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Ann. Univ. Paedagog. Crac. Stud. Math. 14 (2015), 5-25
DOI: 10.1515/aupcsm-2015-0001

FOLIA 160

Annales Universitatis Paedagogicae Cracoviensis
Studia Mathematica XIV (2015)

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# The multi-morphisms and their properties and applications

**Abstract.** In this paper a new class of multi-valued mappings (multi-morphisms) is defined as a version of a strongly admissible mapping, and its properties and applications are presented.

## 1. Introduction

In 1976, L. Górniewicz (see [3]) introduced the notion of strongly admissible multi-valued mappings and proved that the composition of strongly admissible mappings is also a strongly admissible mapping. In 1981 it was L. Górniewicz (see [3, 4, 5, 6]) as well that introduced the notion of a morphism, i.e. some other version of strongly admissible mappings. Morphisms, as opposed to strongly-admissible mappings, together with metric spaces create a category on which a functor of Čech homology is extended. In 1994, W. Kryszewski (see [8]) introduced the notion of a morphism essentially different from the morphism in the sense of Górniewicz in regard to some important applications of their properties. In this paper a new type of morphisms (multi-morphisms) is defined and its properties and applications are presented.

## 2. Preliminaries

Throughout this paper all topological spaces are assumed to be metrizable. Let $X$ and $Y$ be two spaces and assume that for every $x \in X$ a non-empty and compact subset $\varphi(x)$ of $Y$ is given. In such a case we say that $\varphi\colon X \multimap Y$ is a multi-valued mapping. For a multi-valued mapping $\varphi\colon X \multimap Y$ and a subset $A \subset Y$, we let

$$\varphi^{-1}(A) = \{x \in X : \ \varphi(x) \subset A\}.$$

AMS (2010) Subject Classification: 32A12, 55P57.

If for every open $U \subset Y$ the set $\varphi^{-1}(U)$ is open, then $\varphi$ is called an upper semi-continuous mapping; we shall write that $\varphi$ is u.s.c. Let $H_*$ be the Čech homology functor with compact carriers and coefficients in the field of rational numbers $\mathbb{Q}$ from the category of Hausdorff topological spaces and continuous maps to the category of graded vector spaces and linear maps of degree zero. Thus $H_*(X) = \{H_q(X)\}$ is a graded vector space, $H_q(X)$ being the $q$-dimensional Čech homology group with compact carriers of $X$. For a continuous map $f\colon X \to Y$, $H_*(f)$ is the induced linear map $f_* = \{f_q\}$, where $f_q\colon H_q(X) \to H_q(Y)$ ([3]). A space $X$ is acyclic if

(i) $X$ is non-empty,

(ii) $H_q(X) = 0$ for every $q \geq 1$ and

(iii) $H_0(X) \approx \mathbb{Q}$.

Let $X$ and $Y$ be Hausdorff topological spaces. A continuous and closed mapping $f\colon X \to Y$ is called proper if for every compact set $K \subset Y$ the set $f^{-1}(K)$ is nonempty and compact. A proper map $p\colon X \to Y$ is called Vietoris provided for every $y \in Y$ the set $p^{-1}(y)$ is acyclic.

Let $u\colon E \to E$ be an endomorphism of an arbitrary vector space. Let us put $N(u) = \{x \in E : \ u^n(x) = 0 \ \text{ for some } n\}$, where $u^n$ is the $n$-th iterate of $u$ and $\widetilde{E} = E/N(u)$. Since $u(N(u)) \subset N(u)$, we have the induced endomorphism $\widetilde{u}\colon \widetilde{E} \to \widetilde{E}$ defined by $\widetilde{u}([x]) = [u(x)]$. We call $u$ admissible provided $\dim \widetilde{E} < \infty$.

Let $u = \{u_q\}\colon E \to E$ be an endomorphism of degree zero of graded vector spaces $E = \{E_q\}$. We call $u$ a Leray endomorphism if

(i) all $u_q$ are admissible,

(ii) almost all $\widetilde{E_q}$ are trivial. For such $u$, we define the (generalized) Lefschetz number $\Lambda(u)$ of $u$ by putting

$$\Lambda(u) = \sum_q (-1)^q \operatorname{tr}(\widetilde{u_q}),$$

where $\operatorname{tr}(\widetilde{u_q})$ is the ordinary trace of $\widetilde{u_q}$ (comp. [3]).

The symbol $D(X, Y)$ will denote the set of all diagrams of the form

$$X \xleftarrow{\ p\ } Z \xrightarrow{\ q\ } Y,$$

where $p\colon Z \to X$ denotes a Vietoris map and $q\colon Z \to Y$ denotes a continuous map. Each such diagram will be denoted by $(p, q)$.

Definition 2.1 (see [3])
Let $(p_1, q_1) \in D(X, Y)$ and $(p_2, q_2) \in D(Y, T)$. The composition of the diagrams

$$X \xleftarrow{\ p_1\ } Z_1 \xrightarrow{\ q_1\ } Y \xleftarrow{\ p_2\ } Z_2 \xrightarrow{\ q_2\ } T,$$

is called a diagram $(p, q) \in D(X, T)$

$$X \xleftarrow{\ p\ } Z_1 \triangle_{q_1 p_2} Z_2 \xrightarrow{\ q\ } T,$$

where

$$Z_1 \triangle_{q_1p_2} Z_2 = \{(z_1, z_2) \in Z_1 \times Z_2 : \ q_1(z_1) = p_2(z_2)\},$$

$$p = p_1 \circ f_1, \qquad q = q_2 \circ f_2,$$

$$Z_1 \xleftarrow{f_1} Z_1 \triangle_{q_1p_2} Z_2 \xrightarrow{f_2} Z_2,$$

$f_1(z_1, z_2) = z_1$ (Vietoris map), $\quad f_2(z_1, z_2) = z_2 \qquad$ for each $(z_1, z_2) \in Z_1 \triangle_{q_1p_2} Z_2$.

It shall be written

$$(p, q) = (p_2, q_2) \circ (p_1, q_1).$$

From the Theorems (40.5), (40.6) in [3, p. 201, 202] it also results that in Definition 2.1 the composition of the diagrams satisfies the condition

$$\text{for each } x \in X \ \ q(p^{-1}(x)) = q_2(p_2^{-1}(q_1(p_1^{-1}(x))))). \tag{1}$$

Recall that if $p\colon X \to Y$ is a Vietoris map then $p_*\colon H_*(X) \to H_*(Y)$ is an isomorphism. Let $(p, q) \in D(X, Y)$. We have the following diagram

$$H_*(X) \xleftarrow{p_*} H_*(Z) \xrightarrow{q_*} H_*(Y). \tag{2}$$

Definition 2.2
Let $(p_1, q_1), (p_2, q_2) \in D(X, Y)$. The equivalency relation in the set $D(X, Y)$ is called a constructor of abstract morphisms (it is denoted as $\sim_a$), if the following conditions are satisfied:

(2.2.1) $((p_1, q_1) \sim_a (p_2, q_2)) \Longrightarrow (\text{for each } x \in X \ \ q_1(p_1^{-1}(x)) = q_2(p_2^{-1}(x)))$,

(2.2.2) $((p_1, q_1) \sim_a (p_2, q_2)) \Longrightarrow (q_{1*} \circ p_{1*}^{-1} = q_{2*} \circ p_{2*}^{-1})$,

(2.2.3) Let $(p_3, q_3), (p_4, q_4) \in D(Y, T)$. Then

$$\left.\begin{array}{l}(p_1, q_1) \sim_a (p_2, q_2)\\ (p_3, q_3) \sim_a (p_4, q_4)\end{array}\right\} \Longrightarrow (((p_3, q_3) \circ (p_1, q_1)) \sim_a ((p_4, q_4) \circ (p_2, q_2))).$$

The condition (2.2.1) will be called an axiom of topological equality, the condition (2.2.2) – an axiom of homological equality, and the condition (2.2.3) – an axiom of composition.

