$ are among so constructed 3-secant lines of $k_1 \cup k_2 \cup k_3$. Repeating the construction for them recovers a $\mathbb{P}^1 \times \mathbb{P}^1$ structure on a surface containing all six initial lines, exactly as in the Segre embedding (1). This surface is the unique smooth quadric.

Moving on with the proof, We first note that the points $P_{12},\ P_{23}$ and P_{31} are not collinear. Indeed, if they were collinear, the line through them would meet the quadric Q in three distinct points. Hence it would be contained in Q. But every line contained in a smooth quadric belongs to one of the two rulings, whereas the points $P_{12},\ P_{23},\ P_{31}$ do not lie on a common line of either ruling. Thus

$$H = \langle P_{12}, P_{23}, P_{31} \rangle$$

is a well-defined plane. It is a *Pascal plane of the grid*.

We now compute the intersections $F_i \cap G_i$. For $i = 1$ we have

$$F_1 = \langle k_1, m_1 \rangle, \quad G_1 = \langle k_3, m_2 \rangle.$$

The point

$$P_{12} = k_1 \cap m_2$$

belongs to both F_1 and G_1, since $k_1 \subset F_1$ and $m_2 \subset G_1$. Similarly,

$$P_{31} = k_3 \cap m_1$$

belongs to both F_1 and G_1. Hence the line

$$\langle P_{12}, P_{31} \rangle$$

is contained in $F_1 \cap G_1$. The planes F_1 and G_1 are distinct; indeed, otherwise a single plane would contain the skew lines k_1 and k_3. Therefore their intersection is a line, and so

$$\ell_{11} = F_1 \cap G_1 = \langle P_{12}, P_{31} \rangle.$$

The other two diagonal intersections are obtained in the same way. Namely,

$$F_2 = \langle k_2, m_2 \rangle, \quad G_2 = \langle k_1, m_3 \rangle.$$

Both planes contain

$$P_{12} = k_1 \cap m_2$$

and

$$P_{23} = k_2 \cap m_3,$$

hence

$$\ell_{22} = F_2 \cap G_2 = \langle P_{12}, P_{23} \rangle.$$

Similarly,

$$F_3 = \langle k_3, m_3 \rangle, \quad G_3 = \langle k_2, m_1 \rangle$$

both contain

$$P_{23} = k_2 \cap m_3$$

and

$$P_{31} = k_3 \cap m_1,$$

and therefore

$$\ell_{33} = F_3 \cap G_3 = \langle P_{23}, P_{31} \rangle.$$

Consequently,

$$\ell_{11}, \ell_{22}, \ell_{33} \subset \langle P_{12}, P_{23}, P_{31} \rangle = H.$$

It remains to check the off-diagonal intersections. We have

$$G_{i+1} = \langle k_i, m_{i+2} \rangle,$$

where indices are read modulo 3. Thus both $F_i = \langle k_i, m_i \rangle$ and G_{i+1} contain the line k_i. These two planes are distinct, since otherwise one plane would contain the two skew lines m_i and m_{i+2}. Hence

$$F_i \cap G_{i+1} = k_i.$$

Similarly,

$$G_{i-1} = \langle k_{i-2}, m_i \rangle.$$

Both $F_i = \langle k_i, m_i \rangle$ and G_{i-1} contain the line m_i, and the two planes are distinct, since otherwise one plane would contain the two skew lines k_i and k_{i-2}. Therefore

$$F_i \cap G_{i-1} = m_i.$$

This proves all the asserted incidences.

The plane H is determined by the choice of one of the two cyclic diagonals of the 3×3 grid of points. Choosing instead the cyclic diagonal twisted by a permutation $\sigma \in S_3$,

$$P_{1\sigma(1)},\ P_{2\sigma(2)},\ P_{3\sigma(3)},$$

gives an analogous configuration and, in general, a different Pascal plane. We come back to that in Section 4.

The preceding theorem shows that the grid formed by the two triples of lines carries a natural Pascal-type incidence structure. There is also a more direct spatial analogue of the usual Pascal construction. Namely, in the plane Pascal theorem one joins two points on the conic by a line and then intersects two such lines. In the present setting the points are replaced by lines on a quadric, so the joining lines are replaced by the planes spanned by pairs of intersecting lines from the two rulings. The following corollary records the resulting incidence.

