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The Littlewood-Paley g -function associated with the Riemann-Liouville operator

Abstract. First, we study the Gauss and Poisson semigroups connected with the Riemann-Liouville operator. Next, we define and study the Littlewood-Paley g -function associated with the Riemann-Liouville operator for which we prove the L^p -boundedness for $p \in]1, 2]$.

1. Introduction

The usual Littlewood-Paley g -function is defined in the Euclidian space [21] by

$$\forall x \in \mathbb{R}^n; \quad g(f)(x) = \left(\int_0^{+\infty} |\nabla P^t f(x)|^2 t \, dt \right)^{\frac{1}{2}},$$

where $(P^t)_{t>0}$ is the usual Poisson semigroup defined by

$$P^t f(x) = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \int_{\mathbb{R}^n} \frac{t f(y)}{(t^2 + |x - y|^2)^{\frac{n+1}{2}}} dy,$$

and ∇ is the gradient given by

$$\nabla = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial t} \right).$$

It is well known (see for example [21]) that the mapping

$$f \longmapsto g(f)$$

is bounded from the Lebesgue space $L^p(\mathbb{R}^n, dx)$, $p \in]1, +\infty[$ into itself. Moreover, the Littlewood-Paley theory plays an important role in the study of many function spaces as the Hardy space H^p . Many aspects of the Littlewood-Paley g -function connected with several hypergroups are studied [1, 2, 19, 23]. The authors have been especially interested by the boundedness of such operator when acting on the Lebesgue space L^p ; $p \in]1, +\infty[$.

In [3], the second author with the others define the so-called Riemann-Liouville operator $\mathcal{R}_\alpha$; $\alpha \geq 0$ by setting

$$\mathcal{R}_\alpha(f)(r, x) = \begin{cases} \frac{\alpha}{\pi} \int_{-1}^1 \int_{-1}^1 f(rs\sqrt{1-t^2}, x+rt)(1-t^2)^{\alpha-\frac{1}{2}} \\ \quad \times (1-s^2)^{\alpha-1} dt ds, & \text{if } \alpha > 0, \\ \frac{1}{\pi} \int_{-1}^1 f(r\sqrt{1-t^2}, x+rt) \frac{dt}{\sqrt{(1-t^2)}}, & \text{if } \alpha = 0, \end{cases}$$

where f is a continuous function on $\mathbb{R}^2$, even with respect to the first variable.

The Fourier transform associated with the operators $\mathcal{R}_\alpha$ is defined by;

$$\forall (\mu, \lambda) \in \Upsilon; \mathcal{F}_\alpha(f)(\mu, \lambda) = \int_0^{+\infty} \int_{\mathbb{R}} f(r, x) j_\alpha(r\sqrt{\mu^2 + \lambda^2}) e^{-i\lambda x} d\nu_\alpha(r, x),$$

where

- $\Upsilon = \mathbb{R}^2 \cup \{(i\mu, \lambda); (\mu, \lambda) \in \mathbb{R}^2; |\mu| \leq |\lambda|\}$
- $d\nu_\alpha$ is the measure defined on $[0, +\infty[\times \mathbb{R}$ by

$$d\nu_\alpha(r, x) = \frac{r^{2\alpha+1} dr}{2^\alpha \Gamma(\alpha+1)} \otimes \frac{dx}{(2\pi)^{\frac{1}{2}}}.$$

- j_α is a modified Bessel function that will be defined in the second section.

Many harmonic analysis results related to the Fourier transform $\mathcal{F}_\alpha$ have been established [3, 4, 5, 18]. Also, the uncertainty principles play an important role in harmonic analysis [6, 7, 8, 12, 13, 15], for this reason, many of such principles are established for the Fourier transform $\mathcal{F}_\alpha$ [16, 17].

The aim of this work is to define and study the g -function associated with the Riemann-Liouville operator $\mathcal{R}_\alpha$. For this, we need first to define the Gauss and Poisson semigroups that will be denoted respectively by $(\mathcal{G}^t)_{t>0}$ and $(\mathcal{P}^t)_{t>0}$. The Poisson semigroup $(\mathcal{P}^t)_{t>0}$ allows us to define the Littlewood-Paley g -function by

$$\forall (r, x) \in [0, +\infty[\times \mathbb{R}; g(f)(r, x) = \left(\int_0^{+\infty} |\nabla \mathcal{P}^t f(r, x)|^2 t dt \right)^{\frac{1}{2}},$$

where

$$\nabla = \left(\frac{\partial}{\partial r}, \frac{\partial}{\partial x}, \frac{\partial}{\partial t} \right).$$

Then, we have established the main result of this paper. Namely, for every $f \in L^p(d\nu_\alpha)$, $p \in]1, 2]$, the function $g(f)$ belongs to the space $L^p(d\nu_\alpha)$ and we have

$$\|g(f)\|_{p, \nu_\alpha} \leq 2^{\frac{2-p}{2}} \left(\frac{p}{p-1} \right)^{\frac{1}{p}} \|f\|_{p, \nu_\alpha},$$

where

$$\|f\|_{p,\nu_\alpha} = \left(\int_0^{+\infty} \int_{\mathbb{R}} |f(r,x)|^p d\nu_\alpha(r,x) \right)^{\frac{1}{p}}.$$

This paper is arranged as follows.

In the second section, we recall some harmonic analysis results related to the Fourier transform $\mathcal{F}_\alpha$. In the third section, we define and study the Gauss semigroup $(\mathcal{G}^t)_{t>0}$ and the Poisson semigroup $(\mathcal{P}^t)_{t>0}$ and we give their mutual connexion. The last section is devoted to establish the boundedness of the Littlewood-Paley g -function from $L^p(d\nu_\alpha)$; $p \in]1, 2]$, into it self.

We want to add that in a coming paper; we will establish a principle of the maximum for the operator

$$\Delta_\alpha = \frac{\partial^2}{\partial r^2} + \frac{2\alpha+1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial t^2}.$$

We use this principle of the maximum to prove that for every $p \in [4, +\infty[$; there is $A_p > 0$ such that for every $f \in L^p(d\nu_\alpha)$; we have

$$\|g(f)\|_{p,\nu_\alpha} \leq A_p \|f\|_{p,\nu_\alpha}.$$

Using Marcinkiewisz interpolation theorem's; we deduce that for every $p \in]1, +\infty[$; there is $C_p > 0$ satisfying

$$\forall f \in L^p(d\nu_\alpha); \quad \frac{1}{C_p} \|f\|_{p,\nu_\alpha} \leq \|g(f)\|_{p,\nu_\alpha} \leq C_p \|f\|_{p,\nu_\alpha}.$$

2. The Riemann-Liouville transform

In this section, we recall some harmonic analysis results related to the convolution product and the Fourier transform associated with Riemann-Liouville operator. For more details see [3, 4, 5, 18].

Let D and Ξ be the singular partial differential operators defined by

$$\begin{cases} D = \frac{\partial}{\partial x}; \\ \Xi = \frac{\partial^2}{\partial r^2} + \frac{2\alpha+1}{r} \frac{\partial}{\partial r} - \frac{\partial^2}{\partial x^2}; \end{cases} \quad (r,x) \in]0, +\infty[\times \mathbb{R}, \quad \alpha \geq 0.$$

For all $(\mu, \lambda) \in \mathbb{C}^2$; the system

$$\begin{cases} Du(r,x) = -i\lambda u(r,x), \\ \Xi u(r,x) = -\mu^2 u(r,x), \\ u(0,0) = 1, \\ \frac{\partial u}{\partial r}(0,x) = 0; \quad \forall x \in \mathbb{R}, \end{cases}$$

admits a unique solution $\varphi_{\mu,\lambda}$ given by

$$\forall (r,x) \in [0, +\infty[\times \mathbb{R}; \quad \varphi_{\mu,\lambda}(r,x) = j_\alpha(r\sqrt{\mu^2 + \lambda^2}) e^{-i\lambda x}, \quad (2.1)$$

where j_α is the modified Bessel function defined by

$$j_\alpha(z) = 2^\alpha \Gamma(\alpha + 1) \frac{J_\alpha(z)}{z^\alpha} = \Gamma(\alpha + 1) \sum_{k=0}^{+\infty} \frac{(-1)^k}{k! \Gamma(\alpha + k + 1)} \left(\frac{z}{2}\right)^{2k},$$

and J_α is the Bessel function of first kind and index α [10, 11, 14, 26]. The modified Bessel function j_α has the integral representation

$$j_\alpha(z) = \frac{\Gamma(\alpha + 1)}{\sqrt{\pi} \Gamma(\alpha + \frac{1}{2})} \int_{-1}^1 (1 - t^2)^{\alpha - \frac{1}{2}} \exp(-izt) dt.$$

Consequently, for every $k \in \mathbb{N}$ and $z \in \mathbb{C}$; we have

$$|j_\alpha^{(k)}(z)| \leq e^{|\operatorname{Im}(z)|}. \quad (2.2)$$

The eigenfunction $\varphi_{\mu,\lambda}$ satisfies the following properties

- The function $\varphi_{\mu,\lambda}$ is bounded on $\mathbb{R}^2$ if, and only if $(\mu, \lambda) \in \Upsilon$, where Υ is the set defined by

$$\Upsilon = \mathbb{R}^2 \cup \{(i\mu, \lambda); (\mu, \lambda) \in \mathbb{R}^2; |\mu| \leq |\lambda|\}$$

and in this case

$$\sup_{(r,x) \in \mathbb{R}^2} |\varphi_{\mu,\lambda}(r, x)| = 1. \quad (2.3)$$

- The function $\varphi_{\mu,\lambda}$ has the following Mehler integral representation

$$\varphi_{\mu,\lambda}(r, x) = \begin{cases} \frac{\alpha}{\pi} \int_{-1}^1 \int_{-1}^1 \cos(\mu r s \sqrt{1 - t^2}) \exp(-i\lambda(x + rt)) \\ \quad \times (1 - t^2)^{\alpha - \frac{1}{2}} (1 - s^2)^{\alpha - 1} dt ds, & \text{if } \alpha > 0, \\ \frac{1}{\pi} \int_{-1}^1 \cos(r\mu \sqrt{1 - t^2}) \exp(-i\lambda(x + rt)) \frac{dt}{\sqrt{1 - t^2}}, & \text{if } \alpha = 0. \end{cases}$$

The precedent integral representation allows us to define the Riemann-Liouville transform $\mathcal{R}_\alpha$ associated with the operators D and Ξ by

$$\mathcal{R}_\alpha(f)(r, x) = \begin{cases} \frac{\alpha}{\pi} \int_{-1}^1 \int_{-1}^1 f(rs \sqrt{1 - t^2}, x + rt) (1 - t^2)^{\alpha - \frac{1}{2}} \\ \quad \times (1 - s^2)^{\alpha - 1} dt ds, & \text{if } \alpha > 0, \\ \frac{1}{\pi} \int_{-1}^1 f(r \sqrt{1 - t^2}, x + rt) \frac{dt}{\sqrt{1 - t^2}}, & \text{if } \alpha = 0, \end{cases}$$

where f is any continuous function on $\mathbb{R}^2$, even with respect to the first variable.

