

Annales Universitatis Paedagogicae Cracoviensis Studia Mathematica XIV (2015)

Renata Malejki

Stability of a generalization of the Fréchet functional equation

Abstract. We prove some stability and hyperstability results for a generalization of the well known Fréchet functional equation, stemming from one of the characterizations of the inner product spaces. As the main tool we use a fixed point theorem for some function spaces. We end the paper with some new inequalities characterizing the inner product spaces.

1. Introduction

The following theorem has been proved in [2] ($\mathbb{N}$ and $\mathbb{Z}$ stand, as usual, for the sets of all positive integers and integers, respectively; moreover, $\mathbb{Z}_0 := \mathbb{Z} \setminus \{0\}$).

THEOREM 1

Let $(X, +)$ be a commutative group, $X_0 := X \setminus \{0\}$, Y be a Banach space, and $f: X \rightarrow Y$, $c: \mathbb{Z}_0 \rightarrow [0, \infty)$ and $L: X_0^3 \rightarrow [0, \infty)$ satisfy the following three conditions:

$$\begin{aligned} \mathcal{M} &:= \{m \in \mathbb{Z}_0 : c(-2m) + 2c(m+1) + 2c(-m) + c(2m+1) < 1\} \neq \emptyset, \\ L(kx, ky, kz) &\leq c(k)L(x, y, z), \quad x, y, z \in X_0, m \in \mathcal{M}, \\ k &\in \{-2m, m+1, -m, 2m+1\}, \end{aligned} \quad (1)$$

$\|f(x+y+z) + f(x) + f(y) + f(z) - f(x+y) - f(x+z) - f(y+z)\| \leq L(x, y, z)$,
for all $x, y, z \in X_0$. Then there is a unique function $F: X \rightarrow Y$ satisfying

$$F(x+y+z) + F(x) + F(y) + F(z) = F(x+y) + F(x+z) + F(y+z) \quad (2)$$

for all $x, y, z \in X$ and such that

$$\|f(x) - F(x)\| \leq \rho_L(x), \quad x \in X_0,$$

where

$$\rho_L(x) := \inf_{m \in \mathcal{M}} \frac{L((2m+1)x, -mx, -mx)}{1 - c(-2m) - 2c(m+1) - 2c(-m) - c(2m+1)}, \quad x \in X_0.$$

Equation (2) is sometimes called the Fréchet functional equation. The reason for this is that M. Fréchet [12] used it to characterize the inner product spaces in a similar way as Jordan and von Neumann [15] did using the parallelogram law. Namely, he proved that a normed space $(X, \|\cdot\|)$ is an inner product space if and only if, for all $x, y, z \in X$,

$$\|x + y + z\|^2 + \|x\|^2 + \|y\|^2 + \|z\|^2 = \|x + y\|^2 + \|x + z\|^2 + \|y + z\|^2. \quad (3)$$

For more information we refer to [1, 10, 21, 22, 23].

Theorem 1 yields the subsequent characterization of inner product spaces (see [2, Corollary 5(i)]).

COROLLARY 2

Let X be a normed space and $X_0 := X \setminus \{0\}$. Write

$$D(x, y, z) := \|\|x + y + z\|^2 + \|x\|^2 + \|y\|^2 + \|z\|^2 - \|x + y\|^2 - \|x + z\|^2 - \|y + z\|^2\|$$

for $x, y, z \in X$. Assume that there exist $w_0, \alpha_i, s_i \in \mathbb{R}$ such that $\alpha_i > 0$ and $w_0 s_i < 0$ for $i = 1, 2, 3$ and

$$\sup_{x, y, z \in X_0} \frac{D(x, y, z)}{(\alpha_1 \|x\|^{s_1} + \alpha_2 \|y\|^{s_2} + \alpha_3 \|z\|^{s_3})^{w_0}} < \infty.$$

Then X is an inner product space.

Actually it is assumed in [2, Corollary 5(i)] that $w_0 s_i > 0$ for $i = 1, 2, 3$, but it is a mistake; the inequality should be as in Corollary 2.

