

*Alfred Witkowski***On Levinson's inequality**

Abstract. We give a very simple proof of the classical Levinson inequality and generalize the result by Mercer.

1. Introduction

In 1964 Norman Levinson ([3]) used the Taylor expansion to prove the following inequality.

THEOREM 1.1 ([3])

Suppose that $f: [0, c] \rightarrow \mathbb{R}$ has a nonnegative third derivative, for $i = 1, \dots, n$ $p_i > 0$, $0 \leq x_i \leq \frac{c}{2}$, $y_i = c - x_i$ and $\sum_{i=1}^n p_i = 1$, then the inequality

$$\sum_{i=1}^n p_i f(x_i) - f(\bar{x}) \leq \sum_{i=1}^n p_i f(y_i) - f(\bar{y}) \quad (1)$$

holds. Here $\bar{x} = \sum_{i=1}^n p_i x_i$ and $\bar{y} = \sum_{i=1}^n p_i y_i$ denote the weighted arithmetic means.

The same year Tiberiu Popoviciu generalized it by showing that for (1) to hold it is enough that f be 3-convex ([6]), and Peter S. Bullen ([2]) gave an alternative proof using mathematical induction. There are also other versions of this inequality, see e.g. [5]. By rescaling axes, the classic Levinson inequality can be restated in the following way.

THEOREM 1.2 ([2])

If $f: [a, b] \rightarrow \mathbb{R}$ is 3-convex, $a \leq x_i$, $y_i \leq b$, $x_i + y_i = c$, $p_i > 0$ for $i = 1, \dots, n$, $\sum_{i=1}^n p_i = 1$ and

$$\max(x_1, \dots, x_n) \leq \min(y_1, \dots, y_n), \quad (2)$$

then (1) holds.

As we can see, both versions assume that x 's and y 's add up to the same number. Recently, Mercer made a significant improvement in [4].

THEOREM 1.3

If $f: [a, b] \rightarrow \mathbb{R}$ satisfies $f''' \geq 0$, $p_i > 0$ for $i = 1, \dots, n$, $\sum_{i=1}^n p_i = 1$, and $a \leq x_i$, $y_i \leq b$ are such that (2) holds and

$$\sum_{i=1}^n p_i (x_i - \bar{x})^2 = \sum_{i=1}^n p_i (y_i - \bar{y})^2, \quad (3)$$

then (1) holds.

It is natural to try to get similar result for 3-convex function. In this paper we shall give a very simple proof of Theorem 1.2 and generalize the Mercer's result in the following way.

THEOREM 1.4

Let I be an open subinterval of $\mathbb{R}$ (bounded or unbounded), $f: I \rightarrow \mathbb{R}$ be a 3-convex function and $X, Y: (\Omega, \mu) \rightarrow I$ be two random variables satisfying

V1) $\mathbf{E}[X^2], \mathbf{E}[Y^2], \mathbf{E}[f(X)], \mathbf{E}[f(Y)], \mathbf{E}[f'(X)], \mathbf{E}[f'(Y)], \mathbf{E}[Xf'(X)], \mathbf{E}[Yf'(Y)]$ are finite,

V2) $\text{ess sup } X \leq \text{ess inf } Y$,

V3) $\mathbf{Var } X = \mathbf{Var } Y$,

then

$$\mathbf{E}[f(X)] - f(\mathbf{E}[X]) \leq \mathbf{E}[f(Y)] - f(\mathbf{E}[Y]). \quad (4)$$

We also provide some conditions for (4) to hold if the condition V3) is not satisfied.

2. Definitions and basic properties

DEFINITION 2.1

For a function $f: I \rightarrow \mathbb{R}$ its n^{th} divided difference is defined inductively by the formula

$$[x_0, \dots, x_n; f] = \begin{cases} f(x_0) & \text{if } n = 0, \\ \frac{[x_0, \dots, x_{n-2}, x_n; f] - [x_0, \dots, x_{n-2}, x_{n-1}; f]}{x_n - x_{n-1}} & \text{if } n > 0, \end{cases}$$

where x_i 's are pairwise distinct.