- From the precedent integral representation of the eigenfunction $\varphi_{\mu,\lambda}$, we deduce that

$$\forall (r, x) \in [0, +\infty[\times \mathbb{R}; \varphi_{\mu,\lambda}(r, x) = \mathcal{R}_\alpha(\cos(\mu \cdot) e^{-i\lambda \cdot})(r, x).$$

In the following, we will define the convolution product and the Fourier transform associated with the Riemann-Liouville operator. For this, we need the coming notation

- $L^p(d\nu_\alpha)$; $p \in [1, +\infty]$, is the Lebesgue space formed by the measurable functions f on $[0, +\infty[\times \mathbb{R}$ such that $\|f\|_{p,\nu_\alpha} < +\infty$, where

$$\|f\|_{p,\nu_\alpha} = \begin{cases} \left(\int_0^{+\infty} \int_{\mathbb{R}} |f(r, x)|^p d\nu_\alpha(r, x) \right)^{\frac{1}{p}}, & \text{if } p \in [1, +\infty[, \\ \text{ess sup}_{(r,x) \in [0, +\infty[\times \mathbb{R}} |f(r, x)|, & \text{if } p = +\infty. \end{cases}$$

DEFINITION 2.1

- For every $(r, x) \in [0, +\infty[\times \mathbb{R}$, the translation operator $\tau_{(r,x)}$ associated with Riemann-Liouville operator is defined on $L^p(d\nu_\alpha)$, $p \in [1, +\infty]$, by

$$\begin{aligned} \tau_{(r,x)} f(s, y) &= \frac{\Gamma(\alpha + 1)}{\sqrt{\pi} \Gamma(\alpha + \frac{1}{2})} \int_0^\pi f(\sqrt{r^2 + s^2 + 2rs \cos \theta}, x + y) \sin^{2\alpha}(\theta) d\theta. \end{aligned} \quad (2.4)$$

- The convolution product of $f, g \in L^1(d\nu_\alpha)$ is defined for every $(r, x) \in [0, +\infty[\times \mathbb{R}$, by

$$f * g(r, x) = \int_0^{+\infty} \int_{\mathbb{R}} \tau_{(r,-x)}(\check{f})(s, y) g(s, y) d\nu_\alpha(s, y), \quad (2.5)$$

where $\check{f}(s, y) = f(s, -y)$.

We have the following properties

- The eigenfunction $\varphi_{\mu,\lambda}$ satisfies the product formula

$$\tau_{(r,x)}(\varphi_{\mu,\lambda})(s, y) = \varphi_{\mu,\lambda}(r, x) \varphi_{\mu,\lambda}(s, y).$$

- For every $f \in L^p(d\nu_\alpha)$, $1 \leq p \leq +\infty$, and for every $(r, x) \in [0, +\infty[\times \mathbb{R}$, the function $\tau_{(r,x)}(f)$ belongs to $L^p(d\nu_\alpha)$ and we have

$$\|\tau_{(r,x)}(f)\|_{p,\nu_\alpha} \leq \|f\|_{p,\nu_\alpha}.$$

- For every $f \in L^p(d\nu_\alpha)$, $p \in [1, +\infty[$, we have

$$\lim_{(r,x) \rightarrow (0,0)} \|\tau_{(r,x)}(f) - f\|_{p,\nu_\alpha} = 0. \quad (2.6)$$

- For $f, g \in L^1(d\nu_\alpha)$, the function $f * g$ belongs to $L^1(d\nu_\alpha)$; the convolution product is commutative, associative and we have

$$\|f * g\|_{1,\nu_\alpha} \leq \|f\|_{1,\nu_\alpha} \|g\|_{1,\nu_\alpha}.$$

Moreover, if $1 \leq p, q, r \leq +\infty$ are such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$ and if $f \in L^p(d\nu_\alpha)$, $g \in L^q(d\nu_\alpha)$, then the function $f * g$ belongs to $L^r(d\nu_\alpha)$, and we have the Young's inequality

$$\|f * g\|_{r,\nu_\alpha} \leq \|f\|_{p,\nu_\alpha} \|g\|_{q,\nu_\alpha}. \quad (2.7)$$

- Let φ be a nonnegative measurable function on $\mathbb{R} \times \mathbb{R}$, even with respect to the first variable, such that

$$\int_0^{+\infty} \int_{\mathbb{R}} \varphi(r, x) d\nu_\alpha(r, x) = 1.$$

Then by relation (2.6), the family $(\varphi_t)_{t>0}$ defined by

$$\forall (r, x) \in \mathbb{R} \times \mathbb{R}; \varphi_t(r, x) = \frac{\varphi(\frac{r}{t}, \frac{x}{t})}{t^{2\alpha+3}},$$

is an approximation of the identity in $L^p(d\nu_\alpha)$; $p \in [1, +\infty]$, that is for every $f \in L^p(d\nu_\alpha)$, we have

$$\lim_{t \rightarrow 0^+} \|\varphi_t * f - f\|_{p,\nu_\alpha} = 0. \quad (2.8)$$

In the sequel, we use the following notations

- Υ_+ is the subset of Υ given by

$$\Upsilon_+ = \mathbb{R}_+ \times \mathbb{R} \cup \{(it, x); (t, x) \in \mathbb{R}^2; 0 \leq t \leq |x|\}.$$

- $\mathcal{B}_{\Upsilon_+}$ is the σ -algebra defined on Υ_+ by

$$\mathcal{B}_{\Upsilon_+} = \{\theta^{-1}(B), B \in \mathcal{B}_{\text{or}}([0, +\infty[\times \mathbb{R})\},$$

where θ is the bijective function defined on the set Υ_+ by

$$\theta(\mu, \lambda) = (\sqrt{\mu^2 + \lambda^2}, \lambda). \quad (2.9)$$

- $d\gamma_\alpha$ is the measure defined on $\mathcal{B}_{\Upsilon_+}$ by

$$\forall A \in \mathcal{B}_{\Upsilon_+}; \gamma_\alpha(A) = \nu_\alpha(\theta(A)).$$

- $L^p(d\gamma_\alpha)$; $p \in [1, +\infty]$, is the space of measurable functions f on Υ_+ , such that

$$\|f\|_{p,\gamma_\alpha} < +\infty.$$

PROPOSITION 2.2

i. For all non negative measurable function g on Υ_+ , we have

$$\begin{aligned} & \iint_{\Upsilon_+} g(\mu, \lambda) d\gamma_\alpha(\mu, \lambda) \\ &= \frac{1}{2^\alpha \Gamma(\alpha + 1) \sqrt{2\pi}} \left(\int_0^{+\infty} \int_{\mathbb{R}} g(\mu, \lambda) (\mu^2 + \lambda^2)^\alpha \mu d\mu d\lambda \right. \\ & \quad \left. + \int_{\mathbb{R}} \int_0^{|\lambda|} g(i\mu, \lambda) (\lambda^2 - \mu^2)^\alpha \mu d\mu d\lambda \right). \end{aligned}$$

ii. For all non negative measurable function f on $[0, +\infty[\times \mathbb{R}$ (respectively integrable on $[0, +\infty[\times \mathbb{R}$ with respect to the measure $d\nu_\alpha$) $f \circ \theta$ is a nonnegative measurable function on Υ_+ (respectively integrable on Υ_+ with respect to the measure $d\gamma_\alpha$) and we have

$$\iint_{\Upsilon_+} (f \circ \theta)(\mu, \lambda) d\gamma_\alpha(\mu, \lambda) = \int_0^{+\infty} \int_{\mathbb{R}} f(r, x) d\nu_\alpha(r, x). \quad (2.10)$$

DEFINITION 2.3

The Fourier transform associated with the Riemann-Liouville operator is defined on $L^1(d\nu_\alpha)$ by

$$\forall (\mu, \lambda) \in \Upsilon; \quad \mathcal{F}_\alpha(f)(\mu, \lambda) = \int_0^{+\infty} \int_{\mathbb{R}} f(r, x) \varphi_{\mu, \lambda}(r, x) d\nu_\alpha(r, x),$$

where $\varphi_{\mu, \lambda}$ is the eigenfunction given by relation (2.1).

We have the following properties

- From relation (2.3), we deduce that for $f \in L^1(d\nu_\alpha)$ the function $\mathcal{F}_\alpha(f)$ belongs to the space $L^\infty(d\gamma_\alpha)$ and we have

$$\|\mathcal{F}_\alpha(f)\|_{\infty, \gamma_\alpha} \leq \|f\|_{1, \nu_\alpha}. \quad (2.11)$$

- For $f \in L^1(d\nu_\alpha)$, we have

$$\widetilde{\mathcal{F}}_\alpha(f)(\mu, \lambda) = \widetilde{\mathcal{F}}_\alpha(f) \circ \theta(\mu, \lambda), \quad (2.12)$$

where for every $(\mu, \lambda) \in \mathbb{R}^2$,

$$\widetilde{\mathcal{F}}_\alpha(f)(\mu, \lambda) = \int_0^{+\infty} \int_{\mathbb{R}} f(r, x) j_\alpha(r\mu) \exp(-i\lambda x) d\nu_\alpha(r, x),$$

and θ is the function defined by relation (2.9).