The set $M_a(X, Y) = D(X, Y)_{/\sim_a}$ will be called abstract morphisms ($a$-morphisms). Definition 2.2 (condition (2.2.1) leads to the following:

Definition 2.3
Let $(p, q) \in D(X, Y)$. For any $\varphi_a \in M_a(X, Y)$ the set $\varphi(x) = q(p^{-1}(x))$, where $\varphi_a = [(p, q)]_a$ is called an image of the point $x$ in the $a$-morphism $\varphi_a$.

We denote by

$$\varphi\colon X \to_a Y \tag{3}$$

a multi-valued map determined by an a-morphism $\varphi_a = [(p, q)]_a \in M_a(X, Y)$ and it will be called an abstract multi-valued map.

Let **TOP** denote categories in which Hausdorff topological spaces are objects and continuous mappings are category mappings. Let $\mathbf{TOP_a}$ denote categories in which Hausdorff topological spaces are objects and abstract multi-valued maps (see (3)) are category mappings. According to Definition 2.2 (2.2.3) the category of $\mathbf{TOP_a}$ is well defined and $\mathbf{TOP} \subset \mathbf{TOP_a}$. Let $\mathbf{VECT_G}$ denote categories in which linear graded vector spaces are objects and linear mappings of degree zero are category mappings.

THEOREM 2.4 (SEE [9])
*The mapping* $\widetilde{\mathbf{H}}_*\colon \mathbf{TOP_a} \to \mathbf{VECT_G}$ *given by the formula*

$$\widetilde{\mathbf{H}}_*(\varphi) = q_* \circ p_*^{-1},$$

*where* $\varphi$ *is an abstract multi-valued map determined by* $\varphi_a = [(p,q)]_a$ *is a functor and the extension of the functor of the Čech homology* $\mathbf{H}_*\colon \mathbf{TOP} \to \mathbf{VECT_G}$.

DEFINITION 2.5
Let $X$ be an ANR and let $X_0 \subset X$ be a closed subset. We say that $X_0$ is movable in $X$ provided every neighborhood $U$ of $X_0$ admits a neighborhood $U'$ of $X_0$, $U' \subset U$, such that for every neighborhood $U''$ of $X_0$, $U'' \subset U$, there exists a homotopy $H\colon U' \times [0,1] \to U$ with $H(x,0) = x$ and $H(x,1) \in U''$, for any $x \in U'$.

DEFINITION 2.6
Let $X$ be a compact metric space. We say that $X$ is movable provided there exists $Z \in ANR$ and an embedding $e\colon X \to Z$ such that $e(X)$ is movable in $Z$.

A map $\varphi\colon X \multimap Y$ is compact, if $\overline{\varphi(X)} \subset Y$ is a compact set. Let $(p,q) \in D(X,X)$, where $p,q\colon Z \to X$. We say that $p$ and $q$ have a coincidence point if there exists a point $z \in Z$ such that $p(z) = q(z)$.

THEOREM 2.7 ([3])
*Consider a diagram*

$$X \xleftarrow{\ p\ } Z \xrightarrow{\ q\ } X,$$

*in which* $X \in ANR$, $p$ *is Vietoris and* $q$ *is compact. Then* $q_* \circ p_*^{-1}$ *is a Leray endomorphism and* $\Lambda(q_* \circ p_*^{-1}) \neq 0$ *implies that* $p$ *and* $q$ *have a coincidence point.*

## 3. Multi-morphisms

We recall that the composition of two Vietoris mappings is a Vietoris mapping. Let $Id$ be an identical map. In the set of all diagrams $D(X,Y)$, the following relation is introduced:

DEFINITION 3.1
Let $(p_1,q_1),(p_2,q_2) \in D(X,Y)$.

$$(p_1,q_1) \sim_m (p_2,q_2)$$

if and only if there exist spaces $Z$, $Z_1$ and $Z_2$, Vietoris maps $p_3\colon Z \to Z_1$,

$p_4\colon Z \to Z_2$ such that the following diagram is commutative

$$\begin{array}{ccccc}
X & \xleftarrow{p_1} & Z_1 & \xrightarrow{q_1} & Y \\
\uparrow{\scriptstyle Id} & & \uparrow{\scriptstyle p_3} & & \uparrow{\scriptstyle Id} \\
X & \xleftarrow{p} & Z & \xrightarrow{q} & Y \\
\downarrow{\scriptstyle Id} & & \downarrow{\scriptstyle p_4} & & \downarrow{\scriptstyle Id} \\
X & \xleftarrow{p_2} & Z_2 & \xrightarrow{q_2} & Y,
\end{array}$$

that is

$$p = p_1 \circ p_3 = p_2 \circ p_4, \qquad q = q_1 \circ p_3 = q_2 \circ p_4.$$

PROPOSITION 3.2
*The relation in the set $D(X,Y)$ introduced in Definition* 3.1 *is an equivalence relation.*

*Proof.* In the proof of reflexivity of the relation, it is enough to assume that $Z = Z_1 = Z_2$ and $p_3 = p_4 = Id$. It is obvious that the relation is symmetrical. It shall be now proven that the relation is transitive. Suppose that $(p_1, q_1) \sim_m (p_2, q_2)$ and $(p_2, q_2) \sim_m (p_3, q_3)$. Then from the assumption we have the following commutative diagram

$$\begin{array}{ccccc}
X & \xleftarrow{p_1} & Z_1 & \xrightarrow{q_1} & Y \\
\uparrow{\scriptstyle Id} & & \uparrow{\scriptstyle p_3} & & \uparrow{\scriptstyle Id} \\
X & \xleftarrow{p} & Z & \xrightarrow{q} & Y \\
\downarrow{\scriptstyle Id} & & \downarrow{\scriptstyle p_4} & & \downarrow{\scriptstyle Id} \\
X & \xleftarrow{p_2} & Z_2 & \xrightarrow{q_2} & Y \\
\uparrow{\scriptstyle Id} & & \uparrow{\scriptstyle p_5} & & \uparrow{\scriptstyle Id} \\
X & \xleftarrow{p'} & Z' & \xrightarrow{q'} & Y \\
\downarrow{\scriptstyle Id} & & \downarrow{\scriptstyle p_6} & & \downarrow{\scriptstyle Id} \\
X & \xleftarrow{p_3} & Z_3 & \xrightarrow{q_3} & Y,
\end{array}$$

that is

$$p = p_1 \circ p_3 = p_2 \circ p_4, \qquad q = q_1 \circ p_3 = q_2 \circ p_4$$

and

$$p' = p_2 \circ p_5 = p_3 \circ p_6, \qquad q' = q_2 \circ p_5 = q_3 \circ p_6.$$

Let $f\colon Z \triangle_{p_4 p_5} Z' \to Z$, $f'\colon Z \triangle_{p_4 p_5} Z' \to Z'$, $f(z,z') = z$, $f'(z,z') = z'$ for each $(z,z') \in Z \triangle_{p_4 p_5} Z'$ (see Definition 2.1). We observe that $f$ and $f'$ are Vietoris

maps and $p_4 \circ f = p_5 \circ f'$. We have the following diagram

$$\begin{array}{ccccc}
X & \xleftarrow{p_1} & Z_1 & \xrightarrow{q_1} & Y \\
\uparrow{\scriptstyle Id} & & \uparrow{\scriptstyle p_7} & & \uparrow{\scriptstyle Id} \\
X & \xleftarrow{r} & Z \triangle_{p_4 p_5} Z' & \xrightarrow{s} & Y \\
\downarrow{\scriptstyle Id} & & \downarrow{\scriptstyle p_8} & & \downarrow{\scriptstyle Id} \\
X & \xleftarrow{p_3} & Z_3 & \xrightarrow{q_3} & Y,
\end{array}$$

where $p_7 = p_3 \circ f$, $p_8 = p_6 \circ f'$. The above diagram is commutative. Indeed

$$\begin{aligned}
r &= p_1 \circ p_7 = p_1 \circ (p_3 \circ f) = (p_1 \circ p_3) \circ f = (p_2 \circ p_4) \circ f = p_2 \circ (p_4 \circ f) \\
&= p_2 \circ (p_5 \circ f') = (p_2 \circ p_5) \circ f' = (p_3 \circ p_6) \circ f' = p_3 \circ (p_6 \circ f') \\
&= p_3 \circ p_8
\end{aligned}$$

and similarly

$$\begin{aligned}
s &= q_1 \circ p_7 = q_1 \circ (p_3 \circ f) = (q_1 \circ p_3) \circ f = (q_2 \circ p_4) \circ f = q_2 \circ (p_4 \circ f) \\
&= q_2 \circ (p_5 \circ f') = (q_2 \circ p_5) \circ f' = (q_3 \circ p_6) \circ f' = q_3 \circ (p_6 \circ f') \\
&= q_3 \circ p_8.
\end{aligned}$$

and the proof is complete.