DEFINITION 2.2

A function $f: I \rightarrow \mathbb{R}$ is called n -convex if

$$[x_0, \dots, x_n; f] \geq 0$$

for any choice of arguments, and n -concave if the inverse inequality holds.

The 1-convex (resp. concave) functions are the increasing (resp. decreasing) functions, while the 2-convex (resp. concave) functions are convex (resp. concave) functions, i.e. the functions satisfying the inequality

$$f(tx + (1-t)y) \leq (\text{resp. } \geq) tf(x) + (1-t)f(y)$$

for all $x, y \in I$ and $0 < t < 1$.

The following property follows immediately from definition.

PROPERTY 2.3

The function g is convex if and only if its divided difference $h(x, y) = \frac{g(x)-g(y)}{x-y}$, $x \neq y$ increases in both variables.

This implies immediately

PROPERTY 2.4

If g is convex, then its right and left derivatives exist in the interior of its domain and for $x < y$ the inequalities

$$g'_-(x) \leq g'_+(x) \leq g'_-(y) \leq g'_+(y)$$

hold.

Another useful property of symmetric sum follows also from Property 2.3.

PROPERTY 2.5

If g is convex, then the symmetric sum $g(a+x) + g(a-x)$ increases for $x > 0$.

Boas and Widder gave the following characterization of n -convex functions.

PROPERTY 2.6 ([1])

If a function g is n -convex, then it is $n-2$ times differentiable and $g^{(n-2)}$ is convex.

At the end of this section let us remind the Leibniz Rule.

LEMMA 2.7 (DIFFERENTIATION UNDER INTEGRAL)

Let I be an open subinterval of $\mathbb{R}$ and (Ω, μ) be a measure space. Suppose a function $h: I \times \Omega \rightarrow \mathbb{R}$ satisfies the following conditions:

- L1) *for every $t \in I$ the function $h(t, \cdot)$ is integrable,*
- L2) *for every $t \in I$ the derivative h_t exists μ a.e.*
- L3) *there exists an integrable function $\theta: \Omega \rightarrow \mathbb{R}$ such that for every t $|h_t(t, \omega)| \leq \theta(\omega)$.*

Then for every $t \in I$

$$\frac{d}{dt} \int_{\Omega} h(t, \omega) \mu(d\omega) = \int_{\Omega} h_t(t, \omega) \mu(d\omega).$$

3. Simple proof of Theorem 1.2

We can rewrite (1) as

$$f\left(c - \sum_{i=1}^n p_i x_i\right) - f\left(\sum_{i=1}^n p_i x_i\right) \leq \sum_{i=1}^n p_i f(c - x_i) - \sum_{i=1}^n p_i f(x_i), \quad (5)$$

which is the Jensen inequality for the function $g(x) = f(c-x) - f(x)$. The function g is differentiable by Property 2.6. To show its convexity, note that

$$g'(x) = -[f'(c-x) + f'(x)] = -\left[f'\left(\frac{c}{2} + \left(\frac{c}{2} - x\right)\right) + f'\left(\frac{c}{2} - \left(\frac{c}{2} - x\right)\right)\right].$$

As x increases, $\frac{c}{2} - x$ decreases, and by Property 2.5, the expression in square brackets decreases. Thus g' increases, so g is convex and we are done.

4. Proof of Theorem 1.4

For $0 \leq t \leq 1$ define two new random variables

$$X_t = tX + (1-t)\mathbf{E}[X] \quad \text{and} \quad Y_t = tY + (1-t)\mathbf{E}[Y]$$

and consider the function

$$U(t, \omega) = f(Y_t(\omega)) - f(\mathbf{E}[Y]) - [f(X_t(\omega)) - f(\mathbf{E}[X])]. \quad (6)$$

Our goal will be to show that $V(t) = \int_{\Omega} U(t, \omega) \mu(d\omega)$ is a nonnegative, increasing, convex function of t .

Let d be any number satisfying $\text{ess sup } X \leq d \leq \text{ess inf } Y$. The proof is split into several steps.

4.1. Behavior near the endpoints of the interval $I = (a, b)$

The functions f and f' are continuous, thus bounded on every compact subinterval of (a, b) . In order to obtain some global bounds we need to know how they behave near the ends.