- Let $f \in L^1(d\nu_\alpha)$ such that the function $\mathcal{F}_\alpha(f)$ belongs to the space $L^1(d\gamma_\alpha)$, then we have the following inversion formula for $\mathcal{F}_\alpha$, for almost every $(r, x) \in [0, +\infty[\times \mathbb{R}$,

$$f(r, x) = \iint_{\Upsilon_+} \mathcal{F}_\alpha(f)(\mu, \lambda) \overline{\varphi_{\mu, \lambda}(r, x)} d\gamma_\alpha(\mu, \lambda)$$

- Let $f \in L^1(d\nu_\alpha)$. For every $(r, x) \in [0, +\infty[\times \mathbb{R}$, we have

$$\forall (\mu, \lambda) \in \Upsilon; \mathcal{F}_\alpha(\tau_{(r, x)}(f))(\mu, \lambda) = \overline{\varphi_{\mu, \lambda}(r, x)} \mathcal{F}_\alpha(f)(\mu, \lambda).$$

- For $f, g \in L^1(d\nu_\alpha)$, we have

$$\forall (\mu, \lambda) \in \Upsilon; \mathcal{F}_\alpha(f * g)(\mu, \lambda) = \mathcal{F}_\alpha(f)(\mu, \lambda) \mathcal{F}_\alpha(g)(\mu, \lambda).$$

- Let $p \in [1, +\infty]$. From relation (2.10), the function f belongs to $L^p(d\nu_\alpha)$ if, and only if the function $f \circ \theta$ belongs to the space $L^p(d\gamma_\alpha)$ and we have

$$\|f \circ \theta\|_{p, \gamma_\alpha} = \|f\|_{p, \nu_\alpha}. \quad (2.13)$$

Since the mapping $\widetilde{\mathcal{F}_\alpha}$ is an isometric isomorphism from $L^2(d\nu_\alpha)$ onto itself [24, 25], then the relations (2.12) and (2.13) show that the Fourier transform $\mathcal{F}_\alpha$ is an isometric isomorphism from $L^2(d\nu_\alpha)$ into $L^2(d\gamma_\alpha)$. Namely, for every $f \in L^2(d\nu_\alpha)$, the function $\mathcal{F}_\alpha(f)$ belongs to the space $L^2(d\gamma_\alpha)$ and we have

$$\|\mathcal{F}_\alpha(f)\|_{2, \gamma_\alpha} = \|f\|_{2, \nu_\alpha}. \quad (2.14)$$

PROPOSITION 2.4

For every f in $L^p(d\nu_\alpha)$, $p \in [1, 2]$; the function $\mathcal{F}_\alpha(f)$ lies in $L^{p'}(d\gamma_\alpha)$, $p' = \frac{p}{p-1}$, and we have

$$\|\mathcal{F}_\alpha(f)\|_{p', \gamma_\alpha} \leq \|f\|_{p, \nu_\alpha}.$$

Proof. The result follows from relations (2.11), (2.14) and the Riesz-Thorin theorem's [20, 22].

We denote by

- $\mathcal{S}_e(\mathbb{R}^2)$ the space of infinitely differentiable functions on $\mathbb{R}^2$, rapidly decreasing together with all their derivatives, even with respect to the first variable. The space $\mathcal{S}_e(\mathbb{R}^2)$ is endowed with the topology generated by the family of norms

$$\rho_m(\varphi) = \sup_{\substack{(r, x) \in [0, +\infty[\times \mathbb{R} \\ k+|\beta| \leq m}} (1+r^2+x^2)^k |D^\beta(\varphi)(r, x)|; \quad m \in \mathbb{N}. \quad (2.15)$$

- $\mathcal{D}_e(\mathbb{R}^2)$ the subspace of $\mathcal{S}_e(\mathbb{R}^2)$ of functions with compact support.

3. Gauss and Poisson semigroups associated with the Riemann-Liouville operator

In this section, we will define and study the Gauss and Poisson semigroups. Also, the maximal functions connected with these semigroups are checked.

DEFINITION 3.1

The Gauss kernel g_t , $t > 0$, associated with the Riemann-Liouville operator is defined on $\mathbb{R}^2$ by

$$\begin{aligned} g_t(r, x) &= \frac{e^{-\frac{(r^2+x^2)}{4t}}}{(2t)^{\alpha+\frac{3}{2}}} = \iint_{\Upsilon_+} e^{-t(\mu^2+2\lambda^2)} \overline{\varphi_{\mu,\lambda}(r, x)} d\gamma_\alpha(\mu, \lambda) \\ &= \widetilde{\mathcal{F}_\alpha}^{-1}(e^{-t(s^2+y^2)})(r, x). \end{aligned} \quad (3.16)$$

LEMMA 3.2

The family $(g_t)_{t>0}$ is an approximation of the identity in the space $\mathcal{S}_e(\mathbb{R}^2)$; that is for every $f \in \mathcal{S}_e(\mathbb{R}^n)$; and every $t > 0$; the function $g_t * f$ belongs to $\mathcal{S}_e(\mathbb{R}^2)$ and for every $m \in \mathbb{N}$;

$$\lim_{t \rightarrow 0^+} \rho_m(g_t * f - f) = 0,$$

where ρ_m is the norm defined by relation (2.15).

Proof. Since the Schwartz space $\mathcal{S}_e(\mathbb{R}^2)$ is stable under convolution product, we deduce that for every $f \in \mathcal{S}_e(\mathbb{R}^2)$; and every $t > 0$; the function $g_t * f$ belongs to the space $\mathcal{S}_e(\mathbb{R}^2)$. On the other hand, the transform $\widetilde{\mathcal{F}_\alpha}$ is a topological isomorphism from $\mathcal{S}_e(\mathbb{R}^2)$ onto itself which satisfies

$$\widetilde{\mathcal{F}_\alpha}(f * g) = \widetilde{\mathcal{F}_\alpha}(f) \widetilde{\mathcal{F}_\alpha}(g). \quad (3.17)$$

By relation (3.16), we get $\widetilde{\mathcal{F}_\alpha}(g_t)(r, x) = e^{-t(r^2+x^2)}$. So, we must show that for every $(k, \beta) \in \mathbb{N} \times \mathbb{N}^2$ and every $f \in \mathcal{S}_e(\mathbb{R}^2)$,

$$\lim_{t \rightarrow 0^+} \|(1 + r^2 + x^2)^k D^\beta(e^{-t(r^2+x^2)} f - f)\|_{\infty, \nu_\alpha} = 0.$$

Applying Leibniz formula, we get

$$\begin{aligned} D^\beta(e^{-t(r^2+x^2)} f(r, x)) &= \sum_{\gamma \leq \beta} \frac{\beta!}{\gamma!(\beta-\gamma)!} D^\gamma(e^{-t(r^2+x^2)}) D^{\beta-\gamma}(f)(r, x) \\ &= \sum_{\gamma \leq \beta} \frac{\beta!}{\gamma!(\beta-\gamma)!} (-1)^{|\gamma|} \sqrt{t}^{|\gamma|} H_\gamma(r\sqrt{t}, x\sqrt{t}) e^{-t(r^2+x^2)} D^{\beta-\gamma}(f)(r, x), \end{aligned}$$

where H_γ is the Hermite polynomial on $\mathbb{R}^2$ with index γ .

Consequently,

$$\begin{aligned} & D^\beta(e^{-t(r^2+x^2)}f(r, x) - f(r, x)) \\ &= \sum_{\substack{\gamma \leq \beta \\ \gamma \neq 0}} \frac{\beta!}{\gamma!(\beta-\gamma)!} (-1)^{|\gamma|} \sqrt{t}^{|\gamma|} H_\gamma(r\sqrt{t}, x\sqrt{t}) e^{-t(r^2+x^2)} D^{\beta-\gamma}(f)(r, x) \\ &+ (e^{-t(r^2+x^2)} - 1) D^\beta(f)(r, x). \end{aligned}$$

Thus, for every t , $0 \leq t < 1$;

$$\begin{aligned} & \|(1+r^2+x^2)^k D^\beta(e^{-t(r^2+x^2)}f - f)\|_{\infty, \nu_\alpha} \\ & \leq \sqrt{t} \left[\sum_{\gamma \leq \beta} \frac{\beta!}{\gamma!(\beta-\gamma)!} \|H_\gamma e^{-(r^2+x^2)}\|_{\infty, \nu_\alpha} \|(1+r^2+x^2)^k D^{\beta-\gamma}(f)\|_{\infty, \nu_\alpha} \right. \\ & \quad \left. + \|(1+r^2+x^2)^{k+1} D^\beta(f)\|_{\infty, \nu_\alpha} \right]. \end{aligned}$$

The last inequality shows that for every $(k, \beta) \in \mathbb{N} \times \mathbb{N}^2$,

$$\lim_{t \rightarrow 0^+} \|(1+r^2+x^2)^k D^\beta(\widetilde{\mathcal{F}_\alpha}(g_t)f - f)\|_{\infty, \nu_\alpha} = 0.$$

The proof is complete.

PROPOSITION 3.3

For every $f \in \mathcal{S}_e(\mathbb{R}^2)$; the function $\mathcal{V}(f)$ defined by

$$\mathcal{V}(f)(r, x, t) = g_t * f(r, x), \quad \forall (r, x, t) \in \mathbb{R}^2 \times]0, +\infty[,$$

is infinitely differentiable on $\mathbb{R}^2 \times]0, +\infty[$ and satisfies the following equation

$$\begin{cases} \Lambda_\alpha(\mathcal{V}(f)) = \frac{\partial}{\partial t}(\mathcal{V}(f)), \\ \lim_{t \rightarrow 0^+} \mathcal{V}(f)(\cdot, \cdot, t) = f \quad \text{uniformly.} \end{cases}$$

Where

$$\Lambda_\alpha = \frac{\partial^2}{\partial r^2} + \frac{2\alpha+1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial x^2}. \quad (3.18)$$

Proof. For every $t > 0$; the function g_t belongs to $\mathcal{S}_e(\mathbb{R}^2)$ and consequently, for every $f \in \mathcal{S}_e(\mathbb{R}^2)$, the function

$$(r, x) \mapsto g_t * f(r, x)$$

belongs to the space $\mathcal{S}_e(\mathbb{R}^2)$ and for every $(\mu, \lambda) \in \mathbb{R}^2$;

$$\widetilde{\mathcal{F}_\alpha}(g_t * f)(\mu, \lambda) = \widetilde{\mathcal{F}_\alpha}(\mathcal{V}(f)(\cdot, \cdot, t))(\mu, \lambda) = e^{-t(\mu^2+\lambda^2)} \widetilde{\mathcal{F}_\alpha}(f)(\mu, \lambda).$$

This implies that for every $(r, x, t) \in \mathbb{R}^2 \times]0, +\infty[$, we have

$$\mathcal{V}(f)(r, x, t) = \int_0^\infty \int_{\mathbb{R}} e^{-t(\mu^2+\lambda^2)} \widetilde{\mathcal{F}_\alpha}(f)(\mu, \lambda) j_\alpha(r\mu) e^{i\lambda x} d\nu_\alpha(\mu, \lambda).$$

From this equality; it follows that the function

$$(r, x, t) \longmapsto \mathcal{V}(f)(r, x, t)$$

is infinitely differentiable on $\mathbb{R}^2 \times]0, +\infty[$ and we have

$$\begin{aligned} \frac{\partial}{\partial t}(\mathcal{V}(f))(r, x, t) &= - \int_0^\infty \int_{\mathbb{R}} (\mu^2 + \lambda^2) e^{-t(\mu^2 + \lambda^2)} \widetilde{\mathcal{F}_\alpha}(f)(\mu, \lambda) j_\alpha(r\mu) e^{i\lambda x} d\nu_\alpha(r, x) \\ &= \Lambda_\alpha(\mathcal{V}(f))(r, x, t), \end{aligned}$$

because $(\frac{\partial^2}{\partial r^2} + \frac{2\alpha+1}{\partial r})(j_\alpha(r\mu)) = -\mu^2 j_\alpha(r\mu)$ and $\frac{\partial^2}{\partial x^2}(e^{i\lambda x}) = -\lambda^2 e^{i\lambda x}$.