PROPOSITION 3.3
*The equivalence relation $\sim_m$ is a constructor of morphisms (see Definition 2.2) in the set $D(X, Y)$.*

*Proof.* First, the axiom of topological equality shall be proven. Assume that $(p_1, q_1) \sim_m (p_2, q_2)$, where $(p_1, q_1), (p_2, q_2) \in D(X, Y)$. From Definition 3.1 we get the following commutative diagram

$$\begin{array}{ccccc}
X & \xleftarrow{p_1} & Z_1 & \xrightarrow{q_1} & Y \\
\uparrow{\scriptstyle Id} & & \uparrow{\scriptstyle p_3} & & \uparrow{\scriptstyle Id} \\
X & \xleftarrow{p} & Z & \xrightarrow{q} & Y \\
\downarrow{\scriptstyle Id} & & \downarrow{\scriptstyle p_4} & & \downarrow{\scriptstyle Id} \\
X & \xleftarrow{p_2} & Z_2 & \xrightarrow{q_2} & Y
\end{array}$$

that is

$$p = p_1 \circ p_3 = p_2 \circ p_4, \qquad q = q_1 \circ p_3 = q_2 \circ p_4.$$

Let $x \in X$. We have

$$q(p^{-1}(x)) = (q_1 \circ p_3)((p_1 \circ p_3)^{-1}(x)) = q_1(p_3(p_3^{-1}(p_1^{-1}(x)))) = q_1(p_1^{-1}(x))$$

and similarly

$$q(p^{-1}(x)) = (q_2 \circ p_4)((p_2 \circ p_4)^{-1}(x)) = q_2(p_4(p_4^{-1}(p_2^{-1}(x)))) = q_2(p_2^{-1}(x)).$$

Hence

$$q_1(p_1^{-1}(x)) = q_2(p_2^{-1}(x)).$$

Now the axiom of homological equality will be proven. From the properties of homologies we get:

$$p_* = p_{1*} \circ p_{3*} = p_{2*} \circ p_{4*}, \qquad q_* = q_{1*} \circ p_{3*} = q_{2*} \circ p_{4*}.$$

We have

$$\begin{aligned} q_* \circ p_*^{-1} &= (q_1 \circ p_3)_* \circ (p_1 \circ p_3)_*^{-1} = (q_{1*} \circ p_{3*}) \circ (p_{1*} \circ p_{3*})^{-1} \\ &= (q_{1*} \circ p_{3*}) \circ (p_{3*}^{-1} \circ p_{1*}^{-1}) \\ &= q_{1*} \circ p_{1*}^{-1} \end{aligned}$$

and similarly

$$\begin{aligned} q_* \circ p_*^{-1} &= (q_2 \circ p_4)_* \circ (p_2 \circ p_4)_*^{-1} = (q_{2*} \circ p_{4*}) \circ (p_{2*} \circ p_{4*})^{-1} \\ &= (q_{2*} \circ p_{4*}) \circ (p_{4*}^{-1} \circ p_{2*}^{-1}) \\ &= q_{2*} \circ p_{2*}^{-1}. \end{aligned}$$

Hence

$$q_{1*} \circ p_{1*}^{-1} = q_{2*} \circ p_{2*}^{-1}.$$

Now it will be shown that the relation $\sim_m$ satisfies the axiom of composition. Let $(p_1, q_1), (p_2, q_2) \in D(X, Y)$, $(p_3, q_3), (p_4, q_4) \in D(Y, T)$ and let the diagrams $(p, q), (p', q') \in D(X, T)$ be the compositions of the diagrams $(p_1, q_1), (p_3, q_3)$ and $(p_2, q_2), (p_4, q_4)$, respectively (see Definition 2.1). It must be proven that

$$(((p_1, q_1) \sim_m (p_2, q_2))) \text{ and } ((p_3, q_3) \sim_m (p_4, q_4)) \implies ((p, q) \sim_m (p', q')).$$

We have the following commutative diagram

$$\begin{array}{ccccccccc} X & \xleftarrow{p_1} & Z_1 & \xrightarrow{q_1} & Y & \xleftarrow{p_3} & Z_3 & \xrightarrow{q_3} & T \\ \uparrow Id & & \uparrow p_5 & & \uparrow Id & & \uparrow p_7 & & \uparrow Id \\ X & \xleftarrow{u_1} & Z & \xrightarrow{v_1} & Y & \xleftarrow{u_2} & Z' & \xrightarrow{v_2} & T \\ \downarrow Id & & \downarrow p_6 & & \downarrow Id & & \downarrow p_8 & & \downarrow Id \\ X & \xleftarrow{p_2} & Z_2 & \xrightarrow{q_2} & Y & \xleftarrow{p_4} & Z_4 & \xrightarrow{q_4} & T \end{array}$$

that is

$$u_1 = p_1 \circ p_5 = p_2 \circ p_6, \qquad v_1 = q_1 \circ p_5 = q_2 \circ p_6$$

and

$$u_2 = p_3 \circ p_7 = p_4 \circ p_8, \qquad v_2 = q_3 \circ p_7 = q_4 \circ p_8.$$

We make the following diagram (see Definition 2.1)

$$\begin{array}{ccccc}
X & \xleftarrow{p} & Z_1 \triangle_{q_1 p_3} Z_3 & \xrightarrow{q} & T \\
\uparrow Id & & \uparrow r_1 & & \uparrow Id \\
X & \xleftarrow{r} & Z \triangle_{v_1 u_2} Z' & \xrightarrow{s} & T \\
\downarrow Id & & \downarrow r_2 & & \downarrow Id \\
X & \xleftarrow{p'} & Z_2 \triangle_{q_2 p_4} Z_4 & \xrightarrow{q'} & T,
\end{array}$$

where $(r,s) = (u_2, v_2) \circ (u_1, v_1)$, $r_1 = p_5 \times p_7$ and $r_2 = p_6 \times p_8$. First we need to prove that the mappings $r_1$, $r_2$ are well defined. For this we need to show that for each $(z, z') \in Z \triangle_{v_1 u_2} Z'$

$$q_1(p_5(z)) = p_3(p_7(z')) \quad \text{and} \quad q_2(p_6(z)) = p_4(p_8(z')).$$

The first of the above equations will be proven as the second is proven in a similar way. Let $(z, z') \in Z \triangle_{v_1 u_2} Z'$. We have

$$q_1(p_5(z)) = v_1(z) = u_2(z') = p_3(p_7(z')).$$

It is clear that $r_1$ and $r_2$ are Vietoris mappings. We shall now show that the above diagram is commutative. Let $f_1 \colon Z_1 \triangle_{q_1 p_3} Z_3 \to Z_1$, $f_3 \colon Z_1 \triangle_{q_1 p_3} Z_3 \to Z_3$, $f_2 \colon Z_2 \triangle_{q_2 p_4} Z_4 \to Z_2$, $f_4 \colon Z_2 \triangle_{q_2 p_4} Z_4 \to Z_4$, $f \colon Z \triangle_{v_1 u_2} Z' \to Z$, $f' \colon Z \triangle_{v_1 u_2} Z' \to Z'$ are projections (see Definition 2.1). Note that $f_1$, $f_2$, $f$ are Vietoris mappings. We recall that by Definition 2.1 we have: $p = p_1 \circ f_1$, $q = q_3 \circ f_3$, $p' = p_2 \circ f_2$, $q' = q_4 \circ f_4$, $r = u_1 \circ f$, $s = v_2 \circ f'$. Let $(z, z') \in Z \triangle_{v_1 u_2} Z'$. Thus

$$\begin{aligned}
p(r_1(z,z')) &= p_1(f_1((p_5(z), p_7(z')))) = p_1(p_5(z)) = u_1(z) = u_1(f(z,z')) = r(z,z'), \\
p'(r_2(z,z')) &= p_2(f_2((p_6(z), p_8(z')))) = p_2(p_6(z)) = u_1(z) = u_1(f(z,z')) = r(z,z')
\end{aligned}$$

and similarly

$$\begin{aligned}
q(r_1(z,z')) &= q_3(f_3((p_5(z), p_7(z')))) = q_3(p_7(z')) = v_2(z') = v_2(f'(z,z')) \\
&= s(z,z'), \\
q'(r_2(z,z')) &= q_4(f_4((p_6(z), p_8(z')))) = q_4(p_8(z')) = v_2(z') = v_2(f'(z,z')) \\
&= s(z,z')
\end{aligned}$$

and the proof is complete.