Equation (2) can also be written in the form

$$\Delta_{x, y, z} f(0) = 0 \quad \text{and} \quad f(0) = 0, \quad (4)$$

where Δ denotes the Fréchet difference operator defined by

$$\begin{aligned} \Delta_y f(x) &= \Delta_y^1 f(x) := f(x + y) - f(x), & x, y \in S, \\ \Delta_{t, z} &:= \Delta_t \circ \Delta_z, & \Delta_t^2 := \Delta_{t, t}, & t, z \in S, \\ \Delta_{t, u, z} &:= \Delta_t \circ \Delta_u \circ \Delta_z, & \Delta_t^3 := \Delta_{t, t, t}, & t, u, z \in S \end{aligned}$$

for functions mapping a commutative group $(S, +)$ into a group (see [2]). Moreover, (2) can be written as

$$C^2 f(x, y, z) = 0,$$

where

$$C^2 f(x, y, z) = Cf(x, y + z) - Cf(x, y) - Cf(x, z)$$

and

$$Cf(x, y) = f(x + y) - f(x) - f(y),$$

i.e. $C^2 f$ is the Cauchy difference of f of the second order.

It is known (see [2, 18]) that every solution f of (2), mapping a commutative group $(G, +)$ into a real linear space X , has the form $f = a + q$ with an additive function $a: G \rightarrow X$ and a quadratic function $q: G \rightarrow X$.

In this paper we show that results analogous to Theorem 1 can be proved for the following more general functional equation

$$\begin{aligned} A_1 f(x + y + z) + A_2 f(x) + A_3 f(y) + A_4 f(z) \\ = A_5 f(x + y) + A_6 f(x + z) + A_7 f(y + z), \end{aligned} \quad (5)$$

in the class of functions mapping a commutative group X into a Banach space Y over a field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, where $A_1, \dots, A_7 \in \mathbb{K}$ are fixed ($\mathbb{R}$ and $\mathbb{C}$ denote the sets of real and complex numbers, respectively). It is easily seen that (5) becomes (2) with $A_1 = \dots = A_7 = 1$.

The results we prove correspond also to some outcomes in [8, 11, 16, 19, 25].

The results in [2] as well as our main theorem have been motivated by the notion of hyperstability of functional equations (see, e.g., [3, 4, 5, 13, 20]), introduced in connection with the issue of stability of functional equations (for more details see, e.g., [14, 17]).

2. Auxiliary fixed point result

We need the subsequent fixed point theorem proved for function spaces in [6]; it will be the main tool in the proof of our main theorem ($\mathbb{R}_+$ stands for the set of nonnegative reals and A^B denotes the family of all functions mapping a set $B \neq \emptyset$ into a set $A \neq \emptyset$). For related outcomes we refer to [7, 9]; a similar approach to stability of functional equations has been already applied in [5, 24].

THEOREM 3

Let the following three hypotheses be valid.

(H1) S is a nonempty set, E is a Banach space, and functions $f_1, \dots, f_k: S \rightarrow S$ and $L_1, \dots, L_k: S \rightarrow \mathbb{R}_+$ are given.

(H2) $\mathcal{T}: E^S \rightarrow E^S$ is an operator satisfying the inequality

$$\|\mathcal{T}\xi(x) - \mathcal{T}\mu(x)\| \leq \sum_{i=1}^k L_i(x) \|\xi(f_i(x)) - \mu(f_i(x))\|, \quad \xi, \mu \in E^S, x \in S.$$

(H3) $\Lambda: \mathbb{R}_+^S \rightarrow \mathbb{R}_+^S$ is defined by

$$\Lambda\delta(x) := \sum_{i=1}^k L_i(x) \delta(f_i(x)), \quad \delta \in \mathbb{R}_+^S, x \in S.$$

Assume that functions $\varepsilon: S \rightarrow \mathbb{R}_+$ and $\varphi: S \rightarrow E$ fulfil the following two conditions

$$\|\mathcal{T}\varphi(x) - \varphi(x)\| \leq \varepsilon(x), \quad x \in S,$$

$$\varepsilon^*(x) := \sum_{n=0}^{\infty} \Lambda^n \varepsilon(x) < \infty, \quad x \in S.$$

Then there exists a unique fixed point ψ of $\mathcal{T}$ with

$$\|\varphi(x) - \psi(x)\| \leq \varepsilon^*(x), \quad x \in S.$$

Moreover,

$$\psi(x) := \lim_{n \rightarrow \infty} \mathcal{T}^n \varphi(x), \quad x \in S.$$

3. The main result

The next theorem is the main result of the paper.