On the other hand; for $(r, x, t) \in \mathbb{R}^2 \times]0, +\infty[$,

$$\begin{aligned} f(r, x) - \mathcal{V}(f)(r, x, t) &= \int_0^\infty \int_{\mathbb{R}} (1 - e^{-t(\mu^2 + \lambda^2)}) \widetilde{\mathcal{F}_\alpha}(f)(\mu, \lambda) j_\alpha(r\mu) e^{i\lambda x} d\nu_\alpha(r, x). \end{aligned}$$

So

$$\|f - \mathcal{V}(f)(\cdot, \cdot, t)\|_{\infty, \nu_\alpha} \leq t \int_0^\infty \int_{\mathbb{R}} (\mu^2 + \lambda^2) |\widetilde{\mathcal{F}_\alpha}(f)(\mu, \lambda)| d\nu_\alpha(\mu, \lambda),$$

which means that

$$\lim_{t \rightarrow 0^+} \|\mathcal{V}(f)(\cdot, \cdot, t) - f\|_{\infty, \nu_\alpha} = 0.$$

PROPOSITION 3.4

i. For every $p \in [1, +\infty]$; the operator $\mathcal{G}^t$, $t > 0$, defined by

$$\mathcal{G}^t(f) = g_t * f \tag{3.19}$$

is a bounded positive operator from $L^p(d\nu_\alpha)$ into itself and for every $f \in L^p(d\nu_\alpha)$, we have

$$\|\mathcal{G}^t(f)\|_{p, \nu_\alpha} \leq \|f\|_{p, \nu_\alpha}.$$

ii. For every $p \in [1, +\infty[$, the family $(\mathcal{G}^t)_{t>0}$ is a strongly continuous semigroup of operators on $L^p(d\nu_\alpha)$, that is

- For $s, t > 0$; $\mathcal{G}^s \circ \mathcal{G}^t = \mathcal{G}^{s+t}$,
- For every $f \in L^p(d\nu_\alpha)$, $\lim_{t \rightarrow 0^+} \|\mathcal{G}^t(f) - f\|_{p, \nu_\alpha} = 0$.

The family $(\mathcal{G}^t)_{t>0}$ is called Gauss semigroup associated with the Riemann-Liouville operator $\mathcal{R}_\alpha$.

Proof. i. Let $g(r, x) = e^{-\frac{r^2 + x^2}{2}}$, g is a measurable positive function and we have

$$g_t(r, x) = \frac{g(\frac{r}{\sqrt{2t}}, \frac{x}{\sqrt{2t}})}{(\sqrt{2t})^{2\alpha+3}}.$$

So

$$\int_0^\infty \int_{\mathbb{R}} g_t(r, x) d\nu_\alpha(r, x) = \int_0^\infty \int_{\mathbb{R}} g(r, x) d\nu_\alpha(r, x) = 1.$$

From relation (2.7), for every $f \in L^p(d\nu_\alpha)$; and every $t > 0$, the function $\mathcal{G}^t(f) = g_t * f$ belongs to $L^p(d\nu_\alpha)$ and we have

$$\|\mathcal{G}^t(f)\|_{p, \nu_\alpha} \leq \|g_t\|_{1, \nu_\alpha} \|f\|_{p, \nu_\alpha} = \|f\|_{p, \nu_\alpha}.$$

ii. From relation (3.16), we have

$$\forall (\mu, \lambda) \in \mathbb{R}^2; \quad \widetilde{\mathcal{F}_\alpha}(g_t)(\mu, \lambda) = e^{-t(\mu^2 + \lambda^2)}.$$

So, from relation (3.17); for $s, t > 0$; we get

$$\widetilde{\mathcal{F}_\alpha}(g_t * g_s)(\mu, \lambda) = e^{-(t+s)(\mu^2 + \lambda^2)} = \widetilde{\mathcal{F}_\alpha}(g_{t+s})(\mu, \lambda),$$

and consequently; $g_s * g_t = g_{s+t}$ which involves that for every $f \in L^p(d\nu_\alpha)$;

$$\mathcal{G}^s(\mathcal{G}^t(f)) = \mathcal{G}^{s+t}(f).$$

Moreover, from relation (2.8),

$$\lim_{t \rightarrow 0^+} \|\mathcal{G}^t(f) - f\|_{\infty, \nu_\alpha} = 0.$$

The proof is complete.

PROPOSITION 3.5

For every $f \in \mathcal{D}_e(\mathbb{R}^2)$, the maximal function $\mathcal{M}(f)$ defined on $\mathbb{R}^2$ by

$$\mathcal{M}(f)(r, x) = \sup_{s>0} \frac{1}{s} \left| \int_0^s \mathcal{G}^t(f)(r, x) dt \right|, \quad (3.20)$$

belongs to the space $L^p(d\nu_\alpha)$, $p \in]1, +\infty[$, moreover

$$\|\mathcal{M}(f)\|_{p, \nu_\alpha} \leq 2 \left(\frac{p}{p-1} \right)^{\frac{1}{p}} \|f\|_{p, \nu_\alpha}.$$

Proof. The result follows immediately from [9, theorem 7, pp 693].

DEFINITION 3.6

For every $t > 0$, the Poisson kernel p_t associated with the Riemann-Liouville operator is defined on $\mathbb{R}^2$ by

$$\begin{aligned} p_t(r, x) &= \int_{\Upsilon_+} e^{-t\sqrt{s^2+2y^2}} \varphi_{s,y}(r, x) d\gamma_\alpha(s, y) = \mathcal{F}_\alpha^{-1}(e^{-t\sqrt{s^2+2y^2}})(r, x) \\ &= \widetilde{\mathcal{F}_\alpha}^{-1}(e^{-t\sqrt{s^2+y^2}})(r, x). \end{aligned} \quad (3.21)$$

LEMMA 3.7

For every $(r, x, t) \in \mathbb{R}^2 \times]0, +\infty[$, we have

$$p_t(r, x) = \frac{2^{\alpha+\frac{3}{2}} \Gamma(\alpha+2)}{\sqrt{\pi}} \frac{t}{(t^2 + r^2 + x^2)^{\alpha+2}}.$$

Proof. We know that for every $x \in \mathbb{R}$; we have

$$\frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} e^{-\frac{x^2}{4u}} du = e^{-|x|}.$$

From Definition 3.6, and applying Fubini's theorem, we get

$$\begin{aligned} p_t(r, x) &= \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} \left(\iint_{\Upsilon_+} e^{-\frac{t^2}{4u}(s^2+2y^2)} \overline{\varphi_{s,y}(r, x)} d\gamma_\alpha(s, y) \right) \\ &= \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} g_{\frac{t^2}{4u}}(r, x) du \\ &= \frac{2^{\alpha+\frac{3}{2}}}{\sqrt{\pi}} t^{-2\alpha-3} \int_0^\infty e^{-\frac{u}{t^2}(r^2+x^2+t^2)} u^{\alpha+1} du \\ &= \frac{2^{\alpha+\frac{3}{2}} \Gamma(\alpha+2)}{\sqrt{\pi}} \frac{t}{(t^2 + r^2 + x^2)^{\alpha+2}}. \end{aligned} \tag{3.22}$$

PROPOSITION 3.8

Let $f \in \mathcal{S}_e(\mathbb{R}^2)$, the function $\mathcal{U}(f)$ defined on $\mathbb{R}^2 \times]0, +\infty[$ by

$$\mathcal{U}(f)(r, x) = p_t * f(r, x)$$

is infinitely differentiable and satisfies the equation

$$\begin{cases} \Lambda_\alpha(\mathcal{U}(f)) + \frac{\partial^2}{\partial t^2}(\mathcal{U}(f)) = 0, \\ \lim_{t \rightarrow 0^+} \mathcal{U}(f)(\cdot, \cdot, t) = f \quad \text{uniformly.} \end{cases}$$

Proof. From relation (3.21), for every $(\mu, \lambda) \in \mathbb{R}^2$, we have

$$\widetilde{\mathcal{F}}_\alpha(\mathcal{U}(f)(\cdot, \cdot, t)) = \widetilde{\mathcal{F}}_\alpha(p_t)(\mu, \lambda) \widetilde{\mathcal{F}}_\alpha(f)(\mu, \lambda) = e^{-t\sqrt{\mu^2+\lambda^2}} \widetilde{\mathcal{F}}_\alpha(f)(\mu, \lambda).$$

So, for every $(r, x, t) \in \mathbb{R}^2 \times]0, +\infty[$;

$$\begin{aligned} \mathcal{U}(f)(r, x, t) &= \widetilde{\mathcal{F}}_\alpha^{-1}(e^{-t\sqrt{\mu^2+\lambda^2}} \widetilde{\mathcal{F}}_\alpha(f))(r, x) \\ &= \int_0^\infty \int_{\mathbb{R}} e^{-t\sqrt{\mu^2+\lambda^2}} \widetilde{\mathcal{F}}_\alpha(f)(\mu, \lambda) j_\alpha(r\mu) e^{i\lambda x} d\nu_\alpha(\mu, \lambda). \end{aligned}$$

From relation (2.2) and the fact that the function $\widetilde{\mathcal{F}}_\alpha(f)$ belongs to the space $\mathcal{S}_e(\mathbb{R}^2)$; we deduce that the function $\mathcal{U}(f)$ is infinitely differentiable on

$\mathbb{R}^2 \times]0, +\infty[$. Moreover,

$$\begin{aligned} \Lambda_\alpha(\mathcal{U}(f))(r, x, t) &= - \int_0^\infty \int_{\mathbb{R}} (\mu^2 + \lambda^2) e^{-t\sqrt{\mu^2 + \lambda^2}} \widetilde{\mathcal{F}_\alpha(f)}(\mu, \lambda) j_\alpha(r\mu) e^{i\lambda x} d\nu_\alpha(\mu, \lambda) \\ &= - \frac{\partial^2}{\partial t^2} (\mathcal{U}(f))(r, x, t). \end{aligned}$$

On the other hand; for every $(r, x, t) \in \mathbb{R}^2 \times]0, +\infty[$; we get

$$\begin{aligned} |f(r, x) - \mathcal{U}(f)(r, x, t)| &\leq \int_0^\infty \int_{\mathbb{R}} |1 - e^{-t\sqrt{\mu^2 + \lambda^2}}| |\widetilde{\mathcal{F}_\alpha(f)}(\mu, \lambda)| d\nu_\alpha(\mu, \lambda) \\ &\leq t \int_0^\infty \int_{\mathbb{R}} \sqrt{\mu^2 + \lambda^2} |\widetilde{\mathcal{F}_\alpha(f)}(\mu, \lambda)| d\nu_\alpha(\mu, \lambda), \end{aligned}$$

which means that

$$\|\mathcal{U}(f)(\cdot, \cdot, t) - f\|_{\infty, \nu_\alpha} \leq t \int_0^\infty \int_{\mathbb{R}} \sqrt{\mu^2 + \lambda^2} |\widetilde{\mathcal{F}_\alpha(f)}(\mu, \lambda)| d\nu_\alpha(\mu, \lambda),$$

and proves that

$$\lim_{t \rightarrow 0^+} \|\mathcal{U}(f)(\cdot, \cdot, t) - f\|_{\infty, \nu_\alpha} = 0.$$

PROPOSITION 3.9

i. For every $p \in [1, +\infty]$; the operator $\mathcal{P}^t$, $t > 0$, defined by

$$\mathcal{P}^t(f) = p_t * f$$

is a bounded positive operator from $L^p(d\nu_\alpha)$ into itself and for every $f \in L^p(d\nu_\alpha)$, we have

$$\|\mathcal{P}^t(f)\|_{p, \nu_\alpha} \leq \|f\|_{p, \nu_\alpha}.$$

ii. For every $p \in [1, +\infty]$, the family $(\mathcal{P}^t)_{t>0}$ is a strongly continuous semigroup of operators on $L^p(d\nu_\alpha)$, that is

- For $s, t > 0$; $\mathcal{P}^s \circ \mathcal{P}^t = \mathcal{P}^{s+t}$,
- For every $f \in L^p(d\nu_\alpha)$, $\lim_{t \rightarrow 0^+} \|\mathcal{P}^t(f) - f\|_{p, \nu_\alpha} = 0$.