The set of the class of the abstraction of the above relation will be denoted by the symbol

$$M_m(X,Y) = D(X,Y)_{/\sim_m}.$$

The elements of the set $M_m(X,Y)$ will be called multi-morphisms and denoted by: $\varphi_m, \psi_m, \ldots$. The following denotation is assumed

$$\varphi_m = [(p,q)]_m \quad (\text{we write } (p,q) \in \varphi_m),$$

where the diagram $(p, q)$ is representative of the class of the abstraction $[(p, q)]_m$ in the relation $\sim_m$.

It shall be noticed that if the two diagrams $(p_1, q_1), (p_2, q_2) \in D(X, Y)$ are in a relation in the sense of Kryszewski (see [8]), then

$$(p_1, q_1) \sim_m (p_2, q_2).$$

An example that the inverse conclusion is not true will be provided now. Let $\mathbb{R}$ be a real number set and let $[0, 1] \subset \mathbb{R}$ be an interval.

Example 3.4
Let $\psi\colon [0, 1] \multimap [0, 1]$ be a map given by

$$\psi(x) = \begin{cases} 0 & \text{for } x < \frac{1}{2}, \\ [0, 1] & \text{for } x = \frac{1}{2}, \\ 1 & \text{for } x > \frac{1}{2}. \end{cases}$$

The mapping $\psi$ is u.s.c. and of compact and convex images. It shall be noticed that $\psi$ does not have a continuous selector, that is, there does not exist a continuous mapping $f\colon [0, 1] \to [0, 1]$ such that for every $x \in [0, 1]$ $f(x) \in \psi(x)$. Let $\Gamma_\psi = \{(x, y) \in [0, 1] \times [0, 1];\ y \in \psi(x)\}$. Then the set $\Gamma_\psi$ is homeomorphic with the set $[0, 1]$. It results in the following commutative diagram

$$\begin{array}{ccccc} [0,1] & \xleftarrow{p} & \Gamma_\psi & \xrightarrow{p} & [0,1] \\ \uparrow{\scriptstyle Id} & & \uparrow{\scriptstyle Id} & & \uparrow{\scriptstyle Id} \\ [0,1] & \xleftarrow{p} & \Gamma_\psi & \xrightarrow{p} & [0,1] \\ \downarrow{\scriptstyle Id} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle Id} \\ [0,1] & \xleftarrow{Id} & [0,1] & \xrightarrow{Id} & [0,1], \end{array}$$

where $p(x, y) = x$ (Vietoris map) for every $(x, y) \in \Gamma_\psi$. It should be noticed that

$$(p, p) \sim_m (Id, Id),$$

but the diagrams $(p, p), (Id, Id) \in D([0, 1], [0, 1])$ are not in a relation either in the sense of Kryszewski or in the sense of Górniewicz (see [4]). Let's assume that there exists a continuous mapping (not necessarily a homeomorphism) $h\colon [0, 1] \to \Gamma_\psi$ such that $p \circ h = Id$. Then for every $x \in [0, 1]$ $h(x) \in p^{-1}(x)$. Let $q\colon \Gamma_\psi \to [0, 1]$ be given by formula $q(x, y) = y$ for every $(x, y) \in \Gamma_\psi$. Then the mapping $f\colon [0, 1] \to [0, 1]$ given by formula $f = q \circ h$ would be a continuous selector of the mapping $\psi$ but it is impossible.

The above example shows that the relation $\sim_m$ is essentially different from the relations $\sim_k$ and $\sim_g$. For single-valued mappings, there is the following fact:

Proposition 3.5
*Let $f\colon X \to Y$ be a continuous mapping and let $(p,q) \in D(X,Y)$, where*

$$X \xleftarrow{p} Z \xrightarrow{q} Y.$$

*Then the following conditions are equivalent:*

(3.5.1) $q = f \circ p$,

(3.5.2) $(p,q) \sim_m (Id, f)$,

(3.5.3) $q(p^{-1}(x)) = f(x)$ *for each* $x \in X$.

*Proof.* (3.5.1) $\Rightarrow$ (3.5.2)
There is the following commutative diagram:

$$\begin{array}{ccccc}
X & \xleftarrow{p} & Z & \xrightarrow{q} & Y \\
\uparrow{\scriptstyle Id} & & \uparrow{\scriptstyle Id} & & \uparrow{\scriptstyle Id} \\
X & \xleftarrow{p} & Z & \xrightarrow{q} & Y \\
\downarrow{\scriptstyle Id} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle Id} \\
X & \xleftarrow{Id} & X & \xrightarrow{f} & Y.
\end{array}$$

Hence $(p,q) \sim_m (Id, f)$.

(3.5.2) $\Rightarrow$ (3.5.3)
This implication is the result of the axiom of topological equality (see Proposition 3.3).

(3.5.3) $\Rightarrow$ (3.5.1)
Let $(p,q) \in D(X,Y)$ such that for each $x \in X$ $q(p^{-1}(x)) = f(x)$ and let $z \in Z$. Then there exists a point $x_1 \in X$ such that $z \in p^{-1}(x_1)$. Hence we get

$$q(z) = f(x_1) = f(p(z)),$$

and the proof is complete.

From the last fact it results that the relation $\sim_m$ orders single-valued multi-morphisms and separates them from multi-valued multi-morphisms.

## 4. The homotopy of multi-morphisms

First, we define the homotopy diagrams in the set $D(X,Y)$ and prove that there is an equivalence relation. At the beginning we prove the following fact:

Proposition 4.1
*Let $(p_1,q_1),(p_2,q_2) \in D(X,Y)$, where*

$$X \xleftarrow{p_1} Z_1 \xrightarrow{q_1} Y, \qquad X \xleftarrow{p_2} Z_2 \xrightarrow{q_2} Y.$$

*Then there exists $(p,q),(p,q') \in D(X,Y)$ such that*

$$(p_1,q_1) \sim_m (p,q) \quad \textit{and} \quad (p_2,q_2) \sim_m (p,q').$$

*Proof.* Let $Z = Z_1 \triangle_{p_1 p_2} Z_2$ (see Definition 2.1) and let

$$f_1 \colon Z \to Z_1,\ f_1(z_1, z_2) = z_1, \qquad f_2 \colon Z \to Z_2,\ f_2(z_1, z_2) = z_2$$

for each $(z_1, z_2) \in Z$. We observe that $f_1$ and $f_2$ are Vietoris maps and $p_1 \circ f_1 = p_2 \circ f_2$. Let

$$p = p_1 \circ f_1 = p_2 \circ f_2, \qquad q = q_1 \circ f_1, \qquad q' = q_2 \circ f_2. \tag{4}$$

We have the following commutative diagrams:

$$\begin{array}{ccccc}
X & \xleftarrow{p_1} & Z_1 & \xrightarrow{q_1} & Y \\
\uparrow{\scriptstyle Id} & & \uparrow{\scriptstyle f_1} & & \uparrow{\scriptstyle Id} \\
X & \xleftarrow{p} & Z & \xrightarrow{q} & Y \\
\downarrow{\scriptstyle Id} & & \downarrow{\scriptstyle Id} & & \downarrow{\scriptstyle Id} \\
X & \xleftarrow{p} & Z & \xrightarrow{q} & Y,
\end{array}
\qquad
\begin{array}{ccccc}
X & \xleftarrow{p_2} & Z_2 & \xrightarrow{q_2} & Y \\
\uparrow{\scriptstyle Id} & & \uparrow{\scriptstyle f_2} & & \uparrow{\scriptstyle Id} \\
X & \xleftarrow{p} & Z & \xrightarrow{q'} & Y \\
\downarrow{\scriptstyle Id} & & \downarrow{\scriptstyle Id} & & \downarrow{\scriptstyle Id} \\
X & \xleftarrow{p} & Z & \xrightarrow{q'} & Y.
\end{array}$$

Hence we get

$$(p_1, q_1) \sim_m (p, q) \quad \text{and} \quad (p_2, q_2) \sim_m (p, q')$$

and the proof is complete.

