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A note on preserving the spark of a matrix

Abstract. Let $\mathcal{M}_{m \times n}(\mathbb{F})$ be the vector space of all $m \times n$ matrices over a field $\mathbb{F}$. In the case where $m \geq n$, $\text{char}(\mathbb{F}) \neq 2$ and $\mathbb{F}$ has at least five elements, we give a complete characterization of linear maps $\Phi: \mathcal{M}_{m \times n}(\mathbb{F}) \rightarrow \mathcal{M}_{m \times n}(\mathbb{F})$ such that $\text{spark}(\Phi(A)) = \text{spark}(A)$ for any $A \in \mathcal{M}_{m \times n}(\mathbb{F})$.

1. Preliminaries and introduction

Throughout the text, m and n stand for positive integers, and $\mathbb{F}$ denotes a field. We define $\mathcal{M}_{m \times n}(\mathbb{F})$ to be the vector space of all $m \times n$ matrices over $\mathbb{F}$. The $m \times n$ zero matrix will be denoted by $O_{m \times n}$ and the n th full linear group over $\mathbb{F}$ by $\mathcal{GL}_n(\mathbb{F})$ (i.e., $\mathcal{GL}_n(\mathbb{F}) = \{V \in \mathcal{M}_{n \times n}(\mathbb{F}) : \det(V) \neq 0\}$). Finally, if $x_1, \dots, x_n$ are the components of a (row or column) vector $x \in \mathbb{F}^n$, then the *Hamming weight* of x is defined by

$$\|x\|_0 = \#\{j \in \{1, \dots, n\} : x_j \neq 0\}.$$

In [3], Donoho and Elad introduced the concept of spark of a matrix into the mathematical theory of compressed sensing. Let us recall the definition.

DEFINITION 1.1

Suppose that $C_1, \dots, C_n \in \mathcal{M}_{m \times 1}(\mathbb{F})$ are the columns of a matrix $A \in \mathcal{M}_{m \times n}(\mathbb{F})$. The spark of A is defined to be the infimum of the set of all positive integers ℓ with the property that

$$\exists j_1, \dots, j_\ell \in \{1, \dots, n\} : \begin{cases} j_1 < \dots < j_\ell, \\ C_{j_1}, \dots, C_{j_\ell} \text{ are linearly dependent.} \end{cases}$$

The following facts about the spark are well known and easy to prove.

PROPOSITION 1.2

Let $A \in \mathcal{M}_{m \times n}(\mathbb{F})$ and $U \in \mathcal{GL}_m(\mathbb{F})$. Then

- (i) $\text{spark}(A) \in \{1, \dots, n\} \cup \{+\infty\}$,
- (ii) $\text{spark}(A) = +\infty$ if and only if $\text{rank}(A) = n$,
- (iii) $\text{spark}(A) = 1$ if and only if A has a zero column,
- (iv) $\text{spark}(A) \leq \text{rank}(A) + 1$ whenever $\text{spark}(A) \neq +\infty$,
- (v) $\text{spark}(UA) = \text{spark}(A)$.

When dealing with a reasonable map f defined on $\mathcal{M}_{m \times n}(\mathbb{F})$, it is always of interest to know what linear endomorphisms $\Phi: \mathcal{M}_{m \times n}(\mathbb{F}) \rightarrow \mathcal{M}_{m \times n}(\mathbb{F})$ have the property that $f(\Phi(A)) = f(A)$ for any $A \in \mathcal{M}_{m \times n}(\mathbb{F})$. Such endomorphisms are called *linear preservers* of the map f . The theory of linear preserver problems dates back to 1890s (Frobenius' theorem on linear preservers of the determinant function) and still attracts the attention of many mathematicians. We refer to [2] for a nice overview of results.

This note provides some remarks on linear preservers of the function

$$\mathcal{M}_{m \times n}(\mathbb{F}) \ni A \mapsto \text{spark}(A) \in \{1, \dots, n\} \cup \{+\infty\}.$$

We will need the following technical definition.