The family $(\mathcal{P}^t)_{t>0}$ is called Poisson semigroup associated with the Riemann-Liouville operator $\mathcal{R}_\alpha$.

Proof. i. Let $p(r, x) = \frac{2^{\alpha+\frac{3}{2}} \Gamma(\alpha+2)}{\sqrt{\pi}} \frac{1}{(1+r^2+x^2)^{\alpha+2}}$, p is a measurable positive function and we have

$$p_t(r, x) = \frac{1}{t^{2\alpha+3}} p\left(\frac{r}{t}, \frac{x}{t}\right).$$

So,

$$\int_0^\infty \int_{\mathbb{R}} p_t(r, x) d\nu_\alpha(r, x) = \int_0^\infty \int_{\mathbb{R}} p(r, x) d\nu_\alpha(r, x) = 1. \quad (3.23)$$

From relation (2.7), for every $f \in L^p(d\nu_\alpha)$; and every $t > 0$, the function $\mathcal{P}^t(f) = p_t * f$ belongs to $L^p(d\nu_\alpha)$ and we have

$$\|\mathcal{P}^t(f)\|_{p, \nu_\alpha} \leq \|p_t\|_{1, \nu_\alpha} \|f\|_{p, \nu_\alpha} = \|f\|_{p, \nu_\alpha}.$$

ii. From relation (3.21), we have

$$\forall (\mu, \lambda) \in \mathbb{R}^2; \quad \widetilde{\mathcal{F}_\alpha}(p_t)(\mu, \lambda) = e^{-t\sqrt{\mu^2 + \lambda^2}}.$$

So, from relation (3.17); for $s, t > 0$; we get

$$\widetilde{\mathcal{F}_\alpha}(p_t * p_s)(\mu, \lambda) = e^{-(t+s)\sqrt{\mu^2 + \lambda^2}} = \widetilde{\mathcal{F}_\alpha}(p_{t+s})(\mu, \lambda),$$

and consequently; $p_s * p_t = p_{s+t}$ which involves that for every $f \in L^p(d\nu_\alpha)$;

$$\mathcal{P}^s(\mathcal{P}^t(f)) = \mathcal{P}^{s+t}(f).$$

Moreover, from relations (2.8) and (3.23),

$$\lim_{t \rightarrow 0^+} \|\mathcal{P}^t(f) - f\|_{p, \nu_\alpha} = 0.$$

This finishes the proof.

LEMMA 3.10

We have the following connexion between the Gauss and Poisson semigroups, that is

$$\mathcal{P}^t(f)(r, x) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} \mathcal{G}_{\frac{t^2}{4u}}(f)(r, x) du.$$

Proof. Let $f \in L^p(d\nu_\alpha)$, $p \in [1, +\infty]$; for every $(r, x, t) \in \mathbb{R}^2 \times]0, +\infty[$, the integral

$$\frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} \mathcal{G}_{\frac{t^2}{4u}}(f)(r, x) du$$

is well defined.

Moreover, from relations (2.5), (3.19) and applying Fubini's theorem, we get

$$\begin{aligned} & \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} \mathcal{G}_{\frac{t^2}{4u}}(f)(r, x) du \\ &= \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} \left(\int_0^\infty \int_{\mathbb{R}} \tau_{(r, -x)}(\check{f})(s, y) g_{\frac{t^2}{4u}}(s, y) d\nu_\alpha(s, y) \right) du \\ &= \int_0^\infty \int_{\mathbb{R}} \tau_{(r, -x)}(\check{f})(s, y) \left(\frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} g_{\frac{t^2}{4u}}(s, y) du \right) d\nu_\alpha(s, y). \end{aligned}$$

By relation (3.22), we obtain

$$\begin{aligned} \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} \mathcal{G}^{\frac{t^2}{4u}}(f)(r, x) du &= \int_0^\infty \int_{\mathbb{R}} \tau_{(r, -x)}(\check{f})(s, y) p_t(s, y) d\nu_\alpha(s, y) \\ &= \mathcal{P}^t(f)(r, x). \end{aligned}$$

PROPOSITION 3.11

For every $f \in \mathcal{D}_e(\mathbb{R}^2)$, the maximal function f^* defined on $\mathbb{R}^2$ by

$$f^*(r, x) = \sup_{t>0} |\mathcal{P}^t(f)(r, x)| \quad (3.24)$$

belongs to the space $L^p(d\nu_\alpha)$; $p \in]1, +\infty[$, and we have

$$\|f^*\|_{p, \nu_\alpha} \leq 2 \left(\frac{p}{p-1} \right)^{\frac{1}{p}} \|f\|_{p, \nu_\alpha}. \quad (3.25)$$

Proof. Let $f \in \mathcal{D}_e(\mathbb{R}^2)$. From Lemma 3.10, for every $(r, x, t) \in \mathbb{R}^2 \times]0, +\infty[$, we have

$$\mathcal{P}^t(f)(r, x) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} \mathcal{G}^{\frac{t^2}{4u}}(f)(r, x) du = \frac{t}{2\sqrt{\pi}} \int_0^\infty \frac{e^{-\frac{t^2}{4s}}}{s^{\frac{3}{2}}} \mathcal{G}^s(f)(r, x) ds.$$

Integrating by parts and using the fact that for every $s > 0$, $|\int_0^s \mathcal{G}^u(f)(r, x) du| \leq s \|f\|_{\infty, \nu_\alpha}$, we get

$$\mathcal{P}^t(f)(r, x) = -\frac{t}{2\sqrt{\pi}} \int_0^\infty s \frac{d}{ds} \left[\frac{e^{-\frac{t^2}{4s}}}{s^{\frac{3}{2}}} \right] \left[\frac{1}{s} \int_0^s \mathcal{G}^u(f)(r, x) du \right] ds.$$

Thus, for every $(r, x, t) \in \mathbb{R}^2 \times]0, +\infty[$; we have

$$|\mathcal{P}^t(f)(r, x)| \leq \mathcal{M}(f)(r, x) \left| \frac{t}{2\sqrt{\pi}} \int_0^\infty s \frac{d}{ds} \left(\frac{e^{-\frac{t^2}{4s}}}{s^{\frac{3}{2}}} \right) ds \right| = \mathcal{M}(f)(r, x).$$

So; for every $(r, x) \in \mathbb{R}^2$; $f^*(r, x) \leq \mathcal{M}(f)(r, x)$; where $\mathcal{M}(f)$ is the maximal function defined by relation (3.20). Using Proposition 3.5, we deduce that

$$\|f^*\|_{p, \nu_\alpha} \leq 2 \left(\frac{p}{p-1} \right)^{\frac{1}{p}} \|f\|_{p, \nu_\alpha}.$$

4. The Littlewood-Paley g -function associated with the Riemann-Liouville operator

This section is devoted to study the boundedness of the g -function. We start this section by some intermediate results.

LEMMA 4.1

Let f be a function of $\mathcal{S}_e(\mathbb{R}^2)$; and let $\mathcal{U}(f)$ be the function defined on $\mathbb{R}^2 \times]0, +\infty[$ by

$$\mathcal{U}(f)(r, x, t) = \mathcal{P}^t(f)(r, x) = p_t * f(r, x).$$

Then for every $k \in \mathbb{N}$, and $(r, x, t) \in \mathbb{R}^2 \times]0, +\infty[$, we have

$$\left| \left(\frac{\partial}{\partial t} \right)^k (\mathcal{U}(f))(r, x, t) \right| \leq \frac{\Gamma(2\alpha + k + 3)}{2^{\alpha + \frac{1}{2}} \Gamma(\alpha + \frac{3}{2})} \frac{\|f\|_{1, \nu_\alpha}}{t^{2\alpha + k + 3}}. \quad (4.26)$$

Proof. From the proof of Proposition 3.8 and for every $(r, x, t) \in \mathbb{R}^2 \times]0, +\infty[$, we have

$$\mathcal{U}(f)(r, x, t) = \int_0^\infty \int_{\mathbb{R}} e^{-t\sqrt{\mu^2 + \lambda^2}} \widetilde{\mathcal{F}_\alpha}(f)(\mu, \lambda) j_\alpha(r\mu) e^{i\lambda x} d\nu_\alpha(\mu, \lambda). \quad (4.27)$$

So, for every $k \in \mathbb{N}$,

$$\begin{aligned} & \left(\frac{\partial}{\partial t} \right)^k (\mathcal{U}(f))(r, x, t) \\ &= (-1)^k \int_0^\infty \int_{\mathbb{R}} (\mu^2 + \lambda^2)^{\frac{k}{2}} e^{-t\sqrt{\mu^2 + \lambda^2}} \widetilde{\mathcal{F}_\alpha}(f)(\mu, \lambda) j_\alpha(r\mu) e^{i\lambda x} d\nu_\alpha(\mu, \lambda). \end{aligned}$$