From the last fact it results that every two different multi-morphisms have a common Vietoris mapping. It means that only continuous mappings $q_1$, $q_2$ decide about the differential of multi-morphisms $(p_1, q_1) \in \varphi_m$ and $(p_2, q_2) \in \psi_m$. With the recent Proposition we can introduce the following definition of homotopy diagrams.

Definition 4.2
Let $(p_1, q_1), (p_2, q_2) \in D(X, Y)$, where

$$X \xleftarrow{p_1} Z_1 \xrightarrow{q_1} Y, \qquad X \xleftarrow{p_2} Z_2 \xrightarrow{q_2} Y.$$

We say that the diagrams $(p_1, q_1)$ and $(p_2, q_2)$ are homotopic which is denoted by

$$(p_1, q_1) \sim_{HD} (p_2, q_2)$$

if there exists a space $Z$ and Vietoris maps $p_3 \colon Z \to Z_1$ and $p_4 \colon Z \to Z_2$ such that the following conditions are satisfied:

(4.2.1) $p_1 \circ p_3 = p_2 \circ p_4$,

(4.2.2) $q_1 \circ p_3 \sim_h q_2 \circ p_4$ that is, the mappings $q_1 \circ p_3, q_2 \circ p_4 \colon Z \to Y$ are homotopic.

Proposition 4.3
*The homotopy relation introduced in the Definition* 4.2 *is an equivalence relation in the set of all diagrams* $D(X, Y)$.

*Proof.* Let $(p,q) \in D(X,Y)$, where

$$X \xleftarrow{p} Z \xrightarrow{q} Y.$$

It is clear that the relation is reflexive that is, $(p,q) \sim_{HD} (p,q)$. Indeed, it is sufficient to adopt $p_3 = p_4 = Id\colon Z \to Z$. It is also evident that the relation is symmetric. We shall now show that the relation is transitive. Let $(p_1,q_1),(p_2,q_2),(p_3,q_3) \in D(X,Y)$, where

$$X \xleftarrow{p_1} Z_1 \xrightarrow{q_1} Y, \quad X \xleftarrow{p_2} Z_2 \xrightarrow{q_2} Y, \quad X \xleftarrow{p_3} Z_3 \xrightarrow{q_3} Y.$$

Assume that

$$(p_1,q_1) \sim_{HD} (p_2,q_2) \quad \text{and} \quad (p_2,q_2) \sim_{HD} (p_3,q_3).$$

We have the following diagram

$$\begin{array}{ccccc}
X & \xleftarrow{p_1} & Z_1 & \xrightarrow{q_1} & Y \\
\uparrow{\scriptstyle Id} & & \uparrow{\scriptstyle p_3} & & \\
X & \xleftarrow{r} & Z & & \\
\downarrow{\scriptstyle Id} & & \downarrow{\scriptstyle p_4} & & \\
X & \xleftarrow{p_2} & Z_2 & \xrightarrow{q_2} & Y \\
\uparrow{\scriptstyle Id} & & \uparrow{\scriptstyle p_5} & & \\
X & \xleftarrow{r'} & Z' & & \\
\downarrow{\scriptstyle Id} & & \downarrow{\scriptstyle p_6} & & \\
X & \xleftarrow{p_3} & Z_3 & \xrightarrow{q_3} & Y,
\end{array}$$

where

$$r = p_1 \circ p_3 = p_2 \circ p_4, \qquad r' = p_2 \circ p_5 = p_3 \circ p_6,$$
$$q_1 \circ p_3 \sim_h q_2 \circ p_4, \qquad q_2 \circ p_5 \sim_h q_3 \circ p_6.$$

Let

$$f\colon Z \triangle_{p_4p_5} Z' \to Z,\ f(z,z') = z, \qquad f'\colon Z \triangle_{p_4p_5} Z' \to Z',\ f'(z,z') = z'$$

for each $(z,z') \in Z \triangle_{p_4p_5} Z'$. We observe that $f$ and $f'$ are Vietoris mappings and $p_4 \circ f = p_5 \circ f'$. We get the following diagram

$$
\begin{array}{ccccc}
X & \xleftarrow{p_1} & Z_1 & \xrightarrow{q_1} & Y \\
\uparrow{\scriptstyle Id} & & \uparrow{\scriptstyle p_3} & & \\
X & & Z & & \\
\uparrow{\scriptstyle Id} & & \uparrow{\scriptstyle f} & & \\
X & \xleftarrow{s} & Z \triangle_{p_4p_5} Z' & & \\
\downarrow{\scriptstyle Id} & & \downarrow{\scriptstyle f'} & & \\
X & & Z' & & \\
\downarrow{\scriptstyle Id} & & \downarrow{\scriptstyle p_6} & & \\
X & \xleftarrow{p_3} & Z_3 & \xrightarrow{q_3} & Y,
\end{array}
$$

where

$$s = (p_1 \circ p_3) \circ f = (p_2 \circ p_4) \circ f = p_2 \circ (p_4 \circ f) = p_2 \circ (p_5 \circ f') = (p_2 \circ p_5) \circ f' = (p_3 \circ p_6) \circ f'.$$

Let $p_7 = p_3 \circ f$, $p_8 = p_6 \circ f'$, then $p_1 \circ p_7 = p_3 \circ p_8$. We define a homotopy $h\colon Z \triangle_{p_4p_5} Z' \times [0,1] \to Y$ given by the formula

$$h(z, z', t) = \begin{cases} h_1(f(z, z'), 2t) & \text{for } t \in [0, \frac{1}{2}], \\ h_2(f'(z, z'), 2t - 1) & \text{for } t \in [\frac{1}{2}, 1], \end{cases}$$

where $h_1\colon Z \times [0,1] \to Y$ is a homotopy between the mappings $q_1 \circ p_3$ and $q_2 \circ p_4$ and $h_2\colon Z' \times [0,1] \to Y$ is a homotopy between the mappings $q_2 \circ p_5$ and $q_3 \circ p_6$. Let $(z, z') \in Z \triangle_{p_4p_5} Z'$. We observe that for $t = \frac{1}{2}$

$$h_1(f(z, z'), 1) = q_2(p_4(f(z, z'))) = q_2(p_5(f'(z, z'))) = h_2(f'(z, z'), 0).$$

Hence the map $h$ is well defined. Furthermore, we have

$$h(z, z', 0) = h_1(f(z, z'), 0) = q_1(p_3(f(z, z'))) = q_1(p_7(z, z'))$$

and

$$h(z, z', 1) = h_2(f'(z, z'), 1) = q_3(p_6(f'(z, z'))) = q_3(p_8(z, z'))$$

and the proof is complete.

Another simple fact does not require proof.

PROPOSITION 4.4
*Let* $(p_1, q_1), (p_2, q_2) \in D(X, Y)$ *and let* $(p_1, q_1) \sim_m (p_2, q_2)$, *then* $(p_1, q_1) \sim_{HD} (p_2, q_2)$.