THEOREM 4

Let $(X, +)$ be a commutative group, $\widehat{X} := X^3 \setminus \{(0, 0, 0)\}$, Y be a Banach space, and $A_1, \dots, A_7 \in \mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ such that $A_1 \neq 0$ and

$$A_2 + A_3 + A_4 = A_5 + A_6 + A_7. \quad (6)$$

Assume that $f: X \rightarrow Y$, $c: \mathbb{Z}_0 \rightarrow [0, \infty)$ and $L: \widehat{X} \rightarrow [0, \infty)$ satisfy (1) and the following two conditions:

$$\begin{aligned} \mathcal{M} := \{m \in \mathbb{Z}_0 : & |A_7|c(-2m) + |A_5 + A_6|c(m+1) \\ & + |A_3 + A_4|c(-m) + |A_2|c(2m+1) < |A_1|\} \neq \emptyset, \end{aligned} \quad (7)$$

$$\begin{aligned} & \|A_1 f(x+y+z) + A_2 f(x) + A_3 f(y) + A_4 f(z) - A_5 f(x+y) \\ & - A_6 f(x+z) - A_7 f(y+z)\| \leq L(x, y, x), \quad (x, y, z) \in \widehat{X}. \end{aligned} \quad (8)$$

Then there exists a unique function $F: X \rightarrow Y$ satisfying (5) for all $x, y, z \in X$ and such that

$$\|f(x) - F(x)\| \leq \rho_L(x), \quad x \in X_0 := X \setminus \{0\}, \quad (9)$$

where

$$\rho_L(x) := \inf_{m \in \mathcal{M}} \frac{L((2m+1)x, -mx, -mx)}{|A_1| - \beta_m}, \quad (10)$$

$$\beta_m := |A_7|c(-2m) + |A_5 + A_6|c(m+1) + |A_3 + A_4|c(-m) + |A_2|c(2m+1).$$

Proof. First we consider the case $A_1 = 1$. Replacing x by $(2m+1)x$ and taking $y = z = -mx$ in (8) we obtain

$$\begin{aligned} & \|f(x) + A_2 f((2m+1)x) + (A_3 + A_4)f(-mx) \\ & \quad - (A_5 + A_6)f((m+1)x) - A_7 f(-2mx)\| \\ & \leq L((2m+1)x, -mx, -mx) =: \varepsilon_m(x), \quad x \in X_0, m \in \mathbb{Z}_0. \end{aligned} \quad (11)$$

Next put

$$\begin{aligned} \mathcal{T}_m \xi(x) &:= A_7 \xi(-2mx) + (A_5 + A_6) \xi((m+1)x) \\ & \quad - (A_3 + A_4) \xi(-mx) - A_2 \xi((2m+1)x), \quad \xi \in Y^X, x \in X, m \in \mathbb{Z}_0. \end{aligned}$$

It is easy to notice that, by (6),

$$\mathcal{T}_m^n f(0) = 0, \quad n \in \mathbb{N}, m \in \mathbb{Z}_0, \quad (12)$$

and inequality (11) can be written as

$$\|\mathcal{T}_m f(x) - f(x)\| \leq \varepsilon_m(x), \quad x \in X_0, m \in \mathbb{Z}_0.$$

Define an operator $\Lambda_m: \mathbb{R}_+^{X_0} \rightarrow \mathbb{R}_+^{X_0}$ for $m \in \mathbb{Z}_0$ by

$$\begin{aligned} \Lambda_m \eta(x) &:= |A_7| \eta(-2mx) + |A_5 + A_6| \eta((m+1)x) \\ & \quad + |A_3 + A_4| \eta(-mx) + |A_2| \eta((2m+1)x) \end{aligned}$$

for $\eta \in \mathbb{R}_+^{X_0}$ and $x \in X_0$. Notice that, for each $m \in \mathbb{Z}_0$, the operator $\Lambda := \Lambda_m$ has the form described in (H3) with $k = 4$, $S = X_0$, $E = Y$ and

$$\begin{aligned} f_1(x) &= -2mx, \quad f_2(x) = (m+1)x, \quad f_3(x) = -mx, \quad f_4(x) = (2m+1)x, \\ L_1(x) &= |A_7|, \quad L_2(x) = |A_5 + A_6|, \quad L_3(x) = |A_3 + A_4|, \quad L_4(x) = |A_2|, \quad x \in X_0. \end{aligned}$$