REMARK 4.1

Since a function g is convex or concave in the interval (a, b) , its monotonicity cannot change more than once. If it is not bounded near a , we conclude that there exists $a' > a$ such that g preserves sign and $|g|$ decreases in (a, a') . Similarly, or g is bounded near b or there exists $b' < b$ such that g preserves sign and $|g|$ increases in (b', b) .

Since f is 3-convex, its derivative is convex and its monotonicity changes at most once. Thus the convexity of f changes at most once, and consequently the Remark 4.1 applies to both f and f' .

4.2. Absolute integrability

Let g be any of f , f' or $\text{Id}_{\mathbb{R}} \cdot f'$, where $\text{Id}_{\mathbb{R}}$ is the identity function on $\mathbb{R}$. It

follows from the results of the subsection 4.1. that g is bounded from above or from below on $(a, d]$. Since

$$\int_{\Omega} g(X) \mu(d\omega) = \int_{\{\omega: g(X(\omega)) > 0\}} g(X) \mu(d\omega) + \int_{\{\omega: g(X(\omega)) < 0\}} g(X) \mu(d\omega)$$

we conclude that two integrals in this formula are finite, thus so is the third one, therefore

$$\int_{\Omega} |g(X)| \mu(d\omega) = \int_{\{\omega: g(X(\omega)) > 0\}} g(X) \mu(d\omega) - \int_{\{\omega: g(X(\omega)) < 0\}} g(X) \mu(d\omega)$$

is also finite.

The same holds for $g(Y)$.

4.3. Leibniz Rule

We shall show now that the function U defined by formula (6) satisfies the Leibniz Rule. Consider $f(X_t)$. If f is bounded near a , then clearly the assumption L1) is satisfied. If not, then choose a' as in step 4.1. in such a way that $a' < \mathbf{E}[X]$. Let $M = \max_{x \in [a', d]} |f(x)|$. Then we have

$$|f(X_t(\omega))| \leq \begin{cases} M & \text{if } X_t(\omega) \in [a', d], \\ |f(X(\omega))| & \text{otherwise, since } X_t(\omega) \geq X(\omega). \end{cases}$$

Thus $|f(X_t)| \leq \max(M, f(X))$ and we are done with L1).

By Property 2.6 condition L2) is also satisfied. To show that L3) holds note that

$$\left| \frac{d}{dt} f(X_t) \right| = |(X - \mathbf{E}[X])f'(X_t)| \leq |X| |f'(X_t)| + |\mathbf{E}[X]| |f'(X_t)|.$$

If f' is bounded, then the right-hand side is not greater than $M|X| + N$ for some $M, N > 0$, otherwise the same reasoning as above applied to f' gives the bound $\max(|Xf'(X)| + |\mathbf{E}[X]| |f'(X)|, (\max(|a'|, |d|) + |\mathbf{E}[X]|)M)$.

The same reasoning can be applied to $f(Y_t)$, so we can apply the Leibniz Rule to the function U .

4.4. Final step

Clearly, $U(0, \omega) = 0$, so $V(0) = 0$. The function U is differentiable in t for all ω and

$$U_t(t, \omega) = (Y(\omega) - \mathbf{E}[Y])f'(Y_t(\omega)) - (X(\omega) - \mathbf{E}[X])f'(X_t(\omega)). \quad (7)$$

Applying the Leibniz Rule we obtain

$$V'(0) = \int_{\Omega} U_t(0, \omega) \mu(d\omega) = 0.$$

In virtue of Property 2.6, the convexity of the derivative of a 3-convex functions implies that $f''_-(\text{ess sup } X) \leq f''_+(\text{ess inf } Y)$ ($f''_{\pm}$ denotes here the right and left derivatives of f'). Let A be an arbitrary number from the interval $[f''_-(\text{ess sup } X), f''_+(\text{ess inf } Y)]$.