PROPOSITION 4.5
*Let* $(p_1, q_1), (p_2, q_2) \in D(X, Y)$, *where*

$$X \xleftarrow{p_1} Z_1 \xrightarrow{q_1} Y, \qquad X \xleftarrow{p_2} Z_2 \xrightarrow{q_2} Y.$$

*If* $(p_1, q_1) \sim_{HD} (p_2, q_2)$*, then* $q_{1*} \circ p_{1*}^{-1} = q_{2*} \circ p_{2*}^{-1}$*, where*

$$H_*(X) \xleftarrow{p_{1*}} H_*(Z_1) \xrightarrow{q_{1*}} H_*(Y), \;\; H_*(X) \xleftarrow{p_{2*}} H_*(Z_2) \xrightarrow{q_{2*}} H_*(Y).$$

*Proof.* From the assumption there exist Vietoris maps $p_3 \colon Z \to Z_1$ and $p_4 \colon Z \to Z_2$ such that $p_1 \circ p_3 = p_2 \circ p_4$ and $q_1 \circ p_3 \sim_h q_2 \circ p_4$. With property homology we get

$$p_{1*} \circ p_{3*} = p_{2*} \circ p_{4*} \quad \text{and} \quad q_{1*} \circ p_{3*} = q_{2*} \circ p_{4*}.$$

We have

$$p_{1*} = p_{2*} \circ p_{4*} \circ p_{3*}^{-1} \quad \text{and} \quad q_{1*} = q_{2*} \circ p_{4*} \circ p_{3*}^{-1}.$$

Finally, we get

$$\begin{aligned} q_{1*} \circ p_{1*}^{-1} &= (q_{2*} \circ p_{4*} \circ p_{3*}^{-1}) \circ (p_{2*} \circ p_{4*} \circ p_{3*}^{-1})^{-1} \\ &= (q_{2*} \circ p_{4*} \circ p_{3*}^{-1}) \circ (p_{3*} \circ p_{4*}^{-1} \circ p_{2*}^{-1}) \\ &= q_{2*} \circ p_{2*}^{-1} \end{aligned}$$

and the proof is complete.

Now, using the Propositions 4.3 and 4.4, we can define homotopy multi-morphisms.

Definition 4.6
Let $\varphi_m, \psi_m \in M_m(X, Y)$ be multi-morphisms. We say that the multi-morphisms $\varphi_m$ and $\psi_m$ are homotopic (we write $\varphi_m \sim_{HM} \psi_m$) if there exist diagrams $(p_1, q_1) \in \varphi_m$ and $(p_2, q_2) \in \psi_m$ such that $(p_1, q_1) \sim_{HD} (p_2, q_2)$.

Proposition 4.7
*The homotopy relation introduced in the Definition* 4.6 *is an equivalence relation in the set of all multi-morphisms* $M_m(X, Y)$*.*

*Proof.* It is obvious that the relation is reflexive and symmetric. Transitivity of the relation follows from Proposition 4.3 and 4.4.

Using the Proposition 4.3 and 4.4, note that, in fact, if $\varphi_m \sim_{HM} \psi_m$, where $\varphi_m, \psi_m \in M_m(X, Y)$ are multi-morphisms, then for each $(p_1, q_1) \in \varphi_m$ and $(p_2, q_2) \in \psi_m$

$$(p_1, q_1) \sim_{HD} (p_2, q_2). \tag{5}$$

Let $f \colon X \to Y$ be a single-valued continuous map. The symbol $f_m \in M_m(X, Y)$ we denote a multi-morphism such that for all $(p, q) \in f_m$ and for each $x \in X$

$$q(p^{-1}(x)) = f(x).$$

Proposition 4.8
*Let* $f, g \colon X \to Y$ *be continuous maps. If* $f \sim_h g$*, then*

$$f_m \sim_{HM} g_m.$$

*Proof.* It is clear that $(Id, f) \sim_{HD} (Id, g)$ because for $p_3 = p_4 = Id$ (see Definition 4.2) we have $Id \circ p_3 = Id \circ p_4$ and $f \circ p_3 \sim_h g \circ p_4$. Hence from Definition 4.6 $f_m \sim_{HM} g_m$ and the proof is complete.

PROPOSITION 4.9
*Let $f, g\colon X \to Y$ be continuous maps. $f_m \sim_{HM} g_m$ if and only if there exists a space $Z$ and a Vietoris mapping $p\colon Z \to X$ such that*

$$f \circ p \sim_h g \circ p.$$

*Proof.* Let $f_m \sim_{HM} g_m$. Then from Proposition 3.5 and (5) we have the following diagram

$$\begin{array}{ccccc} X & \xleftarrow{Id} & X & \xrightarrow{f} & Y \\ \uparrow{\scriptstyle Id} & & \uparrow{\scriptstyle p_3} & & \\ X & \xleftarrow{p} & Z & & \\ \downarrow{\scriptstyle Id} & & \downarrow{\scriptstyle p_4} & & \\ X & \xleftarrow{Id} & X & \xrightarrow{g} & Y, \end{array}$$

where $p = Id \circ p_3 = Id \circ p_4$ and $f \circ p = f \circ p_3 \sim_h g \circ p_4 = g \circ p$ and the same proof one way has been completed. Assume now that there exists a space $Z$ and a Vietoris mapping $p\colon Z \to X$ such that $f \circ p \sim_h g \circ p$. Then we get the diagram

$$\begin{array}{ccccc} X & \xleftarrow{Id} & X & \xrightarrow{f} & Y \\ \uparrow{\scriptstyle Id} & & \uparrow{\scriptstyle p} & & \\ X & \xleftarrow{p} & Z & & \\ \downarrow{\scriptstyle Id} & & \downarrow{\scriptstyle p} & & \\ X & \xleftarrow{Id} & X & \xrightarrow{g} & Y. \end{array}$$

Hence $f_m \sim_{HM} g_m$ and the proof is complete.

## 5. The applications

From the axiom of topological equality the correctness of the following definition results:

DEFINITION 5.1
For any $\varphi_m \in M_m(X, Y)$, the set $\varphi(x) = q(p^{-1}(x))$ where $\varphi_m = [(p, q)]_m$ is called an image of point $x$ in a multi-morphism $\varphi_m$.

The concept of multi-contractibility of space in the context of multi-morphisms will be now introduced.

Definition 5.2
Let $X$ be a metrizable space and let $x_0 \in X$. Let $C^{x_0} \colon X \to X$ be a constant map such that $C^{x_0}(x) = x_0$ for each $x \in X$. We say that a space $X$ is multi-contractible to a point $x_0$ in the context of multi-morphisms (we write $X \in MCN_m$) if

$$[(Id, Id)]_m = Id_m \sim_{HM} C_m^{x_0} = [(Id, C^{x_0})]_m.$$

From Proposition 4.9 we get the following fact:

Proposition 5.3
*A space $X \in MCN_m$ if and only if there exists a metrizable space $Z$ and a Vietoris map $p \colon Z \to X$ such that $p \sim_h C_1^{x_0}$, where $C_1^{x_0} \colon Z \to X$ is a constant map, that is for each $z \in Z$ $C_1^{x_0}(z) = x_0$.*

We recall that the space $X$ is contractible to the point $x_0 \in X$ (we write $X \in CN$) if $Id \sim_h C^{x_0}$. It is obvious that if $X \in CN$ then $X \in MCN_m$. We will give an example that the inverse theorem is not true. We know that if $X \in CN$ then the space $X$ is movable (see Definition 2.5). $Q$ shall denote the Hilbert cube.

Example 5.4
Let $X \subset Q$ be a non-movable compact metric space and such that there exists a Vietoris mapping $p \colon Q \to X$ (see [7]). Then $X \in MCN_m$. We define a homotopy $h \colon Q \times [0,1] \to X$ between the $p$ and $C_1^{x_0}$ (see Proposition 5.3), where $x_0 \in X$ given by formula

$$h(z,t) = p((1-t)z + tz_0) \qquad \text{for each } (z,t) \in Q \times [0,1],$$

where $z_0 \in Q$ and $p(z_0) = x_0$. We observe that $X \notin CN$ since $X$ is non-movable.