Moreover, for every $\xi, \mu \in Y^{X_0}$, $x \in X_0$, $m \in \mathbb{Z}_0$,

$$\begin{aligned} & \|\mathcal{T}_m \xi(x) - \mathcal{T}_m \mu(x)\| \\ &= \|A_7 \xi(-2mx) + (A_5 + A_6) \xi((m+1)x) - (A_3 + A_4) \xi(-mx) \\ & \quad - A_2 \xi((2m+1)x) - A_7 \mu(-2mx) - (A_5 + A_6) \mu((m+1)x) \\ & \quad + (A_3 + A_4) \mu(-mx) + A_2 \mu((2m+1)x)\| \\ &\leq |A_7| \|(\xi - \mu)(-2mx)\| + |A_5 + A_6| \|(\xi - \mu)((m+1)x)\| \\ & \quad + |A_3 + A_4| \|(\xi - \mu)(-mx)\| + |A_2| \|(\xi - \mu)((2m+1)x)\| \\ &= \sum_{i=1}^4 L_i(x) \|(\xi - \mu)(f_i(x))\|, \end{aligned}$$

where

$$(\xi - \mu)(y) := \xi(y) - \mu(y), \quad y \in X_0.$$

Note that, in view of (1), we have

$$\Lambda_m \varepsilon_k(x) \leq \beta_m \varepsilon_k(x), \quad k, m \in \mathbb{Z}_0, x \in X_0. \quad (13)$$

By induction it is easy to show that the linearity of Λ_m implies

$$\Lambda_m^n \varepsilon_k(x) \leq (\beta_m)^n \varepsilon_k(x) \quad (14)$$

for $x \in X_0$, $k, n \in \mathbb{N}$. So, we receive the following estimation

$$\varepsilon_m^*(x) := \sum_{n=0}^{\infty} \Lambda_m^n \varepsilon_m(x) \leq \sum_{n=0}^{\infty} (\beta_m)^n \varepsilon_m(x) = \frac{\varepsilon_m(x)}{1 - \beta_m}, \quad m \in \mathcal{M}, x \in X_0.$$

By Theorem 3 (with $S = X_0$ and $E = Y$), for each $m \in \mathcal{M}$ there exists a function $F'_m: X_0 \rightarrow Y$ such that

$$\begin{aligned} F'_m(x) &= A_7 F'_m(-2mx) + (A_5 + A_6) F'_m((m+1)x) \\ &\quad - (A_3 + A_4) F'_m(-mx) - A_2 F'_m((2m+1)x), \quad x \in X_0 \end{aligned}$$

and

$$\|f(x) - F'_m(x)\| \leq \frac{\varepsilon_m(x)}{1 - \beta_m}, \quad x \in X_0.$$

Moreover,

$$F'_m(x) = \lim_{n \rightarrow \infty} \mathcal{T}_m^n f(x), \quad x \in X_0, m \in \mathcal{M}.$$

Now, define $F_m: X \rightarrow Y$ by

$$F_m(0) = 0, \quad F_m(x) := F'_m(x), \quad x \in X_0, m \in \mathcal{M}.$$

Then it is easily seen that, by (12),

$$F_m(x) = \lim_{n \rightarrow \infty} \mathcal{T}_m^n f(x), \quad x \in X, m \in \mathcal{M}.$$

Next, by induction we show that

$$\begin{aligned} &\|\mathcal{T}_m^n f(x+y+z) + A_2 \mathcal{T}_m^n f(x) + A_3 \mathcal{T}_m^n f(y) + A_4 \mathcal{T}_m^n f(z) \\ &\quad - A_5 \mathcal{T}_m^n f(x+y) - A_6 \mathcal{T}_m^n f(x+z) - A_7 \mathcal{T}_m^n f(y+z)\| \\ &\leq (\beta_m)^n L(x, y, z) \end{aligned} \quad (15)$$

for every $(x, y, z) \in \widehat{X}$, $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $m \in \mathcal{M}$.

Fix $m \in \mathcal{M}$. For $n = 0$ condition (15) becomes (8). So, take $l \in \mathbb{N}_0$ and suppose that (15) holds for $n = l$ and $(x, y, z) \in \widehat{X}$. Then we have

$$\begin{aligned} &\|\mathcal{T}_m^{l+1} f(x+y+z) + A_2 \mathcal{T}_m^{l+1} f(x) + A_3 \mathcal{T}_m^{l+1} f(y) + A_4 \mathcal{T}_m^{l+1} f(z) \\ &\quad - A_5 \mathcal{T}_m^{l+1} f(x+y) - A_6 \mathcal{T}_m^{l+1} f(x+z) - A_7 \mathcal{T}_m^{l+1} f(y+z)\| \\ &= \|A_7 \mathcal{T}_m^l f(-2m(x+y+z)) + (A_5 + A_6) \mathcal{T}_m^l f((m+1)(x+y+z)) \\ &\quad - (A_3 + A_4) \mathcal{T}_m^l f(-m(x+y+z)) - A_2 \mathcal{T}_m^l f((2m+1)(x+y+z)) \\ &\quad + A_7 A_2 \mathcal{T}_m^l f(-2mx) + (A_5 + A_6) A_2 \mathcal{T}_m^l f((m+1)x) \\ &\quad - (A_3 + A_4) A_2 \mathcal{T}_m^l f(-mx) - A_2 A_2 \mathcal{T}_m^l f((2m+1)x)\| \end{aligned}$$