We have

$$\begin{aligned}
\frac{U_t(u, \omega) - U_t(v, \omega)}{u - v} &= (Y(\omega) - \mathbf{E}[Y]) \frac{f'(Y_u(\omega)) - f'(Y_v(\omega))}{u - v} \\
&\quad - (X(\omega) - \mathbf{E}[X]) \frac{f'(X_u(\omega)) - f'(X_v(\omega))}{u - v} \\
&= (Y(\omega) - \mathbf{E}[Y])^2 \frac{f'(Y_u(\omega)) - f'(Y_v(\omega))}{Y_u(\omega) - Y_v(\omega)} \\
&\quad - (X(\omega) - \mathbf{E}[X])^2 \frac{f'(X_u(\omega)) - f'(X_v(\omega))}{X_u(\omega) - X_v(\omega)} \quad (8) \\
&= (Y(\omega) - \mathbf{E}[Y])^2 \left[\frac{f'(Y_u(\omega)) - f'(Y_v(\omega))}{Y_u(\omega) - Y_v(\omega)} - A \right] \\
&\quad + (X(\omega) - \mathbf{E}[X])^2 \left[A - \frac{f'(X_u(\omega)) - f'(X_v(\omega))}{X_u(\omega) - X_v(\omega)} \right] \\
&\quad + A[(Y(\omega) - \mathbf{E}[Y])^2 - (X(\omega) - \mathbf{E}[X])^2].
\end{aligned}$$

Property 2.3 applied to f' shows that the expressions in square brackets in (8) are nonnegative a.e., so

$$\frac{U_t(u, \omega) - U_t(v, \omega)}{u - v} \geq A[(Y(\omega) - \mathbf{E}[Y])^2 - (X(\omega) - \mathbf{E}[X])^2] \quad \text{a.e.}$$

Integrating and applying once more the Leibniz Rule one get

$$\frac{V'(u) - V'(v)}{u - v} \geq A(\mathbf{Var } Y - \mathbf{Var } X) = 0. \quad (9)$$

This means V' is increasing, and we conclude that V is nonnegative and convex in the unit interval. In particular $V(0) \leq V(1)$, which is the Levinson inequality.

5. Further results

Let us take a closer look at the proof of Theorem 1.4. We have some freedom in selection of A : this can be any number from the gap between $f''(X)$ and $f''(Y)$. The critical inequality (9) holds not only if $\mathbf{Var } Y = \mathbf{Var } X$ but also in case when $A(\mathbf{Var } Y - \mathbf{Var } X) \geq 0$. This means that if the gap between X 's and Y 's is in the area where the function f is strictly convex/concave (i.e. $A > (<) 0$), then the assumption V3) of Theorem 1.4 can be replaced by a weaker condition $\mathbf{Var } Y \geq (\leq) \mathbf{Var } X$.

If f changes convexity between X 's and Y 's, then we can choose $A = 0$ and the assumption V3) becomes obsolete. In fact, in this case the Levinson inequality follows immediately from the Jensen inequality, because the left-hand side of (4) is nonpositive, while its right-hand side is nonnegative. Thus we obtain the following version of Theorem 1.4.

THEOREM 5.1

Let I be an open subinterval of $\mathbb{R}$ (bounded or unbounded), $f: I \rightarrow \mathbb{R}$ be a 3-convex function and $X, Y: (\Omega, \mu) \rightarrow I$ be two random variables satisfying

- (1) $\mathbf{E}[X^2], \mathbf{E}[Y^2], \mathbf{E}[f(X)], \mathbf{E}[f(Y)], \mathbf{E}[f'(X)], \mathbf{E}[f'(Y)], \mathbf{E}[Xf'(X)], \mathbf{E}[Yf'(Y)]$ are finite,
- (2) $\text{ess sup } X \leq \text{ess inf } Y$,
- (3) $f''_+(\text{ess sup } X) > 0$ and $\mathbf{Var } X \leq \mathbf{Var } Y$, or $f''_-(\text{ess inf } Y) < 0$ and $\mathbf{Var } X \geq \mathbf{Var } Y$, or $f''_+(\text{ess sup } X) \leq 0 \leq f''_-(\text{ess inf } Y)$,

then

$$\mathbf{E}[f(X)] - f(\mathbf{E}[X]) \leq \mathbf{E}[f(Y)] - f(\mathbf{E}[Y])$$

If X and Y are discrete random variables, then we obtain the following corollary.