Corollary 3.2 (Spatial Pascal lines)
Let $Q \subset \mathbb{P}^3$ be a smooth quadric, and let $k_1,\ k_2,\ k_3$ and $m_1,\ m_2,\ m_3$ be two triples of lines contained in the two rulings of Q. Put

$$P_{ij} = k_i \cap m_j$$

and

$$H_{ij} = \langle k_i, m_j \rangle.$$

Thus H_{ij} is the tangent plane to Q at P_{ij}. Define

$$U_3 = H_{12} \cap H_{21}, \quad U_2 = H_{13} \cap H_{31}, \quad U_1 = H_{23} \cap H_{32}.$$

Then the three lines U_1, U_2, U_3 are coplanar. More precisely,

$$U_3 = \langle P_{11}, P_{22} \rangle, \quad U_2 = \langle P_{11}, P_{33} \rangle, \quad U_1 = \langle P_{22}, P_{33} \rangle.$$

Consequently,

$$U_1, U_2, U_3 \subset \langle P_{11}, P_{22}, P_{33} \rangle.$$

Proof. We prove the assertion for U_3; the other two cases are analogous. Since H_{12} contains k_1 and m_2, while H_{21} contains k_2 and m_1, we have

$$P_{11} = k_1 \cap m_1 \in H_{12} \cap H_{21}$$

and

$$P_{22} = k_2 \cap m_2 \in H_{12} \cap H_{21}.$$

Hence the line $\langle P_{11}, P_{22} \rangle$ is contained in $H_{12} \cap H_{21}$. Since H_{12} and H_{21} are distinct planes in $\mathbb{P}^3$, their intersection is a line. Therefore

$$U_3 = H_{12} \cap H_{21} = \langle P_{11}, P_{22} \rangle.$$

Similarly,

$$U_2 = H_{13} \cap H_{31} = \langle P_{11}, P_{33} \rangle,$$

and

$$U_1 = H_{23} \cap H_{32} = \langle P_{22}, P_{33} \rangle.$$

It follows that all three lines are contained in the Pascal plane $\langle P_{11}, P_{22}, P_{33} \rangle$.

Remark 3.3 (Relation with the plane Pascal theorem)
Corollary 3.2 may be viewed as a spatial lift of the classical Pascal theorem. Indeed, let $\Pi \subset \mathbb{P}^3$ be a plane which meets Q in a smooth conic

$$C = Q \cap \Pi$$

and which contains none of the lines k_i, m_j. Set

$$K_i = k_i \cap \Pi, \quad M_j = m_j \cap \Pi.$$

Then the six points

$$K_1,\ K_2,\ K_3,\ M_1,\ M_2,\ M_3$$

lie on the conic C. Moreover,

$$\Pi \cap H_{ij} = \langle K_i, M_j \rangle.$$

Therefore the classical Pascal points are obtained by intersecting the lines U_1, U_2, U_3 with Π:

$$u_3 = \langle K_1, M_2 \rangle \cap \langle K_2, M_1 \rangle = \Pi \cap U_3,$$

$$u_2 = \langle K_1, M_3\rangle \cap \langle K_3, M_1\rangle = \Pi \cap U_2,$$

and

$$u_1 = \langle K_2, M_3\rangle \cap \langle K_3, M_2\rangle = \Pi \cap U_1.$$

Since $U_1,\ U_2,\ U_3$ are contained in the plane

$$\Lambda = \langle P_{11}, P_{22}, P_{33}\rangle,$$

the three points u_1, u_2, u_3 lie on the line

$$\Pi \cap \Lambda.$$

This is precisely the Pascal line of the six points $K_1,\ K_2,\ K_3,\ M_1,\ M_2,\ M_3$ on the conic C.

The plane $\Lambda_{\mathrm{id}} = \langle P_{11}, P_{22}, P_{33}\rangle$ is only one of the Pascal planes obtained from the grid. Indeed, one may choose instead any diagonal

$$P_{1,\sigma(1)},\ P_{2,\sigma(2)},\ P_{3,\sigma(3)}$$

with $\sigma \in S_3$. We study the resulting six planes in the next section.

4. The six Pascal planes and the mirror grid

For $\sigma \in S_3$, set $\Lambda_\sigma = \langle P_{1,\sigma(1)}, P_{2,\sigma(2)}, P_{3,\sigma(3)}\rangle$. We call Λ_σ the Pascal plane associated with σ. Thus the plane appearing in Corollary 3.2 is Λ_{id}.