$$\begin{aligned}
& + A_7 A_3 \mathcal{T}_m^l f(-2my) + (A_5 + A_6) A_3 \mathcal{T}_m^l f((m+1)y) \\
& - (A_3 + A_4) A_3 \mathcal{T}_m^l f(-my) - A_2 A_3 \mathcal{T}_m^l f((2m+1)y) \\
& + A_7 A_4 \mathcal{T}_m^l f(-2mz) + (A_5 + A_6) A_4 \mathcal{T}_m^l f((m+1)z) \\
& - (A_3 + A_4) A_4 \mathcal{T}_m^l f(-mz) - A_2 A_4 \mathcal{T}_m^l f((2m+1)z) \\
& - A_7 A_5 \mathcal{T}_m^l f(-2m(x+y)) - (A_5 + A_6) A_5 \mathcal{T}_m^l f((m+1)(x+y)) \\
& + (A_3 + A_4) A_5 \mathcal{T}_m^l f(-m(x+y)) + A_2 A_5 \mathcal{T}_m^l f((2m+1)(x+y)) \\
& - A_7 A_6 \mathcal{T}_m^l f(-2m(x+z)) - (A_5 + A_6) A_6 \mathcal{T}_m^l f((m+1)(x+z)) \\
& + (A_3 + A_4) A_6 \mathcal{T}_m^l f(-m(x+z)) + A_2 A_6 \mathcal{T}_m^l f((2m+1)(x+z)) \\
& - A_7 A_7 \mathcal{T}_m^l f(-2m(y+z)) - (A_5 + A_6) A_7 \mathcal{T}_m^l f((m+1)(y+z)) \\
& + (A_3 + A_4) A_7 \mathcal{T}_m^l f(-m(y+z)) + A_2 A_7 \mathcal{T}_m^l f((2m+1)(y+z)) \| \\
& \leq (\beta_m)^l (|A_7| L(-2mx, -2my, -2mz) \\
& \quad + |A_5 + A_6| L((m+1)x, (m+1)y, (m+1)z) \\
& \quad + |A_3 + A_4| L(-mx, -my, -mz) + |A_2| L((2m+1)x, (2m+1)y, (2m+1)z)) \\
& \leq (\beta_m)^{l+1} L(x, y, z)
\end{aligned}$$

for every $(x, y, z) \in \widehat{X}$, which ends the proof of (15).

Letting $n \rightarrow \infty$ in (15), we obtain

$$\begin{aligned}
& F_m(x+y+z) + A_2 F_m(x) + A_3 F_m(y) + A_4 F_m(z) \\
& = A_5 F_m(x+y) + A_6 F_m(x+z) + A_7 F_m(y+z), \quad (x, y, z) \in \widehat{X}.
\end{aligned} \tag{16}$$

So, we have proved that, for each $m \in \mathcal{M}$ there exists a function $F_m: X \rightarrow Y$ satisfying the equation (5) for $(x, y, z) \in \widehat{X}$ and such that

$$\|f(x) - F_m(x)\| \leq \frac{\varepsilon_m(x)}{1 - \beta_m}, \quad x \in X_0. \tag{17}$$

Next, we show that $F_m = F_k$ for all $m, k \in \mathcal{M}$. So, fix $m, k \in \mathcal{M}$. Note that F_k satisfies (16) with m replaced by k . Hence, replacing x by $(2m+1)x$ and taking $y = z = -mx$ in (16), we obtain that $\mathcal{T}_m F_j = F_j$ for $j = m, k$ and

$$\|F_m(x) - F_k(x)\| \leq \frac{\varepsilon_m(x)}{1 - \beta_m} + \frac{\varepsilon_k(x)}{1 - \beta_k}, \quad x \in X_0,$$

whence, by the linearity of Λ and (14),

$$\begin{aligned}
\|F_m(x) - F_k(x)\| & = \|\mathcal{T}_m^n F_m(x) - \mathcal{T}_m^n F_k(x)\| \leq \frac{\Lambda_m^n \varepsilon_m(x)}{1 - \beta_m} + \frac{\Lambda_m^n \varepsilon_k(x)}{1 - \beta_k} \\
& \leq \frac{(\beta_m)^n \varepsilon_m(x)}{1 - \beta_m} + \frac{(\beta_k)^n \varepsilon_k(x)}{1 - \beta_k}
\end{aligned}$$

for every $x \in X_0$ and $n \in \mathbb{N}$. Therefore, letting $n \rightarrow \infty$ we get $F_m = F_k =: F$. Thus, in view of (17), we have proved that

$$\|f(x) - F(x)\| \leq \frac{\varepsilon_m(x)}{1 - \beta_m}, \quad x \in X, x \neq 0, m \in \mathcal{M},$$

whence we derive (9).