DEFINITION 1.3

Let $U \in \mathcal{GL}_m(\mathbb{F})$ and $V \in \mathcal{GL}_n(\mathbb{F})$. A map $\Phi: \mathcal{M}_{m \times n}(\mathbb{F}) \rightarrow \mathcal{M}_{m \times n}(\mathbb{F})$ is said to be (U, V) -standard, if either $\Phi(A) = UAV$ for all $A \in \mathcal{M}_{m \times n}(\mathbb{F})$, or $m = n$ and $\Phi(A) = UA^T V$ for all $A \in \mathcal{M}_{m \times n}(\mathbb{F})$.

Notice that the (U, V) -standard map is a linear automorphism of $\mathcal{M}_{m \times n}(\mathbb{F})$.

The note is based on the characterization of rank k preservers given by Beasley and Laffey (see [1]), which we recall below.

THEOREM 1.4

Let k be a positive integer such that $k \leq \min\{m, n\}$. Suppose that the field $\mathbb{F}$ has at least four elements. If a linear automorphism $\Phi: \mathcal{M}_{m \times n}(\mathbb{F}) \rightarrow \mathcal{M}_{m \times n}(\mathbb{F})$ satisfies the condition

$$\forall A \in \mathcal{M}_{m \times n}(\mathbb{F}) : \text{rank}(A) = k \implies \text{rank}(\Phi(A)) = k,$$

then it is a (U, V) -standard map, for some $U \in \mathcal{GL}_m(\mathbb{F})$ and some $V \in \mathcal{GL}_n(\mathbb{F})$.

2. Results

Our main purpose is to prove

THEOREM 2.1

If $\mathbb{F}$ has at least five elements, $\text{char}(\mathbb{F}) \neq 2$, and $m \geq n$, then for a linear endomorphism $\Phi: \mathcal{M}_{m \times n}(\mathbb{F}) \rightarrow \mathcal{M}_{m \times n}(\mathbb{F})$, the following conditions are equivalent:

- (1) $\forall A \in \mathcal{M}_{m \times n}(\mathbb{F}) : \text{spark}(\Phi(A)) = \text{spark}(A)$,
- (2) there exist a matrix $U \in \mathcal{GL}_m(\mathbb{F})$, a diagonal matrix $D \in \mathcal{GL}_n(\mathbb{F})$, and an $n \times n$ permutation matrix P such that $\forall A \in \mathcal{M}_{m \times n}(\mathbb{F}) : \Phi(A) = UADP$.

The proof will use two simple propositions and a lemma. The propositions are of independent interest.

PROPOSITION 2.2

Let $\Phi: \mathcal{M}_{m \times n}(\mathbb{F}) \rightarrow \mathcal{M}_{m \times n}(\mathbb{F})$ be a linear map. Suppose that $\text{spark}(\Phi(A)) = \text{spark}(A)$ for any $A \in \mathcal{M}_{m \times n}(\mathbb{F})$. Then Φ is bijective.

Proof. Pick a matrix $B \in \mathcal{M}_{m \times n}(\mathbb{F})$ such that $\Phi(B) = O_{m \times n}$. It is enough to show that $B = O_{m \times n}$. Since $\text{spark}(B) = \text{spark}(\Phi(B)) = 1$, the matrix B has a zero column. Assume that B has a nonzero column as well (and hence $n \geq 2$). Let $S \in \mathcal{M}_{m \times n}(\mathbb{F})$ be the matrix consisting of n copies of this nonzero column. Then $\text{spark}(S) = 2$ and

$$\text{spark}(\Phi(S)) = \text{spark}(\Phi(S - B)) = \text{spark}(S - B) = 1,$$

a contradiction. Consequently, $B = O_{m \times n}$.