Consequently, for every $(r, x, t) \in \mathbb{R}^2 \times]0, +\infty[$,

$$\begin{aligned} \left| \left(\frac{\partial}{\partial t} \right)^k (\mathcal{U}(f))(r, x, t) \right| &\leq \|\widetilde{\mathcal{F}_\alpha}(f)\|_{\infty, \nu_\alpha} \int_0^\infty \int_{\mathbb{R}} (\mu^2 + \lambda^2)^{\frac{k}{2}} e^{-t\sqrt{\mu^2 + \lambda^2}} d\nu_\alpha(\mu, \lambda) \\ &\leq \|f\|_{1, \nu_\alpha} \int_0^\infty \int_{\mathbb{R}} (\mu^2 + \lambda^2)^{\frac{k}{2}} e^{-t\sqrt{\mu^2 + \lambda^2}} d\nu_\alpha(\mu, \lambda) \\ &= \frac{\Gamma(2\alpha + k + 3) \|f\|_{1, \nu_\alpha}}{2^{\alpha + \frac{1}{2}} \Gamma(\alpha + \frac{3}{2})} \frac{1}{t^{2\alpha + k + 3}}. \end{aligned}$$

LEMMA 4.2

Let f be a function of $\mathcal{D}_e(\mathbb{R}^2)$ and let a be a positive real number such that $\text{supp}(f) \subset B_a = \{(r; x) \in \mathbb{R}^2, r^2 + x^2 \leq a^2\}$. Then for every $(r, x, t) \in \mathbb{R}^2 \times]0, +\infty[$ such that $r^2 + x^2 \geq 4a^2$, we have

$$|\mathcal{U}(f)(r, x, t)| \leq \frac{a^{2\alpha + 3} 2^{2\alpha + 4} \Gamma(\alpha + 2)}{(2\alpha + 3) \Gamma(\alpha + \frac{3}{2}) \sqrt{\pi}} \frac{\|f\|_{\infty, \nu_\alpha}}{(t^2 + r^2 + x^2)^{\alpha + \frac{3}{2}}} \quad (4.28)$$

$$\left| \frac{\partial}{\partial r} (\mathcal{U}(f))(r, x, t) \right| \leq \frac{a^{2\alpha + 3} 2^{2\alpha + 8} \Gamma(\alpha + 3)}{(2\alpha + 3) \Gamma(\alpha + \frac{3}{2}) \sqrt{\pi}} \frac{\|f\|_{\infty, \nu_\alpha}}{(t^2 + r^2 + x^2)^{\alpha + 2}}, \quad (4.29)$$

$$\left| \frac{\partial}{\partial x} (\mathcal{U}(f))(r, x, t) \right| \leq \frac{a^{2\alpha + 3} 2^{2\alpha + 8} \Gamma(\alpha + 3)}{(2\alpha + 3) \Gamma(\alpha + \frac{3}{2}) \sqrt{\pi}} \frac{\|f\|_{\infty, \nu_\alpha}}{(t^2 + r^2 + x^2)^{\alpha + 2}}. \quad (4.30)$$

Proof. From relation (2.4) and Lemma 3.7, we have

$$\begin{aligned} \tau_{(r,-x)}(p_t)(s, y) &= \frac{2^{\alpha+\frac{3}{2}}\Gamma(\alpha+2)}{\sqrt{\pi}} \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})} \int_0^\pi \frac{t \sin^{2\alpha} \theta d\theta}{(t^2 + (r^2 + s^2 + 2rs \cos \theta) + (x-y)^2)^{\alpha+2}} \\ &\leq \frac{2^{\alpha+\frac{3}{2}}\Gamma(\alpha+2)}{\sqrt{\pi}} \frac{t}{(t^2 + (r-s)^2 + (x-y)^2)^{\alpha+2}} \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})} \int_0^\pi \sin^{2\alpha} \theta d\theta \end{aligned} \quad (4.31)$$

and

$$\tau_{(r,-x)}(p_t)(s, y) \leq \frac{2^{\alpha+\frac{3}{2}}\Gamma(\alpha+2)}{\sqrt{\pi}} \frac{t}{(t^2 + (r-s)^2 + (x-y)^2)^{\alpha+2}}. \quad (4.32)$$

Let $f \in \mathcal{D}_e(\mathbb{R}^2)$; $\text{supp}(f) \subset B_a$. We have

$$\begin{aligned} \mathcal{U}(f)(r, x, t) &= p_t * f(r, x) = \int_0^\infty \int_{\mathbb{R}} \tau_{(r,-x)}(p_t)(s, y) f(s, y) d\nu_\alpha(s, y) \\ &= \iint_{B_a^+} \tau_{(r,-x)}(p_t)(s, y) f(s, y) d\nu_\alpha(s, y), \end{aligned}$$

where

$$B_a^+ = \{(r, x); r^2 + x^2 \leq a^2, r \geq 0\}.$$

From relation (4.32), for every $(r, x, t) \in \mathbb{R}^2 \times]0, +\infty[$:

$$\begin{aligned} |\mathcal{U}(f)(r, x, t)| &\leq \frac{2^{\alpha+\frac{3}{2}}\Gamma(\alpha+2)\|f\|_{\infty, \nu_\alpha}}{\sqrt{\pi}} \iint_{B_a^+} \frac{t d\nu_\alpha(s, y)}{(t^2 + (r-s)^2 + (x-y)^2)^{\alpha+2}} \\ &\leq \frac{2^{\alpha+\frac{3}{2}}\Gamma(\alpha+2)\|f\|_{\infty, \nu_\alpha}}{\sqrt{\pi}} \iint_{B_a^+} \frac{d\nu_\alpha(s, y)}{(t^2 + (r-s)^2 + (x-y)^2)^{\alpha+\frac{3}{2}}} \\ &= \frac{2^{\alpha+\frac{3}{2}}\Gamma(\alpha+2)\|f\|_{\infty, \nu_\alpha}}{\sqrt{\pi}} \iint_{B_a^+} \frac{d\nu_\alpha(s, y)}{(t^2 + \|(r, x) - (s, y)\|^2)^{\alpha+\frac{3}{2}}}. \end{aligned}$$

For every $(r, x) \in \mathbb{R}^2$ such that $r^2 + x^2 \geq 4a^2$ and for every $(s, y) \in B_a^+$; we have

$$\|(r, x) - (s, y)\| \geq \|(r, x)\| - \|(s, y)\| = \sqrt{r^2 + x^2} - \sqrt{s^2 + y^2} \geq \frac{1}{2}\|(r, x)\|.$$

This implies that for every $(r, x, t) \in \mathbb{R}^2 \times]0, +\infty[$; $r^2 + x^2 \geq 4a^2$,

$$\begin{aligned} |\mathcal{U}(f)(r, x, t)| &\leq \frac{2^{3\alpha+\frac{9}{2}}\Gamma(\alpha+2)}{\sqrt{\pi}} \frac{\|f\|_{\infty, \nu_\alpha}}{(t^2 + r^2 + x^2)^{\alpha+\frac{3}{2}}} \nu_\alpha(B_a^+) \\ &= \frac{a^{2\alpha+3}2^{2\alpha+4}\Gamma(\alpha+2)}{(2\alpha+3)\Gamma(\alpha+\frac{3}{2})\sqrt{\pi}} \|f\|_{\infty, \nu_\alpha} \frac{1}{(t^2 + r^2 + x^2)^{\alpha+\frac{3}{2}}}. \end{aligned}$$

From relation (4.31); we have

$$\begin{aligned} \frac{\partial}{\partial r}(\tau_{(r,-x)}(p_t)(s,y)) &= \frac{2^{\alpha+\frac{3}{2}}\Gamma(\alpha+2)}{\sqrt{\pi}} \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}(-2(\alpha+2)) \\ &\quad \times \int_0^\pi \frac{t(r+s\cos\theta)\sin^{2\alpha}\theta d\theta}{(t^2+(r^2+s^2+2rs\cos\theta)+(x-y)^2)^{\alpha+3}}, \end{aligned}$$

and consequently,

$$\begin{aligned} \left| \frac{\partial}{\partial r}(\tau_{(r,-x)}(p_t)(s,y)) \right| &\leq \frac{2^{\alpha+\frac{5}{2}}\Gamma(\alpha+3)}{\sqrt{\pi}} \frac{t(r+s)}{(t^2+(r-s)^2+(x-y)^2)^{\alpha+3}} \\ &\leq \frac{2^{\alpha+\frac{5}{2}}\Gamma(\alpha+3)}{\sqrt{\pi}} \frac{r+s}{(t^2+(r-s)^2+(x-y)^2)^{\alpha+\frac{5}{2}}}. \end{aligned}$$

Hence,

$$\begin{aligned} \left| \frac{\partial}{\partial r}(\mathcal{U}(f))(r,s,t) \right| &\leq \iint_{B_a^+} \left| \frac{\partial}{\partial r}(\tau_{(r,-x)}(p_t)(s,y)) \right| |f(s,y)| d\nu_\alpha(s,y) \\ &\leq \frac{2^{\alpha+\frac{5}{2}}\Gamma(\alpha+3)}{\sqrt{\pi}} \|f\|_{\infty,\nu_\alpha} \iint_{B_a^+} \frac{(r+s)d\nu_\alpha(s,y)}{(t^2+(r-s)^2+(x-y)^2)^{\alpha+\frac{5}{2}}}. \end{aligned}$$

But, for every (r,x) ; $r^2+x^2 \geq 4a^2$ and every $(s,y) \in B_a^+$; we have

$$\frac{r+s}{(t^2+(r-s)^2+(x-y)^2)^{\alpha+\frac{5}{2}}} \leq \frac{2\sqrt{r^2+x^2}}{(t^2+\frac{1}{4}(r^2+x^2))^{\alpha+\frac{5}{2}}} \leq \frac{2^{2\alpha+6}}{(t^2+r^2+x^2)^{\alpha+2}}.$$

This implies that

$$\left| \frac{\partial}{\partial r}(\mathcal{U}(f))(r,s,t) \right| \leq \frac{2^{3\alpha+\frac{17}{2}}\Gamma(\alpha+3)}{\sqrt{\pi}} \|f\|_{\infty,\nu_\alpha} \frac{1}{(t^2+r^2+x^2)^{\alpha+2}} \nu_\alpha(B_a^+).$$

Then, the result follows from the fact that

$$\nu_\alpha(B_a^+) = \frac{a^{2\alpha+3}}{(2\alpha+3)2^{\alpha+\frac{1}{2}}\Gamma(\alpha+\frac{3}{2})}.$$

We get the result (4.30) as the same way as the precedent inequality.