Since (in view of (16)) it is easy to notice that F is a solution to (5) (i.e. (5) holds for all $x, y, z \in X$), it remains to prove the statement concerning the uniqueness of F . So, let $G: X \rightarrow Y$ be also a solution of (5) and $\|f(x) - G(x)\| \leq \rho_L(x)$ for $x \in X$, $x \neq 0$. Then

$$\|G(x) - F(x)\| \leq 2\rho_L(x), \quad x \in X, x \neq 0. \quad (18)$$

Further, $\mathcal{T}_m G = G$ for each $m \in \mathbb{Z}_0$. Hence, with a fixed $m \in \mathcal{M}$, by (14) we get

$$\begin{aligned} \|G(x) - F(x)\| &= \|\mathcal{T}_m^n G(x) - \mathcal{T}_m^n F(x)\| \leq 2\Lambda_m^n \rho_L(x) \leq \frac{2\Lambda_m^n \varepsilon_m(x)}{1 - \beta_m} \\ &\leq \frac{2(\beta_m)^n \varepsilon_m(x)}{1 - \beta_m} \end{aligned}$$

for $x \in X_0$ and $n \in \mathbb{N}$. Consequently, letting $n \rightarrow \infty$ we obtain that $G = F$, and that ends the proof in the case $A_1 = 1$.

If $A_1 \neq 1$, then (8) can be rewritten in the form

$$\begin{aligned} &\|f(x + y + z) + A'_2 f(x) + A'_3 f(y) + A'_4 f(z) - A'_5 f(x + y) \\ &\quad - A'_6 f(x + z) - A'_7 f(y + z)\| \leq L'(x, y, z), \quad (x, y, z) \in \widehat{X}, \end{aligned} \quad (19)$$

where

$$A'_i := \frac{A_i}{A_1}, \quad i = 2, \dots, 7,$$

$$L'(x, y, z) := \frac{L(x, y, z)}{|A_1|}, \quad (x, y, z) \in \widehat{X},$$

and it is easily seen that the statement can be easily deduced from the case $A_1 = 1$.

The following hyperstability result can be deduced from Theorem 4. It corresponds to the recent hyperstability outcomes in [5, 24] and some classical stability results concerning the Cauchy equation (see, e.g., [3, p.3], [14, p.15,16] and [17, p.2]).

COROLLARY 5

Let $(X, +)$ be a commutative group, $\widehat{X} := X^3 \setminus \{(0, 0, 0)\}$, Y be a Banach space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $A_1, \dots, A_7 \in \mathbb{K}$, $A_1 \neq 0$ and (6) be valid. Assume that $f: X \rightarrow Y$, $c: \mathbb{Z}_0 \rightarrow [0, \infty)$ and $L: \widehat{X} \rightarrow [0, \infty)$ satisfy conditions (1), (7), (8) and

$$\sup_{m \in \mathcal{M}} \beta_m < |A_1|, \quad \inf_{m \in \mathcal{M}} L((2m+1)x, -mx, -mx) = 0, \quad x \in X, x \neq 0.$$

Then there exists a function $F: X \rightarrow Y$ satisfying (5) for all $x, y, z \in X_0$ such that $f(x) = F(x)$ for $x \in X_0$.

Proof. It is easily seen that $\rho_L(x) = 0$ for each $x \in X_0$, where ρ_L is defined by (10). Hence Theorem 4 implies the statement.

REMARK 6

If, in Theorem 4,

$$L(x, y, z) = (\alpha_1 \|x\|^p + \alpha_2 \|y\|^p + \alpha_3 \|z\|^p)^w, \quad (x, y, z) \in \widehat{X}, \quad (20)$$

with some $\alpha_i, p, w \in \mathbb{R}$ such that $\alpha_i > 0$ for $i = 1, 2, 3$, $p > 0$ and $w < 0$, then it is easily seen that the function c can, for instance, have the form $c(m) = |m|^{pw}$.

The next corollary generalizes Corollary 2 to some extent; it shows possible application of the main result of this paper.