COROLLARY 5.2

Let $f: I \rightarrow \mathbb{R}$ be a 3-convex function, $p_1, \dots, p_m$ and $q_1, \dots, q_n$ be positive real numbers such that $p_1 + \dots + p_m = q_1 + \dots + q_n = 1$. For $x_1, \dots, x_m, y_1, \dots, y_n \in I$ such that

$$\max_{1 \leq i \leq m} x_i \leq \min_{1 \leq i \leq n} y_i, \quad (10)$$

let $\bar{x} = \sum_{i=1}^m p_i x_i$ and $\bar{y} = \sum_{i=1}^n q_i y_i$. If

$$f''_+(\max x_i) > 0 \quad \text{and} \quad \sum_{i=1}^m p_i (x_i - \bar{x})^2 \leq \sum_{i=1}^n q_i (y_i - \bar{y})^2,$$

or

$$f''_-(\min x_i) < 0 \quad \text{and} \quad \sum_{i=1}^m p_i (x_i - \bar{x})^2 \geq \sum_{i=1}^n q_i (y_i - \bar{y})^2,$$

or

$$f''_+(\max_{1 \leq i \leq m} x_i) \leq 0 \leq f''_-(\min_{1 \leq i \leq n} y_i),$$

then

$$\sum_{i=1}^m p_i f(x_i) - f(\bar{x}) \leq \sum_{i=1}^n q_i f(y_i) - f(\bar{y}).$$

The function V used in the proof of Theorem 1.4 is convex, so we can apply the Hermite-Hadamard inequality.

COROLLARY 5.3

If the assumptions of Theorem 1.4 or 5.1 are satisfied and $F(x) = \int f(x) dx$, then

$$\begin{aligned} 0 &\leq \mathbf{E}\left[f\left(\frac{Y + \mathbf{E}[Y]}{2}\right)\right] - f(\mathbf{E}[Y]) - \mathbf{E}\left[f\left(\frac{X + \mathbf{E}[X]}{2}\right)\right] + f(\mathbf{E}[X]) \\ &\leq \mathbf{E}\left[\frac{F(Y) - F(\mathbf{E}[Y])}{Y - \mathbf{E}[Y]}\right] - f(\mathbf{E}[Y]) - \mathbf{E}\left[\frac{F(X) - F(\mathbf{E}[X])}{X - \mathbf{E}[X]}\right] + f(\mathbf{E}[X]) \\ &\leq \frac{1}{2}(\mathbf{E}[f(Y)] - f(\mathbf{E}[Y]) - \mathbf{E}[f(X)] + f(\mathbf{E}[X])). \end{aligned}$$

Its discrete version reads as follows

COROLLARY 5.4

Under the assumptions of Corollary 5.2 the following inequalities are valid:

$$\begin{aligned} 0 &\leq \sum_{i=1}^n q_i f\left(\frac{y_i + \bar{y}}{2}\right) - f(\bar{y}) - \sum_{i=1}^m p_i f\left(\frac{x_i + \bar{x}}{2}\right) + f(\bar{x}) \\ &\leq \sum_{i=1}^n q_i \frac{\int_{\bar{y}}^{y_i} f(t) dt}{y_i - \bar{y}} - f(\bar{y}) - \sum_{i=1}^m p_i \frac{\int_{\bar{x}}^{x_i} f(t) dt}{x_i - \bar{x}} + f(\bar{x}) \\ &\leq \frac{1}{2} \left[\sum_{i=1}^n q_i f(y_i) - f(\bar{y}) - \sum_{i=1}^m p_i f(x_i) + f(\bar{x}) \right]. \end{aligned}$$

Note that the rightmost inequality can be rewritten in a nice symmetric form

$$\sum_{i=1}^m p_i \left(\frac{f(x_i) + f(\bar{x})}{2} - \frac{\int_{\bar{x}}^{x_i} f(t) dt}{x_i - \bar{x}} \right) \leq \sum_{i=1}^n q_i \left(\frac{f(y_i) + f(\bar{y})}{2} - \frac{\int_{\bar{y}}^{y_i} f(t) dt}{y_i - \bar{y}} \right), \quad (11)$$

while the leftmost inequality is

$$\sum_{i=1}^m p_i \left(\frac{\int_{\bar{x}}^{x_i} f(t) dt}{x_i - \bar{x}} - f\left(\frac{x_i + \bar{x}}{2}\right) \right) \leq \sum_{i=1}^n q_i \left(\frac{\int_{\bar{y}}^{y_i} f(t) dt}{y_i - \bar{y}} - f\left(\frac{y_i + \bar{y}}{2}\right) \right). \quad (12)$$