THEOREM 4.3

Let Δ_α be the partial differential operator defined by

$$\Delta_\alpha = \Lambda_\alpha + \frac{\partial^2}{\partial t^2},$$

where Λ_α is given by relation (3.18). Then, for every non negative function $f \in \mathcal{D}_e(\mathbb{R}^2)$ and every $p \in]1, 2]$, we have

$$\int_0^\infty \int_0^\infty \int_{\mathbb{R}} \Delta_\alpha((\mathcal{U}(f))^p)(r, x, t) d\nu_\alpha(r, x) t dt = \|f\|_{p, \nu_\alpha}^p. \quad (4.33)$$

Proof. Let f be a non negative function, $f \in \mathcal{D}_e(\mathbb{R}^2)$. Then $\mathcal{U}(f)$ is a positive function and from Proposition 3.8,

$$\Delta_\alpha(\mathcal{U}(f)) = 0.$$

Moreover; we have

$$\Delta_\alpha((\mathcal{U}(f))^p) = p(p-1)(\mathcal{U}(f))^{p-2} |\nabla(\mathcal{U}(f))|^2 \geq 0, \quad (4.34)$$

where

$$\nabla(\mathcal{U}(f)) = \left(\frac{\partial}{\partial r}(\mathcal{U}(f)), \frac{\partial}{\partial x}(\mathcal{U}(f)), \frac{\partial}{\partial t}(\mathcal{U}(f)) \right).$$

Then, we have

$$\begin{aligned} & \int_0^\infty \int_0^\infty \int_{\mathbb{R}} \Delta_\alpha((\mathcal{U}(f))^p)(r, x, t) d\nu_\alpha(r, x) t dt \\ &= \lim_{A \rightarrow +\infty} \int_0^A \int_0^A \int_{-A}^A \left(\Lambda_\alpha((\mathcal{U}(f))^p)(r, x, t) + \frac{\partial^2}{\partial t^2}((\mathcal{U}(f))^p)(r, x, t) \right) d\nu_\alpha(r, x) t dt. \end{aligned}$$

From relation (4.27); we deduce that for every $(r, x, t) \in \mathbb{R}^2 \times]0, +\infty[$ and for every $k \in \mathbb{N}$; we have

$$\left| \frac{\partial^k}{\partial t^k}(\mathcal{U}(f))(r, x, t) \right| \leq \int_0^\infty \int_{\mathbb{R}} (\mu^2 + \lambda^2)^{\frac{k}{2}} |\widetilde{\mathcal{F}_\alpha}(f)(\mu, \lambda)| d\nu_\alpha(\mu, \lambda) < +\infty.$$

It follows that, the function

$$\begin{aligned} & \frac{\partial^2}{\partial t^2}((\mathcal{U}(f))^p) \\ &= p(p-1)(\mathcal{U}(f))^{p-2} \left(\frac{\partial}{\partial t}(\mathcal{U}(f)) \right)^2 + p(\mathcal{U}(f))^{p-1} \frac{\partial^2(\mathcal{U}(f))}{\partial t^2} \end{aligned}$$

is bounded on $[0, A] \times [-A, A] \times [0, A]$.

As the same way; the function

$$\Lambda_\alpha((\mathcal{U}(f))^p) = \frac{\partial^2}{\partial r^2}((\mathcal{U}(f))^p) + \frac{2\alpha+1}{r} \frac{\partial}{\partial r}((\mathcal{U}(f))^p) + \frac{\partial^2}{\partial x^2}((\mathcal{U}(f))^p)$$

is bounded on $[0, A] \times [-A, A] \times [0, A]$.

Then, by Fubini's theorem; we get

$$\int_0^A \int_0^A \int_{-A}^A \Delta_\alpha((\mathcal{W}(f))^p)(r, x, t) d\nu_\alpha(r, x) t dt = I_1(A) + I_2(A) + I_3(A), \quad (4.35)$$

where

$$\begin{aligned} I_1(A) &= C_\alpha \int_0^A \int_{-A}^A \left(\int_0^A \frac{\partial}{\partial r} \left[r^{2\alpha+1} \frac{\partial}{\partial r} ((\mathcal{W}(f))^p) \right] (r, x, t) dr \right) dx t dt, \\ I_2(A) &= C_\alpha \int_0^A \int_0^A \left(\int_{-A}^A \frac{\partial^2}{\partial x^2} [(\mathcal{W}(f))^p](r, x, t) dx \right) r^{2\alpha+1} dr t dt, \\ I_3(A) &= \int_0^A \int_{-A}^A \left(\int_0^A \left(\frac{\partial}{\partial t} \right)^2 [(\mathcal{W}(f))^p](r, x, t) t dt \right) d\nu_\alpha(r, x), \end{aligned}$$

with $C_\alpha = \frac{1}{2^\alpha \Gamma(\alpha+1) \sqrt{2\pi}}$.

Now,

$$I_1(A) = p C_\alpha \int_0^A \int_{-A}^A A^{2\alpha+1} \frac{\partial}{\partial r} (\mathcal{W}(f))(A, x, t) (\mathcal{W}(f))^{p-1}(A, x, t) dx t dt.$$

Let $a > 0$ such that $\text{supp}(f) \subset B_a$ and let $A \geq 2a$. By relations (4.28) and (4.29), we have

$$|I_1(A)| \leq \frac{C_1 A^{2\alpha+4}}{A^{(2\alpha+3)(p-1)} A^{2\alpha+4}} = \frac{C_1}{A^{(2\alpha+3)(p-1)}};$$

which involves that

$$\lim_{A \rightarrow +\infty} I_1(A) = 0. \quad (4.36)$$

As the same way;

$$\begin{aligned} I_2(A) &= p C_\alpha \int_0^A \int_0^A \left[\frac{\partial}{\partial x} (\mathcal{W}(f))(r, A, t) (\mathcal{W}(f))^{p-1}(r, A, t) \right. \\ &\quad \left. - \frac{\partial}{\partial x} (\mathcal{W}(f))(r, -A, t) (\mathcal{W}(f))^{p-1}(r, -A, t) \right] r^{2\alpha+1} dr t dt, \end{aligned}$$

and by relations (4.28) and (4.30); we obtain

$$|I_2(A)| \leq \frac{C_2 A^{2\alpha+4}}{A^{(2\alpha+3)(p-1)} A^{2\alpha+4}} = \frac{C_2}{A^{(2\alpha+3)(p-1)}};$$

so,

$$\lim_{A \rightarrow +\infty} I_2(A) = 0. \quad (4.37)$$

Let us checking the integral $I_3(A)$. We have

$$\begin{aligned} & \int_0^A \left(\frac{\partial}{\partial t} \right) [(\mathcal{W}(f))^p](r, x, t) t \, dt \\ &= pA \frac{\partial}{\partial t} (\mathcal{W}(f))(r, x, A) (\mathcal{W}(f))^{p-1}(r, x, A) - (\mathcal{W}(f))^p(r, x, A) + f^p(r, x). \end{aligned}$$

However,

$$\begin{aligned} \int_0^A \int_{-A}^A \mathcal{W}^p(f)(r, x, A) \, d\nu_\alpha(r, x) &\leq \int_0^\infty \int_{\mathbb{R}} \mathcal{W}^p(f)(r, x, A) \, d\nu_\alpha(r, x) \\ &= \|p_A * f\|_{p, \nu_\alpha}^p \leq \|p_A\|_{p, \nu_\alpha}^p \|f\|_{1, \nu_\alpha}^p. \end{aligned}$$

By a simple computation and using Lemma 3.7, we deduce that

$$\lim_{A \rightarrow +\infty} \|p_A\|_{p, \nu_\alpha}^p = 0,$$

and then

$$\lim_{A \rightarrow +\infty} \int_0^A \int_{-A}^A \mathcal{W}^p(f)(r, x, A) \, d\nu_\alpha(r, x) = 0.$$

On the other hand, by relation (4.26), we have

$$pA \int_0^A \int_{-A}^A \left| \frac{\partial}{\partial t} (\mathcal{W}(f))(r, x, A) \right| (\mathcal{W}(f))^{p-1}(r, x, A) \, d\nu_\alpha(r, x) \leq \frac{C_3}{A^{(2\alpha+3)(p-1)}},$$

which implies that

$$\lim_{A \rightarrow +\infty} pA \int_0^A \int_{-A}^A \frac{\partial}{\partial t} (\mathcal{W}(f))(r, x, A) (\mathcal{W}(f))^{p-1}(r, x, A) \, d\nu_\alpha(r, x) = 0.$$

hence,

$$\lim_{A \rightarrow +\infty} I_3(A) = \int_0^\infty \int_{\mathbb{R}} (f(r, x))^p \, d\nu_\alpha(r, x) = \|f\|_{p, \nu_\alpha}^p. \quad (4.38)$$

Then, the desired result follows from relations (4.35), (4.36), (4.37) and (4.38).

DEFINITION 4.4

The Littlewood-Paley g -function associated with the Riemann-Liouville operator is defined for $f \in \mathcal{D}_e(\mathbb{R}^2)$ by

$$g(f)(r, x) = \left(\int_0^\infty |\nabla(\mathcal{W}(f))(r, x, t)|^2 t \, dt \right)^{\frac{1}{2}}.$$

Let $\mathcal{C}_{c,e}(\mathbb{R}^2)$ be the space of continuous functions on $\mathbb{R}^2$, even with respect to the first variable and with compact support.

In the following, we need the coming result.

LEMMA 4.5

Let g be a non negative function, $g \in \mathcal{C}_{c,e}(\mathbb{R}^2)$; $\text{supp}(g) \subset B_a$. For every ε ; $0 < \varepsilon < 1$, there exists a non negative function $f \in \mathcal{D}_e(\mathbb{R}^2)$ such that

$$\forall (r, x) \in \mathbb{R}^2; \quad 0 \leq f(r, x) - g(r, x) \leq \varepsilon,$$

with $\text{supp}(f) \subset B_{a+2}$.

Proof. It is well known that for every non negative function h ; $h \in \mathcal{C}_{c,e}(\mathbb{R}^2)$, $\text{supp}(h) \subset B_a$ and for every $\eta > 0$, there is a non negative function $f \in \mathcal{D}_e(\mathbb{R}^2)$, $\text{supp}(f) \subset B_{a+1}$ such that

$$\forall (r, x) \in \mathbb{R}^2; \quad -\eta \leq f(r, x) - h(r, x) \leq \eta. \quad (4.39)$$

Let g be a non negative function in $\mathcal{C}_{c,e}(\mathbb{R}^2)$, $\text{supp}(g) \subset B_a$ and let $\varepsilon \in \mathbb{R}$, $0 < \varepsilon < 1$. We define the function θ by

$$\theta(r, x) = \begin{cases} g(r, x) + \frac{\varepsilon}{2}, & \text{if } r^2 + x^2 \leq a^2; \\ -\sqrt{r^2 + x^2} + a + \frac{\varepsilon}{2}, & \text{if } a^2 \leq r^2 + x^2 \leq \left(a + \frac{\varepsilon}{2}\right)^2; \\ 0, & \text{if } r^2 + x^2 \geq \left(a + \frac{\varepsilon}{2}\right)^2. \end{cases}$$

Then θ is a non negative function, θ belongs to the space $\mathcal{C}_{c,e}(\mathbb{R}^2)$ and $\text{supp}(\theta) \subset B_{a+1}$.