Proposition 4.1 (The six Pascal planes)
Let $Q \subset \mathbb{P}^3$ be a smooth quadric, and let $k_1,\ k_2,\ k_3$ and $m_1,\ m_2,\ m_3$ be two triples of lines contained in the two rulings of Q. Put $P_{ij} = k_i \cap m_j$. For every $\sigma \in S_3$, let

$$\Lambda_\sigma = \langle P_{1,\sigma(1)}, P_{2,\sigma(2)}, P_{3,\sigma(3)}\rangle.$$

Then the three Pascal planes corresponding to even permutations form a pencil of planes, and the three Pascal planes corresponding to odd permutations form another pencil of planes. More precisely,

$$L_{\mathrm{ev}} := \bigcap_{\sigma \in A_3} \Lambda_\sigma \quad \textit{and} \quad L_{\mathrm{odd}} := \bigcap_{\sigma \notin A_3} \Lambda_\sigma$$

are lines in $\mathbb{P}^3$, and

$$\{\Lambda_\sigma : \sigma \in A_3\} \quad \textit{and} \quad \{\Lambda_\sigma : \sigma \notin A_3\}$$

are pencils with axes L_{ev} and L_{odd}, respectively. Moreover, the lines L_{ev} and L_{odd} are skew and polar to one another with respect to the polarity defined by Q.

Proof. The assertion is invariant under projective transformations preserving the quadric. Identifying

$$Q \simeq \mathbb{P}^1 \times \mathbb{P}^1 \tag{1}$$

via the Segre embedding $([u:v],[s:t]) \longrightarrow [us:ut:vs:vt]$, we may choose coordinates
$$[z_0 : z_1 : z_2 : z_3] \text{ in } \mathbb{P}^3$$
such that
$$Q: \ z_0z_3 - z_1z_2 = 0,$$
and such that the three chosen lines in each ruling correspond to the points $0, \infty, 1$ of $\mathbb{P}^1$. So that specifically the coordinates of points P_{ij} are:

$$\begin{aligned} &P_{11} = [0:0:0:1],\ P_{21} = [0:0:1:0],\ P_{31} = [0:0:1:1],\\ &P_{12} = [0:1:0:0],\ P_{22} = [1:0:0:0],\ P_{32} = [1:1:0:0],\\ &P_{13} = [0:1:0:1],\ P_{23} = [1:0:1:0],\ P_{33} = [1:1:1:1]. \end{aligned}$$

In these coordinates the six Pascal planes are as follows:

σ	Λ_σ
id	$z_2 - z_1 = 0$
(123)	$z_0 - z_2 + z_3 = 0$
(132)	$z_0 - z_1 + z_3 = 0$
(12)	$z_3 - z_0 = 0$
(13)	$z_3 - z_1 - z_2 = 0$
(23)	$-z_0 + z_1 + z_2 = 0.$

The first three planes correspond to the even permutations. Their common intersection is the line
$$L_{\mathrm{ev}}: \ z_1 = z_2, \quad z_0 - z_1 + z_3 = 0.$$
The last three planes correspond to the odd permutations. Their common intersection is the line
$$L_{\mathrm{odd}}: \ z_0 = z_3, \quad z_0 - z_1 - z_2 = 0.$$
These two lines are skew.

Finally, a direct computation with the bilinear form associated with
$$Q: \ z_0z_3 - z_1z_2 = 0$$
shows that the polar line of L_{ev} is L_{odd}, and conversely.

Lemma 4.2 (Mixed intersections)
Let $\sigma \in A_3$ and $\tau \notin A_3$. Then the line
$$\Lambda_\sigma \cap \Lambda_\tau$$
passes through exactly one point of the original grid. More precisely, σ and τ agree in a unique position. If
$$\sigma(i) = \tau(i) = j,$$
then
$$P_{ij} \in \Lambda_\sigma \cap \Lambda_\tau.$$

Thus the nine mixed intersections

$$\Lambda_\sigma \cap \Lambda_\tau, \qquad \sigma \in A_3,\ \tau \notin A_3,$$

are naturally indexed by the nine grid points P_{ij}.

Proof. The planes Λ_σ and Λ_τ both contain the point

$$P_{i,\sigma(i)} = P_{i,\tau(i)} = P_{ij}.$$

Since σ and τ have opposite parity, they agree in exactly one position. Therefore each mixed pair determines exactly one grid point in this way. Note that the grid point in the intersection $\Lambda_\sigma \cap \Lambda_\tau$ is unique by construction of these planes (each of them contains exactly three grid points and no grid line). Alternatively, one can use explicit coordinates in the proof of Proposition 4.1 to verify the assertion.