Another important fact is the following:

Proposition 5.5
*If $X \in MCN_m$ then $X$ is path connected.*

*Proof.* From Proposition 5.3 there exists a space $Z$ and a Vietoris map $p \colon Z \to X$ such that $p \sim_h C_1^{x_0}$. Let $x_1 \in X$ and let $p(z_1) = x_1$ for some point $z_1 \in Z$. We define a path $d \colon [0,1] \to X$ between the point $x_0 \in X$ and the point $x_1 \in X$ given by formula

$$d(t) = h(z_1, t) \qquad \text{for each } t \in [0,1],$$

where $h \colon Z \times [0,1] \to X$ is a homotopy between $p$ and $C_1^{x_0}$.

Homotopy multi-morphisms can be defined according to the mathematical literature of homotopy morphisms in the sense of Kryszewski (see [8]). Let $i^j \colon X \times \{j\} \to X \times [0,1]$, $j = 0, 1$ be an inclusion given by formula $i^j(x,j) = (x,j)$ for each $(x,j) \in X \times \{j\}$.

Definition 5.6
We say that the multi-morphisms $\varphi_m, \psi_m \in M_m(X,Y)$ are weakly homotopic (we write $\varphi_m \sim_{HMW} \psi_m$) if there exists a multi-morphism $H_m \in M_m(X \times [0,1], Y)$ such that

$$H_m \circ i_m^0 = \varphi_m, \qquad H_m \circ i_m^1 = \psi_m.$$

We observe that:

PROPOSITION 5.7
*Let $\varphi_m, \psi_m \in M_m(X,Y)$ be multi-morphisms. If $\varphi_m \sim_{HM} \psi_m$ then $\varphi_m \sim_{HMW} \psi_m$.*

*Proof.* From Definition 4.6 there exist diagrams $(p_1, q_1) \in \varphi_m$ and $(p_2, q_2) \in \psi_m$ such that $(p_1, q_1) \sim_{HD} (p_2, q_2)$, where

$$X \xleftarrow{p_1} Z_1 \xrightarrow{q_1} Y, \qquad X \xleftarrow{p_2} Z_2 \xrightarrow{q_2} Y.$$

From Definition 4.2 there exists a space $Z$ and Vietoris mappings $p_3\colon Z \to Z_1$ and $p_4\colon Z \to Z_2$ such that

$$p_1 \circ p_3 = p_2 \circ p_4 \quad \text{and} \quad q_1 \circ p_3 \sim_h q_2 \circ p_4.$$

Let $p = p_1 \circ p_3 = p_2 \circ p_4$ and let $r\colon Z \times [0,1] \to X \times [0,1]$ be a Vietoris map given by formula $r(z,t) = (p(z), t)$ for each $(z,t) \in Z \times [0,1]$. We define a weakly homotopy $H_m \in M_m(X \times [0,1], Y)$ given by formula

$$H_m = h_m \circ \eta_m,$$

where $\eta_m = [(r, Id)]_m$ and $h_m = [(Id, h)]_m$, $h\colon Z \times [0,1] \to Y$ is a homotopy between the mappings $q_1 \circ p_3$ and $q_2 \circ p_4$. We have the following diagram

$$X \times [0,1] \xleftarrow{r} Z \times [0,1] \xrightarrow{Id} Z \times [0,1] \xleftarrow{Id} Z \times [0,1] \xrightarrow{h} Y.$$

It is clear that

$$H_m \circ i_m^0 = \varphi_m, \qquad H_m \circ i_m^1 = \psi_m$$

and the proof is complete.

We can adopt, of course, the following definition.

DEFINITION 5.8
We say that a space $X$ is a weakly multi-contractible to the point $x_0 \in X$ in the context of the multi-morphisms (we write $X \in MCN_{mw}$) if there exists a weakly homotopy $H_m \in M_m(X \times [0,1], X)$ such that

$$Id_m \sim_{HMW} C_m^{x_0}.$$

We will give an example that the inverse theorem to the Proposition 5.7 is not true. We recall that a multi-valued u.s.c. map $\varphi\colon X \multimap Y$ is acyclic if for each $x \in X$ the set $\varphi(x)$ is compact and acyclic. The acyclic map $\varphi$ is determined by a multi-morphism $\varphi_m = [(p_\varphi, q_\varphi)]_m \in M_m(X,Y)$, where

$$X \xleftarrow{p_\varphi} \Gamma_\varphi \xrightarrow{q_\varphi} Y, \tag{6}$$

$\Gamma_\varphi = \{(x,y) \in X \times Y;\ y \in \varphi(x)\}$, $p_\varphi(x,y) = x$ (Vietoris map) $q_\varphi(x,y) = y$ for each $(x,y) \in \Gamma_\varphi$ such that for each $x \in X$

$$q_\varphi(p_\varphi^{-1}(x)) = \varphi(x).$$

In the mathematical literature it is known that if $X$ is compact and of trivial shape in the sense of Borsuk (see [1]) then it is acyclic.

Example 5.9
We define a set $X \subset \mathbb{R}^2$ given by formula

$$X = \{(x,y) \in \mathbb{R}^2;\ y = \sin(1/x),\ x \in (0,1]\} \cup (\{0\} \times [-1,1]).$$

We know (see [2, 3]) that $X$ is compact and of trivial shape. We also know that $X$ is not path connected, so $X \notin MCN_m$ (see Proposition 5.5). Hence we have

$$Id_m \nsim_{HM} C_m^{x_0}.$$

We define a multi-valued map $H\colon X \times [0,1] \multimap X$ given by formula:

$$H(x,t) = \begin{cases} x & \text{for } t \in [0, \frac{1}{2}), \\ X & \text{for } t = \frac{1}{2}, \\ x_0 & \text{for } t \in (\frac{1}{2}, 1]. \end{cases}$$

The map $H$ is acyclic, so the multi-morphism $H_m = [(p_H, q_H)]_m \in M_m(X \times [0,1], X)$ (see (6)) is a weakly homotopy joining $Id_m$ and $C_m^{x_0}$. Hence $X \in MCN_{mw}$.

Note that there are three types of contractibility to the point of a metric space. The first type of contractibility is $CN$, the second type is $MCN_m$, while the third is $MCN_{mw}$. On the basis of the above considerations, we can see that

$$CN \subset MCN_m \subset MCN_{mw}$$

and any of these inclusions can not be reversed. Note that if the space $X$ is compact and acyclic then $X \in MCN_{mw}$. In this case, homotopy can be written as in the Example 5.9. Contractibility of $MCN_m$ type is more general than the one of $CN$ (see Example 5.4) type but it remains in the class of path connected (see Proposition 5.5).

In the second part we will present the application to the theory of coincidence. We introduce the following definitions:

Definition 5.10
Let $(p_1, q_1), (p_2, q_2) \in D(X,Y)$, where

$$X \xleftarrow{p_1} Z_1 \xrightarrow{q_1} Y, \qquad X \xleftarrow{p_2} Z_2 \xrightarrow{q_2} Y.$$

We say that the diagrams $(p_1, q_1)$ and $(p_2, q_2)$ have a coincidence point

$$(\text{we write } \ (p_1, q_1) \sim_{z_0} (p_2, q_2))$$

if there exists a metrizable space $Z$, Vietoris maps $p_3\colon Z \to Z_1$, $p_4\colon Z \to Z_2$ and a point $z_0 \in Z$ such that $p_1 \circ p_3 = p_2 \circ p_4$ and

$$q_1(p_3(z_0)) = q_2(p_4(z_0)).$$

We recommend the articles [5, 6] on the theory of coincidence.

Definition 5.11
Let $\varphi_m, \psi_m \in M_m(X,Y)$. We say that the multi-morphisms $\varphi_m$ and $\psi_m$ have a coincidence point if there exist diagrams $(p_1,q_1) \in \varphi_m$ and $(p_2,q_2) \in \psi_m$ such that $(p_1,q_1) \sim_{z_0} (p_2,q_2)$.