PROPOSITION 2.3

Suppose that $\text{char}(\mathbb{F}) \neq 2$. Then, for a matrix $V \in \mathcal{GL}_n(\mathbb{F})$, the following conditions are equivalent:

- (1) $\forall A \in \mathcal{M}_{m \times n}(\mathbb{F}) : \text{spark}(AV) = \text{spark}(A)$,
- (2) there exist a diagonal matrix $D \in \mathcal{GL}_n(\mathbb{F})$ and an $n \times n$ permutation matrix P such that $V = DP$.

Proof. Let $W = [w_{ij}] \in \mathcal{M}_{n \times n}(\mathbb{F})$ be such that

$$\exists \ell, p, q \in \{1, \dots, n\} : \begin{cases} p \neq q, \\ w_{p\ell} \neq 0, w_{q\ell} \neq 0. \end{cases}$$

We will show that there is a matrix $B = [b_{kj}] \in \mathcal{M}_{m \times n}(\mathbb{F})$ with $\text{spark}(B) = 2$ and $\text{spark}(BW) = 1$.

Let $\lambda = w_{1\ell} + \dots + w_{n\ell}$. Assume that $w_{r\ell} - \lambda \neq 0$ for some $r \in \{p, q\}$. Then it suffices to define

$$b_{kj} = \begin{cases} 1, & \text{if } k = 1 \text{ and } j \neq r, \\ 1 - \lambda w_{r\ell}^{-1}, & \text{if } k = 1 \text{ and } j = r, \\ 0, & \text{otherwise.} \end{cases}$$

Assume, therefore, that $w_{p\ell} - \lambda = 0 = w_{q\ell} - \lambda$. Then $\lambda + (\lambda - w_{p\ell} - w_{q\ell}) = 0$, and hence it suffices to define

$$b_{kj} = \begin{cases} 2, & \text{if } k = 1 \text{ and } j \notin \{p, q\}, \\ 1, & \text{if } k = 1 \text{ and } j \in \{p, q\}, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, if a matrix $V \in \mathcal{GL}_n(\mathbb{F})$ satisfies condition (1), then each column of V (and hence each row) has exactly one element different from 0. Condition (2) follows. Implication (2) $\Rightarrow$ (1) is obvious and holds true over an arbitrary field.

We denote by Σ_n the set of all permutations of $\{1, \dots, n\}$.

COROLLARY 2.4

Suppose that $\text{char}(\mathbb{F}) \neq 2$. Then, for a linear endomorphism $f: \mathcal{M}_{1 \times n}(\mathbb{F}) \rightarrow \mathcal{M}_{1 \times n}(\mathbb{F})$, the following conditions are equivalent:

- (1) $\forall x \in \mathcal{M}_{1 \times n}(\mathbb{F}) : \|f(x)\|_0 = \|x\|_0$,
- (2) $\exists \sigma \in \Sigma_n \exists a_1, \dots, a_n \in \mathbb{F} \setminus \{0\} \forall x = (x_1, \dots, x_n) \in \mathcal{M}_{1 \times n}(\mathbb{F}) : f(x) = (a_1 x_{\sigma(1)}, \dots, a_n x_{\sigma(n)})$,
- (3) $\forall x \in \mathcal{M}_{1 \times n}(\mathbb{F}) : \text{spark}(f(x)) = \text{spark}(x)$.

Proof. If $n = 1$, then there is nothing to do. Assume that $n \geq 2$. Then $\text{spark}(x) \in \{1, 2\}$ for all $x \in \mathcal{M}_{1 \times n}(\mathbb{F})$. Moreover, $\text{spark}(x) = 2$ for some $x \in \mathcal{M}_{1 \times n}(\mathbb{F})$ if and only if $\|x\|_0 = n$. These two properties yield implication (1) $\Rightarrow$ (3). Let us proceed to (3) $\Rightarrow$ (2). If condition (3) is satisfied, then by Proposition 2.2, the endomorphism f is bijective, and hence

$$\exists V \in \mathcal{GL}_n(\mathbb{F}) \forall x \in \mathcal{M}_{1 \times n}(\mathbb{F}) : f(x) = xV.$$

Condition (2) now follows from Proposition 2.3. Implication (2) $\Rightarrow$ (1) is obvious.