For the unique pair $(\sigma, \tau) \in A_3 \times (S_3 \setminus A_3)$ corresponding to P_{ij} in Lemma 4.2, set

$$R_{ij} := \Lambda_\sigma \cap \Lambda_\tau.$$

The line R_{ij} passes through P_{ij}. It is not a line contained in Q: indeed, the only lines on Q passing through P_{ij} are the two ruling lines k_i and m_j, whereas R_{ij} is a mixed intersection of two Pascal planes and is different from both of them. Hence R_{ij} meets Q in a scheme of length two. We denote the residual intersection point by P'_{ij}, so that

$$R_{ij} \cap Q = \{P_{ij}, P'_{ij}\}.$$

The nine points

$$\mathcal{G}' = \{P'_{ij} : 1 \leqslant i, j \leqslant 3\} \subset Q$$

will be called the mirror grid associated with

$$\mathcal{G} = \{P_{ij} : 1 \leqslant i, j \leqslant 3\}.$$

In the normal form used above, where the original grid corresponds to the three values $0, \infty, 1$ in each ruling of $Q \simeq \mathbb{P}^1 \times \mathbb{P}^1$, a direct computation shows that the residual points P'_{ij} form again a 3×3 grid on Q. More precisely, they correspond to the three values

$$\frac{1}{2}, \quad 2, \quad -1$$

in each ruling. Thus the mixed intersection construction does not produce an arbitrary set of nine residual points on Q, but a second grid on the same quadric. This is the mirror grid.

Proposition 4.3 (The mirror construction is an involution)
Let

$$\mathcal{G} = \{P_{ij} : 1 \leqslant i, j \leqslant 3\}$$

be the original grid, and let

$$\mathcal{G}' = \{P'_{ij} : 1 \leqslant i, j \leqslant 3\}$$

be the mirror grid constructed above. Then $\mathcal{G}'$ is again a 3×3 grid on Q. Moreover, the Pascal planes associated with $\mathcal{G}'$ are the same six planes Λ_σ as those associated with $\mathcal{G}$. Consequently, applying the mirror construction to $\mathcal{G}'$ gives back $\mathcal{G}$.

Proof. Again we work in the normal form

$$Q: \ z_0z_3 - z_1z_2 = 0.$$

For the original grid we take the values

$$0, \ \infty, \ 1$$

in both rulings. As computed above, the residual grid is the grid corresponding to the values

$$\frac{1}{2}, \ 2, \ -1$$

in both rulings. Thus, after preserving the labelling, the residual grid is given by

$$P'_{ij} = ([1 : x'_j], [1 : y'_i])$$

under the Segre embedding, where

$$x'_1 = \frac{1}{2}, \quad x'_2 = 2, \quad x'_3 = -1,$$

and similarly

$$y'_1 = \frac{1}{2}, \quad y'_2 = 2, \quad y'_3 = -1.$$

Thus explicitly

$$\begin{array}{lll}
P'_{11} = [1 : \frac{1}{2} : \frac{1}{2} : \frac{1}{4}], & P'_{21} = [1 : 2 : \frac{1}{2} : 1], & P'_{31} = [1 : -1 : \frac{1}{2} : -\frac{1}{2}], \\
P'_{12} = [1 : \frac{1}{2} : 2 : 1], & P'_{22} = [1 : 2 : 2 : 4], & P'_{32} = [1 : -1 : 2 : -2], \\
P'_{13} = [1 : \frac{1}{2} : -1 : -\frac{1}{2}], & P'_{23} = [1 : 2 : -1 : -2], & P'_{33} = [1 : -1 : -1 : 1].
\end{array}$$

A direct computation shows that the six Pascal planes of the residual grid are again

σ	Λ_σ
id	$z_2 - z_1 = 0$
(123)	$z_0 - z_2 + z_3 = 0$
(132)	$z_0 - z_1 + z_3 = 0$
(12)	$z_3 - z_0 = 0$
(13)	$z_3 - z_1 - z_2 = 0$
(23)	$-z_0 + z_1 + z_2 = 0.$

Hence the six Pascal planes are unchanged when we pass from $\mathcal{G}$ to $\mathcal{G}'$. Therefore the mixed intersection lines R_{ij} are also unchanged. Each such line meets Q in precisely the two points P_{ij} and P'_{ij}. Thus, if the construction sends P_{ij} to P'_{ij} in the first step, it sends P'_{ij} back to P_{ij} in the second step. Hence

$$\mathcal{G} \longmapsto \mathcal{G}' \longmapsto \mathcal{G},$$

and the mirror construction is an involution.

Remark 4.4 (The six points on each ruling)
On each ruling line, the original grid and its mirror grid determine the six points

$$\{\infty, 0, 1, -1, \tfrac{1}{2}, 2\} \subset \mathbb{P}^1.$$

This six-tuple is projectively equivalent to the set of sixth roots of unity. Indeed, the change of coordinate

$$t = \frac{2x - 1 - i\sqrt{3}}{2x - 1 + i\sqrt{3}}$$

maps it to the regular hexagon in $\mathbb{P}^1$. Consequently, over $\mathbb{C}$, the double cover of $\mathbb{P}^1$ branched over these six points is isomorphic to the genus two curve

$$y^2 = t^6 - 1.$$

In particular, the mirror construction naturally produces a highly symmetric six-tuple on each ruling.

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