At the beginning we prove the following fact:

Proposition 5.12
*Let $\varphi_m, \psi_m \in M_m(X,Y)$. The multi-morphisms $\varphi_m$ and $\psi_m$ have a coincidence point if and only if there exist a point $x_0 \in X$ such that*

$$\varphi(x_0) \cap \psi(x_0) \neq \emptyset.$$

*Proof.* Assume that the multi-morphisms $\varphi_m$ and $\psi_m$ have a coincidence point. From the Definition 5.11 we get $(p_1,q_1) \in \varphi_m$ and $(p_2,q_2) \in \psi_m$ such that $(p_1,q_1)$ and $(p_2,q_2)$ have a coincidence point, where

$$X \xleftarrow{p_1} Z_1 \xrightarrow{q_1} Y, \qquad X \xleftarrow{p_2} Z_2 \xrightarrow{q_2} Y.$$

In turn, by Definition 5.10 there exists a metrizable space $Z$, Vietoris maps $p_3\colon Z \to Z_1$, $p_4\colon Z \to Z_2$ and a point $z_0 \in Z$ such that $p_1 \circ p_3 = p_2 \circ p_4$ and $q_1(p_3(z_0)) = q_2(p_4(z_0))$. Let $z_1 = p_3(z_0) \in Z_1$, $z_2 = p_4(z_0) \in Z_2$ and let $x_0 = p_1(z_1) = p_2(z_2) \in X$. Then $q_1(z_1) = q_2(z_2)$ and hence

$$q_1(p_1^{-1}(x_0)) \cap q_2(p_2^{-1}(x_0)) \neq \emptyset,$$

so from the axiom of topological equation (see Definition 2.2) $\varphi(x_0) \cap \psi(x_0) \neq \emptyset$ and proof one way has been completed. Assume now that there exists a point $x_0 \in X$ such that $\varphi(x_0) \cap \psi(x_0) \neq \emptyset$. Then there exists $(p_1,q_1) \in \varphi_m$ and $(p_2,q_2) \in \psi_m$ such that $q_1(p_1^{-1}(x_0)) \cap q_2(p_2^{-1}(x_0)) \neq \emptyset$, where

$$X \xleftarrow{p_1} Z_1 \xrightarrow{q_1} Y, \qquad X \xleftarrow{p_2} Z_2 \xrightarrow{q_2} Y.$$

There exist points $z_1^0 \in Z_1$, $z_2^0 \in Z_2$ such that

$$p_1(z_1^0) = p_2(z_2^0) = x_0 \quad \text{and} \quad q_1(z_1^0) = q_2(z_2^0).$$

Let $Z = Z_1 \triangle_{p_1 p_2} Z_2$, $f_1\colon Z \to Z_1$, $f_2\colon Z \to Z_2$ be projections (see Definition 2.1). It is obvious that $f_1$ and $f_2$ are Vietoris maps and $p_1 \circ f_1 = p_2 \circ f_2$. Let $z_0 = (z_1^0, z_2^0)$. Then $z_0 \in Z$ and

$$q_1(f_1(z_0)) = q_1(f_1((z_1^0, z_2^0)) = q_1(z_1^0) = q_2(z_2^0) = q_2(f_2(z_1^0, z_2^0)) = q_2(f_2(z_0))$$

and the proof is complete.

Note that the Definition 5.11, Proposition 5.12 and the axiom of topological equality (see Definition 2.2) show that if the multi-morphisms $\varphi_m$ and $\psi_m$ have a point of coincidence, then for each diagram $(p_1,q_1) \in \varphi_m$ and $(p_2,q_2) \in \psi_m$

$$(p_1,q_1) \sim_{z_0} (p_2,q_2) \quad \text{(see Definition 5.10)}.$$

We say that a multi-morphism $\psi_m = [(p,q)]_m \in M_m(X,Y)$ is compact if $q$ is a compact mapping.

THEOREM 5.13
*Let $\Delta_m \in M_m(X,Y)$ be a multi-morphism such that $\Delta_m = [(p_1,p_2)]_m$, where*

$$X \xleftarrow{p_1} Z_1 \xrightarrow{p_2} Y$$

*and $p_1, p_2$ are Vietoris maps and let $\overleftarrow{\Delta}_m = [(p_2,p_1)]_m \in M_m(Y,X)$. Let $Y \in ANR$ and let $\psi_m \in M_m(X,Y)$ be a compact multi-morphism. Then*

$$(\psi_m \circ \overleftarrow{\Delta}_m)_* \colon H_*(Y) \to H_*(Y)$$

*(see Theorem 2.4) is a Leray endomorphism and if $\Lambda((\psi_m \circ \overleftarrow{\Delta}_m)_*) \neq 0$ then $\psi_m$ and $\Delta_m$ have a point of coincidence.*

*Proof.* From the axiom of composition (see Definition 2.2) and from the assumption $(\psi_m \circ \overleftarrow{\Delta}_m) \in M_m(Y,Y)$ is a compact multi-morphism and $Y \in ANR$, so $(\psi_m \circ \overleftarrow{\Delta}_m)_*$ is a Leray endomorphism. Assume that $\Lambda((\psi_m \circ \overleftarrow{\Delta}_m)_*) \neq 0$. Let $(p_1,p_2) \in \Delta_m$, $(p_3,q_3) \in \psi_m$, where

$$X \xleftarrow{p_1} Z_1 \xrightarrow{p_2} Y, \qquad X \xleftarrow{p_3} Z_2 \xrightarrow{q_3} Y.$$

From Proposition 4.1 we get a metrizable space $Z$, Vietoris maps $p_4 \colon Z \to Z_1$, $p_5 \colon Z \to Z_2$ such that $p = p_1 \circ p_4 = p_3 \circ p_5$ and

$$(p, p_2 \circ p_4) \in \Delta_m, \qquad (p, q_3 \circ p_5) \in \psi_m.$$

It is clear that

$$(p_2 \circ p_4, p) \in \overleftarrow{\Delta}_m.$$

We have

$$\begin{aligned}\Lambda((\psi_m \circ \overleftarrow{\Delta}_m)_*) &= \Lambda((\psi_m)_* \circ (\overleftarrow{\Delta}_m)_*) \\ &\overset{\text{(see Theorem 2.4)}}{=} \Lambda(((q_3 \circ p_5)_* \circ p_*^{-1}) \circ (p_* \circ (p_2 \circ p_4)_*^{-1}) \\ &= \Lambda(((q_3 \circ p_5)_* \circ (p_2 \circ p_4)_*^{-1}) \neq 0.\end{aligned}$$

Hence and from Theorem 2.7 the maps $q_3 \circ p_5$ and $p_2 \circ p_4$ have a point of coincidence and from Definition 5.11 and Proposition 5.12 there exists a point $x_0 \in X$ such that

$$\Delta(x_0) \cap \psi(x_0) \neq \emptyset$$

and the proof is complete.

This paper presents only a few applications but there can be many more. In the opinion the author, the theory of multi-morphisms is a tool to study the properties of sets, spaces and multi-valued mappings. It is suggested that the multi-valued mapping $\varphi \colon X \multimap Y$ determined by the multi-morphisms $\varphi_m \in M_m(X,Y)$ be denoted with

$$\varphi \colon X \to_m Y$$

and called a multi-functions. The notion of a multi-function is already present in the mathematical literature and usually denotes any multi-valued mapping. Because of the single-valued character of multi-morphisms, the use of the name of

a multi-function is completely justified. In the opinion of the author the multi-functions in regard to their numerous applications construct a different class of mappings than Kryszewski's morphisms. It shall be noticed that every strongly admissible mapping in the sense of Górniewicz is determined by some multi-morphism and so it is a multi-function. In the class of multi-functions the notion of homotopy that is an equivalence relation is introduced and its definition bases on the homotopy of single-valued mappings. Multi-functions resulted in some kind of multi-contractibility to a point that is essentially more general than the regular contractibility (it can also pertain to spaces that are not movable), but it remains in the class of path connected spaces. It is also worth noting that every single-valued multi-function is a function, as well as the other way around, every function is a multi-function.

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*Received: July 30, 2014; final version: October 25, 2014;*
*available online: January 2, 2015.*