6. Applications

As an application we provide generalization and refinement of the famous Ky-Fan inequality: if $0 < x_i \leq \frac{1}{2}$ and $y_i = 1 - x_i$, then

$$\frac{A(y_1, \dots, y_i)}{A(x_1, \dots, x_i)} \leq \frac{G(y_1, \dots, y_i)}{G(x_1, \dots, x_i)}.$$

Levinson in [3] noticed, that this inequality (in weighted version) follows from his inequality by taking $f(x) = \log x$. The logarithmic function is concave, so if $\sum_{i=1}^m p_i (x_i - \bar{x})^2 \geq \sum_{i=1}^n q_i (y_i - \bar{y})^2$ and (10) holds, then Corollary 5.4 yields

$$\frac{\sum_{i=1}^n q_i y_i}{\sum_{i=1}^m p_i x_i} \leq \frac{\prod_{i=1}^n A^{q_i}(y_i, \bar{y})}{\prod_{i=1}^m A^{p_i}(x_i, \bar{x})} \leq \frac{\prod_{i=1}^n I^{q_i}(y_i, \bar{y})}{\prod_{i=1}^m I^{p_i}(x_i, \bar{x})} \leq \frac{\prod_{i=1}^n G^{q_i}(y_i, \bar{y})}{\prod_{i=1}^m G^{p_i}(x_i, \bar{x})} \leq \frac{\prod_{i=1}^n y_i^{q_i}}{\prod_{i=1}^m x_i^{p_i}},$$

where $A(x, y) = \frac{x+y}{2}$, $I(x, y) = e^{-1} \left(\frac{x}{y} \right)^{\frac{1}{x-y}}$ and $G(x, y) = \sqrt{xy}$ are the arithmetic, identric and geometric means.

Consider the function $f(x) = -\frac{1}{x}$ for $x > 0$.

If x 's and y 's are the same as in the previous example, then, by (11)

$$\sum_{i=1}^m p_i \left(\frac{1}{H(x_i, \bar{x})} - \frac{1}{L(x_i, \bar{x})} \right) \geq \sum_{i=1}^n q_i \left(\frac{1}{H(y_i, \bar{y})} - \frac{1}{L(y_i, \bar{y})} \right),$$

where $H(x, y) = \frac{2xy}{x+y}$ and $L(x, y) = \frac{x-y}{\log x - \log y}$ denote the harmonic and the logarithmic means. In this case Corollary 5.4 gives

$$\frac{1}{\bar{x}} - \frac{1}{\bar{y}} \geq \sum_{i=1}^m \frac{p_i}{L(x_i, \bar{x})} - \sum_{i=1}^n \frac{q_i}{L(y_i, \bar{y})}.$$

In particular, setting $y_1 = \dots = y_n > \max x_i$ we obtain the inequality

$$\frac{1}{\bar{x}} \geq \sum_{i=1}^m \frac{p_i}{L(x_i, \bar{x})}$$

valid for all positive x 's.

References

- [1] R.P. Boas, D.V. Widder, *Functions with positive differences*, Duke Math. J. **7** (1940), 496–503.
- [2] P.S. Bullen, *An inequality of N. Levinson*, Univ. Beograd Publ. Elektrotehn. Fak. Ser. Mat. Fiz. **412-460** (1973), 109–112.
- [3] N. Levinson, *Generalization of an inequality of Ky Fan*, J. Math. Anal. Appl. **8** (1964), 133–134.
- [4] A.McD. Mercer, *Short proofs of Jensen's and Levinson's inequalities*, Math. Gazette **94** (2010), 492–495. (Note 94.33).
- [5] J.E. Pečarić, *On Levinson's inequality*, Real Anal. Exchange **15** (1989/90), 710–712.
- [6] T. Popoviciu, *Sur une inégalité de N. Levinson*, Mathematica (Cluj) **6** (1964), 301–306.

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