The above equivalence (1) $\Leftrightarrow$ (2) is well known and can be easily proved over an arbitrary field, without involving the concept of spark. Implication (1) $\Rightarrow$ (3) also holds true over an arbitrary field.

EXAMPLE 2.5

The linear endomorphism $g: \mathcal{M}_{1 \times 3}(\mathbb{Z}_2) \ni (x_1, x_2, x_3) \mapsto (x_1, x_1 + x_2 + x_3, x_3) \in \mathcal{M}_{1 \times 3}(\mathbb{Z}_2)$ satisfies the condition

$$\forall x \in \mathcal{M}_{1 \times 3}(\mathbb{Z}_2) : \text{spark}(g(x)) = \text{spark}(x).$$

However, g is not a “Hamming isometry”.

Notice that a linear endomorphism $h: \mathcal{M}_{n \times 1}(\mathbb{F}) \rightarrow \mathcal{M}_{n \times 1}(\mathbb{F})$ is a preserver of the spark if and only if h is bijective.

Let us return to the main purpose of the note.

LEMMA 2.6

If $n \geq 2$, then no matrix $V \in \mathcal{GL}_n(\mathbb{F})$ has the property that $\text{spark}(A^T V) = \text{spark}(A)$ for all $A \in \mathcal{M}_{n \times n}(\mathbb{F})$.

Proof. Assume that $n \geq 2$ and pick a matrix $V \in \mathcal{GL}_n(\mathbb{F})$. Let $C \in \mathcal{M}_{n \times 1}(\mathbb{F})$ be a nonzero column such that every element of $C^T V$ is different from 0. Define $S \in \mathcal{M}_{n \times n}(\mathbb{F})$ to be the matrix whose first column coincides with C and any other column coincides with $O_{n \times 1}$. Then $\text{spark}(S) = 1$ and $\text{spark}(S^T V) = 2$.

Proof of Theorem 2.1. Implication (2) $\Rightarrow$ (1) is obvious (cf. Proposition 1.2; the implication holds true over an arbitrary field and even if $m < n$). Assume that Φ satisfies condition (1). Then it follows from Proposition 2.2 that Φ is bijective. Moreover, if $\text{rank}(A) = n$ for a matrix $A \in \mathcal{M}_{m \times n}(\mathbb{F})$, then $\text{spark}(\Phi(A)) = \text{spark}(A) = +\infty$, and hence $\text{rank}(\Phi(A)) = n$. Theorem 1.4 yields therefore that Φ is a (U, V) -standard map for some $U \in \mathcal{GL}_m(\mathbb{F})$ and some $V \in \mathcal{GL}_n(\mathbb{F})$. Suppose, for a moment, that $m = n \geq 2$ and

$$\forall A \in \mathcal{M}_{m \times n}(\mathbb{F}) : \Phi(A) = U A^T V.$$

Then $\text{spark}(A^T V) = \text{spark}(\Phi(A)) = \text{spark}(A)$ for any $A \in \mathcal{M}_{m \times n}(\mathbb{F})$, which contradicts Lemma 2.6. Consequently, $\Phi(A) = U A V$ for all $A \in \mathcal{M}_{m \times n}(\mathbb{F})$. This implies that for an arbitrary $A \in \mathcal{M}_{m \times n}(\mathbb{F})$, we have $\text{spark}(A V) = \text{spark}(U A V) = \text{spark}(A)$. Thus, by Proposition 2.3, there exist a diagonal matrix $D \in \mathcal{GL}_n(\mathbb{F})$ and an $n \times n$ permutation matrix P such that $V = D P$. The proof is complete.

References

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