COROLLARY 7

Let X be a normed space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $X_0 := X \setminus \{0\}$, $A_1, \dots, A_7 \in \mathbb{K}$, $A_1 \neq 0$, and (6) be valid. Write

$$\begin{aligned} \widehat{D}(x, y, z) := & |A_1 \|x + y + z\|^2 + A_2 \|x\|^2 + A_3 \|y\|^2 + A_4 \|z\|^2 \\ & - A_5 \|x + y\|^2 - A_6 \|x + z\|^2 - A_7 \|y + z\|^2| \end{aligned}$$

for $x, y, z \in X$. Assume that there exist $\alpha_i, w, p \in \mathbb{R}$ such that $p > 0$, $w < 0$, $\alpha_i > 0$ for $i = 1, 2, 3$ and

$$\sup_{(x, y, z) \in \widehat{X}} \frac{\widehat{D}(x, y, z)}{(\alpha_1 \|x\|^p + \alpha_2 \|y\|^p + \alpha_3 \|z\|^p)^w} < \infty.$$

Then X is an inner product space.

Proof. Write $f(x) = \|x\|^2$ for $x \in X$. Then, with L and c of the forms described in Remark 6, from Corollary 5 and (6) we easily derive that f is a solution to equation (5).

We show that $A_1 = \dots = A_7$. Replacing x by αx , y by βx and z by γx in (5), where $\alpha, \beta, \gamma \in \mathbb{K}$, we obtain

$$\begin{aligned} & (A_1(\alpha + \beta + \gamma)^2 + A_2\alpha^2 + A_3\beta^2 + A_4\gamma^2)\|x\|^2 \\ & = (A_5(\alpha + \beta)^2 + A_6(\alpha + \gamma)^2 + A_7(\beta + \gamma)^2)\|x\|^2, \quad x \in X, \end{aligned}$$

whence

$$\begin{aligned} & A_1(\alpha + \beta + \gamma)^2 + A_2\alpha^2 + A_3\beta^2 + A_4\gamma^2 \\ & = A_5(\alpha + \beta)^2 + A_6(\alpha + \gamma)^2 + A_7(\beta + \gamma)^2, \quad \alpha, \beta, \gamma \in \mathbb{K}. \end{aligned} \quad (21)$$

Taking $\alpha = 1$, $\beta = \gamma = 0$ in (21) we have $A_1 + A_2 = A_5 + A_6$, and next, with $\beta = -\alpha = 1$ and $\gamma = 0$ in (21) we obtain the equality $A_2 + A_3 = A_6 + A_7$ and consequently

$$A_1 - A_3 = A_5 - A_7. \quad (22)$$

Analogously, with $\beta = 1$, $\alpha = \gamma = 0$ and $\beta = -\gamma = 1$, $\alpha = 0$ we obtain

$$A_1 - A_4 = A_7 - A_6, \quad (23)$$

and $\gamma = 1$, $\alpha = \beta = 0$ and $\alpha = -\gamma = 1$, $\beta = 0$ gives

$$A_1 - A_2 = A_6 - A_5. \quad (24)$$

Further, inserting, $1 = \alpha = -\beta = -\gamma$, $1 = -\alpha = \beta = -\gamma$ and $1 = -\alpha = -\beta = \gamma$ into (21), we respectively get

$$A_1 + A_2 + A_3 + A_4 = 4A_7,$$

$$A_1 + A_2 + A_3 + A_4 = 4A_6,$$

$$A_1 + A_2 + A_3 + A_4 = 4A_5,$$

whence $A_5 = A_6 = A_7$ and consequently, by (22)–(24), $A_1 = A_2 = A_3 = A_4$. This and (6) finally yield $A_1 = \dots = A_7$.

Thus we have proved that (3) holds, which implies the statement.