From relation (4.39), there exists a non negative function $f \in \mathcal{D}_e(\mathbb{R}^2)$ such that $\text{supp}(f) \subset B_{a+2}$, and

$$\forall (r, x) \in \mathbb{R}^2; \quad -\frac{\varepsilon}{4} \leq f(r, x) - \theta(r, x) \leq \frac{\varepsilon}{4}.$$

Thus, the function f satisfies

$$\forall (r, x) \in \mathbb{R}^2; \quad 0 \leq f(r, x) - g(r, x) \leq \varepsilon,$$

with $\text{supp}(f) \subset B_{a+2}$.

PROPOSITION 4.6

For every $p \in]1, 2]$, and for every function $f \in \mathcal{D}_e(\mathbb{R}^2)$, the function $g(f)$ belongs to the space $L^p(d\nu_\alpha)$ and we have

$$\|g(f)\|_{p, \nu_\alpha} \leq 2^{\frac{2-p}{2}} \left(\frac{p}{p-1} \right)^{\frac{1}{p}} \|f\|_{p, \nu_\alpha}.$$

Proof. Let f be a non negative function; $f \in \mathcal{D}_e(\mathbb{R}^2)$. From relation (4.34), we have

$$|\nabla(\mathcal{U}(f))(r, x, t)|^2 = \frac{1}{p(p-1)} (\mathcal{U}(f))^{2-p}(r, x, t) \Delta_\alpha(\mathcal{U}^p(f))(r, x, t).$$

For $p = 2$ and using relation (4.33), we obtain

$$\begin{aligned} \int_0^\infty \int_{\mathbb{R}} g^2(f)(r, x) d\nu_\alpha(r, x) &= \int_0^\infty \int_{\mathbb{R}} \left(\int_0^\infty |\nabla(\mathcal{U}(f))(r, x, t)|^2 t dt \right) d\nu_\alpha(r, x) \\ &= \frac{1}{2} \int_0^\infty \int_{\mathbb{R}} \int_0^\infty \Delta_\alpha(\mathcal{U}^2(f))(r, x, t) t dt d\nu_\alpha(r, x) \\ &= \frac{1}{2} \int_0^\infty \int_{\mathbb{R}} (f(r, x))^2 d\nu_\alpha(r, x). \end{aligned}$$

This means that

$$\|g(f)\|_{2, \nu_\alpha} = \frac{1}{\sqrt{2}} \|f\|_{2, \nu_\alpha}.$$

For $p \in]1, 2[$, we have

$$\begin{aligned} \int_0^\infty \int_{\mathbb{R}} (g(f))^p(r, x) d\nu_\alpha(r, x) &= \int_0^\infty \int_{\mathbb{R}} \left(\int_0^\infty |\nabla(\mathcal{U}(f))(r, x, t)|^2 t dt \right)^{\frac{p}{2}} d\nu_\alpha(r, x) \\ &= \left(\frac{1}{p(p-1)} \right)^{\frac{p}{2}} \int_0^\infty \int_{\mathbb{R}} \left(\int_0^\infty \mathcal{U}^{2-p}(f)(r, x, t) \Delta_\alpha(\mathcal{U}^p(f))(r, x, t) t dt \right)^{\frac{p}{2}} d\nu_\alpha(r, x) \\ &\leq \left(\frac{1}{p(p-1)} \right)^{\frac{p}{2}} \int_0^\infty \int_{\mathbb{R}} (f^*(r, x))^{(2-p)\frac{p}{2}} \left(\int_0^\infty \Delta_\alpha(\mathcal{U}^p(f))(r, x, t) t dt \right)^{\frac{p}{2}} d\nu_\alpha(r, x), \end{aligned}$$

where f^* is the maximal function defined by relation (3.24).

Using Hölder's inequality and relation (4.33), we get

$$\begin{aligned} \int_0^\infty \int_{\mathbb{R}^n} (g(f))^p(r, x) d\nu_\alpha(r, x) &\leq \left(\frac{1}{p(p-1)} \right)^{\frac{p}{2}} \|f^*\|_{p, \nu_\alpha}^{p \frac{(2-p)}{2}} \left(\int_0^\infty \int_{\mathbb{R}} \int_0^\infty \Delta_\alpha(\mathcal{U}^p(f))(r, x, t) t dt d\nu_\alpha(r, x) \right)^{\frac{p}{2}} \\ &= \left(\frac{1}{p(p-1)} \right)^{\frac{p}{2}} \|f^*\|_{p, \nu_\alpha}^{p \frac{(2-p)}{2}} \|f\|_{p, \nu_\alpha}^{p \frac{p}{2}}, \end{aligned}$$

and by means of relation (3.25),

$$\int_0^\infty \int_{\mathbb{R}^n} (g(f))^p(r, x) d\nu_\alpha(r, x) \leq \left(\frac{1}{p(p-1)} \right)^{\frac{p}{2}} \left(2 \left(\frac{p}{p-1} \right)^{\frac{1}{p}} \right)^{p \frac{(2-p)}{2}} \|f\|_{p, \nu_\alpha}^p,$$

in other words,

$$\|g(f)\|_{p, \nu_\alpha} \leq \frac{2^{\frac{2-p}{2}}}{p} \left(\frac{p}{p-1} \right)^{\frac{1}{p}} \|f\|_{p, \nu_\alpha}. \quad (4.40)$$

Let $f \in \mathcal{D}_e(\mathbb{R}^2)$; $\text{supp}(f) \subset B_a$ and let $f^+ = \frac{f+|f|}{2}$, $f^- = \frac{-f+|f|}{2}$. Then f^+ is a non negative function, $f^+ \in \mathcal{C}_{c,e}(\mathbb{R}^2)$. From Lemma 4.5, for every $\varepsilon \in \mathbb{R}$, $0 < \varepsilon < 1$, there is a non negative function $h_1 \in \mathcal{D}_e(\mathbb{R}^2)$, $\text{supp}(h_1) \subset B_{a+2}$ and

$$\forall (r, x) \in \mathbb{R}^2; \quad 0 \leq h_1(r, x) - f^+(r, x) \leq \varepsilon. \quad (4.41)$$

Now, the function

$$h_2 = h_1 - f = h_1 - f^+ + f^-$$

is non negative, belongs to the space $\mathcal{D}_e(\mathbb{R}^2)$ with $\text{supp}(h_2) \subset B_{a+2}$. Moreover

$$\forall (r, x) \in \mathbb{R}^2; \quad 0 \leq h_2(r, x) - f^-(r, x) = h_1(r, x) - f^+(r, x) \leq \varepsilon,$$

and we have $f = h_1 - h_2$.

Since the mapping $f \mapsto g(f)$ is sub-linear in the sense that $g(f_1 + f_2) \leq g(f_1) + g(f_2)$; we deduce that

$$g(f) \leq g(h_1) + g(h_2),$$

and applying inequality (4.40), we get

$$\|g(f)\|_{p, \nu_\alpha} \leq \|g(h_1)\|_{p, \nu_\alpha} + \|g(h_2)\|_{p, \nu_\alpha} \leq \frac{2^{\frac{2-p}{2}}}{p} \left(\frac{p}{p-1} \right)^{\frac{1}{p}} (\|h_1\|_{p, \nu_\alpha} + \|h_2\|_{p, \nu_\alpha}).$$

On the other hand, from relation (4.41), we obtain

$$\begin{aligned} \|h_1\|_{p, \nu_\alpha} &= \left(\iint_{B_{a+2}^+} (h_1(r, x))^p d\nu_\alpha(r, x) \right)^{\frac{1}{p}} \\ &\leq \left(\iint_{B_{a+2}^+} (f^+(r, x))^p d\nu_\alpha(r, x) \right)^{\frac{1}{p}} + \varepsilon (\nu_\alpha(B_{a+2}^+))^{\frac{1}{p}} \\ &\leq \|f\|_{p, \nu_\alpha} + \varepsilon (\nu_\alpha(B_{a+2}^+))^{\frac{1}{p}}. \end{aligned}$$

As the same way,

$$\|h_2\|_{p, \nu_\alpha} \leq \|f\|_{p, \nu_\alpha} + \varepsilon (\nu_\alpha(B_{a+2}^+))^{\frac{1}{p}}.$$

This means that for every $\varepsilon \in \mathbb{R}$, $0 < \varepsilon < 1$,

$$\|g(f)\|_{p,\nu_\alpha} \leq 2 \frac{2^{\frac{2-p}{2}}}{p} \left(\frac{p}{p-1} \right)^{\frac{1}{p}} (\|f\|_{p,\nu_\alpha} + \varepsilon (\nu_\alpha(B_{a+2}^+))^{\frac{1}{p}}),$$

and consequently,

$$\|g(f)\|_{p,\nu_\alpha} \leq 2 \frac{2^{\frac{2-p}{2}}}{p} \left(\frac{p}{p-1} \right)^{\frac{1}{p}} \|f\|_{p,\nu_\alpha}.$$

The precedent Proposition allows us to prove the followings Theorem, that is the main result of this paper.

THEOREM 4.7

For every $p \in]1, 2]$; the mapping $f \mapsto g(f)$ can be extended to the space $L^p(d\nu_\alpha)$ and for every $f \in L^p(d\nu_\alpha)$, we have

$$\|g(f)\|_{p,\nu_\alpha} \leq 2 \frac{2^{\frac{2-p}{2}}}{2} \left(\frac{p}{p-1} \right)^{\frac{1}{p}} \|f\|_{p,\nu_\alpha}.$$

Proof. Let $f \in L^p(d\nu_\alpha)$, then there exists a sequence $(f_k)_k \subset \mathcal{D}_e(\mathbb{R}^2)$ such that

$$\lim_{k \rightarrow +\infty} \|f_k - f\|_{p,\nu_\alpha} = 0.$$

Since the mapping $f \mapsto g(f)$ is sub-linear; then for every $(k, l) \in \mathbb{N}^2$; we have

$$\begin{aligned} \|g(f_{k+l}) - g(f_k)\|_{p,\nu_\alpha} &\leq \|g(f_{k+l} - f_k)\|_{p,\nu_\alpha} \\ &\leq 2 \frac{2^{\frac{2-p}{2}}}{2} \left(\frac{p}{p-1} \right)^{\frac{1}{p}} \|f_{k+l} - f_k\|_{p,\nu_\alpha}. \end{aligned}$$

Consequently, the sequence $(g(f_k))_k$ is a Cauchy one in $L^p(d\nu_\alpha)$. We put

$$g(f) = \lim_{k \rightarrow +\infty} g(f_k)$$

in $L^p(d\nu_\alpha)$.

It is clear that $g(f)$ is independent of the choice of the sequence $(f_k)_k$ and we have

$$\|g(f)\|_{p,\nu_\alpha} = \lim_{k \rightarrow +\infty} \|g(f_k)\|_{p,\nu_\alpha} \leq 2 \frac{2^{\frac{2-p}{2}}}{p} \left(\frac{p}{p-1} \right)^{\frac{1}{p}} \|f\|_{p,\nu_\alpha}.$$

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