References

- [1] C. Alsina, J. Sikorska, M.S. Tomás, *Norm derivatives and characterizations of inner product spaces*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2010. Cited on 70.
- [2] A. Bahyrycz, J. Brzdęk, M. Piszczek, J. Sikorska, *Hyperstability of the Fréchet equation and a characterization of inner product spaces*, J. Funct. Spaces Appl. 2013, Art. ID 496361, 6 pp. Cited on 69, 70 and 71.
- [3] N. Brillouët-Belluot, J. Brzdęk, K. Ciepliński, *On some recent developments in Ulam's type stability*, Abstr. Appl. Anal. 2012, Art. ID 716936, 41 pp. Cited on 71 and 76.
- [4] J. Brzdęk, *Remarks on hyperstability of the Cauchy functional equation*, Aequationes Math. **86** (2013), no. 3, 255–267. Cited on 71.
- [5] J. Brzdęk, *Hyperstability of the Cauchy equation on restricted domains*, Acta Math. Hungar. **141** (2013), no. 1-2, 58–67. Cited on 71 and 76.
- [6] J. Brzdęk, J. Chudziak, Z. Páles, *A fixed point approach to stability of functional equations*, Nonlinear Anal. **74** (2011), no. 17, 6728–6732. Cited on 71.
- [7] J. Brzdęk, K. Ciepliński, *A fixed point approach to the stability of functional equations in non-Archimedean metric spaces*, Nonlinear Anal. **74** (2011), no. 18, 6861–6867. Cited on 71.
- [8] M. Chudziak, *On solutions and stability of functional equations connected to the Popoviciu inequality*, Ph.D. Thesis (in Polish), Pedagogical University of Cracow (Poland), Cracow 2012. Cited on 71.
- [9] L. Cădariu, L. Găvruta, P. Găvruta, *Fixed points and generalized Hyers-Ulam stability*, Abstr. Appl. Anal. 2012, Art. ID 712743, 10 pp. Cited on 71.
- [10] S.S. Dragomir, *Some characterizations of inner product spaces and applications*, Studia Univ. Babeş-Bolyai Math. **34** (1989), no. 1, 50–55. Cited on 70.
- [11] W. Fechner, *On the Hyers-Ulam stability of functional equations connected with additive and quadratic mappings*, J. Math. Anal. Appl. **322** (2006), no. 2, 774–786. Cited on 71.

- [12] M. Fréchet, *Sur la définition axiomatique d'une classe d'espaces vectoriels distanciés applicables vectoriellement sur l'espace de Hilbert*, Ann. of Math. (2) **36** (1935), no. 3, 705–718. Cited on 70.
- [13] E. Gselmann, *Hyperstability of a functional equation*, Acta Math. Hungar. **124** (2009), no. 1-2, 179–188. Cited on 71.
- [14] D.H. Hyers, G. Isac, Th.M. Rassias, *Stability of functional equations in several variables. Progress in Nonlinear Differential Equations and their Applications*, 34. Birkhäuser Boston, Inc., Boston, MA, 1998. Cited on 71 and 76.
- [15] P. Jordan, J. Von Neumann, *On inner products in linear, metric spaces*, Ann. of Math. (2) **36** (1935), no. 3, 719–723. Cited on 70.
- [16] S.-M. Jung, *On the Hyers-Ulam stability of the functional equation that have the quadratic property*, J. Math. Anal. Appl. **222** (1998), no. 1, 126–137. Cited on 71.
- [17] S.-M. Jung, *Hyers-Ulam-Rassias stability of functional equations in nonlinear analysis*, Springer Optimization and Its Applications, 48, Springer, New York, 2011. Cited on 71 and 76.
- [18] P. Kannappan, *Functional equations and inequalities with applications*, Springer Monographs in Mathematics, Springer, New York, 2009. Cited on 71.
- [19] Y.-H. Lee, *On the Hyers-Ulam-Rassias stability of the generalized polynomial function of degree 2*, Journal of the Chungcheong Mathematical Society **22** (2009), no. 2, 201–209. Cited on 71.
- [20] G. Maksa, Z. Páles, *Hyperstability of a class of linear functional equations*, Acta Math. Acad. Paedagog. Nyházi. (N.S.) **17** (2001), no. 2, 107–112. Cited on 71.
- [21] M.S. Moslehian, J.M. Rassias, *A characterization of inner product spaces concerning an Euler-Lagrange identity*, Commun. Math. Anal. **8** (2010), no. 2, 16–21. Cited on 70.
- [22] K. Nikodem, Z. Pales, *Characterizations of inner product spaces by strongly convex functions*, Banach J. Math. Anal. **5** (2011), no. 1, 83–87. Cited on 70.
- [23] Th. M. Rassias, *New characterizations of inner product spaces*, Bull. Sci. Math. (2) **108** (1984), no. 1, 95–99. Cited on 70.
- [24] M. Piszczek, *Remark on hyperstability of the general linear equation*, Aequationes Math. **88** (2014), no. 1-2, 163–168. Cited on 71 and 76.
- [25] J. Sikorska, *On a direct method for proving the Hyers-Ulam stability of functional equations*, J. Math. Anal. Appl. **372** (2010), no. 1, 99–109. Cited on 71.

*Institute of Mathematics
Pedagogical University
Podchorążych 2
PL-30-084 Kraków
Poland
E-mail: rmalejki@o2.pl
rmalejki@up.krakow.pl*

*Received: April 10, 2015; final version: May 29, 2015;
available online: July 15, 2015.*