

**Annales Universitatis Paedagogicae Cracoviensis
Studia Mathematica XIII (2014)***Monika Herzog***A note on the convergence of partial
Szász-Mirakyan type operators**

Abstract. In this paper we study approximation properties of partial modified Szász-Mirakyan operators for functions from exponential weight spaces. We present some direct theorems giving the degree of approximation for these operators. The considered version of Szász-Mirakyan operators is more useful from the computational point of view.

1. Introduction

Let us denote by $C(\mathbb{R}_0)$ a set of all real-valued functions continuous on $\mathbb{R}_0 = [0; +\infty)$ and let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. In paper [1] we investigated operators of Szász-Mirakyan type defined as follows

$$A_n^\nu(f; x) = \begin{cases} \sum_{k=0}^{\infty} p_{n,k}^\nu(x) f\left(\frac{2k}{n}\right), & x > 0; \\ f(0), & x = 0 \end{cases}$$

and

$$p_{n,k}^\nu(x) = \frac{1}{I_\nu(nx)} \frac{x^{2k+\nu}}{2^{2k+\nu} k! \Gamma(k + \nu + 1)},$$

where Γ is the gamma function and I_ν stands for the modified Bessel function, i.e.

$$I_\nu(z) = \sum_{k=0}^{\infty} \frac{z^{2k+\nu}}{2^{2k+\nu} k! \Gamma(k + \nu + 1)}.$$

We studied the operators in polynomial weight spaces

$$C_p = \{f \in C(\mathbb{R}_0) : w_p f \text{ is uniformly continuous and bounded on } \mathbb{R}_0\},$$

where w_p was the polynomial weight function defined as follows

$$w_p(x) = \begin{cases} 1, & p = 0; \\ \frac{1}{1+x^p}, & p \in \mathbb{N} \end{cases}$$

for $x \in \mathbb{R}_0$. The space C_p is a normed space with the norm

$$\|f\|_p = \sup_{x \in \mathbb{R}_0} w_p(x) |f(x)|.$$

In paper [4] a certain modification of the above operators was introduced

$$B_n^\nu(f, a_n; x) = \begin{cases} \sum_{k=0}^{[n(x+a_n)]} p_{n,k}^\nu(x) f\left(\frac{2k}{n}\right), & x > 0; \\ f(0), & x = 0 \end{cases}$$

for $f \in C_p$, where (a_n) was a sequence of positive numbers such that

$$\lim_{n \rightarrow \infty} a_n \sqrt{n} = \infty \quad (1)$$

and $[n(x+a_n)]$ denotes the integral part of $n(x+a_n)$.

In the paper were studied approximation properties of these operators. Among others there was deduced that

$$\lim_{n \rightarrow \infty} \{B_n^\nu(f, a_n; x) - f(x)\} = 0$$

for every $f \in C_p$, uniformly on every interval $[x_1, x_2]$, $x_2 > x_1 \geq 0$. In that case the crucial assumption was that the sequence of positive numbers (a_n) satisfied the condition (1).

Similar problems for Baskakov type operators were discussed in paper [5] and some generalization of truncated operators we can find in [6].

We shall drive analogous results for the modified Szász-Mirakyan operators for functions from exponential weight spaces

$$E_q = \{f \in C(\mathbb{R}_0) : v_q f \text{ is uniformly continuous and bounded on } \mathbb{R}_0\},$$

where v_q is the exponential weight function defined as follows

$$v_q(x) = e^{-qx}, \quad q \in \mathbb{R}_+$$

for $x \in \mathbb{R}_0$. The space E_q is a normed space with the norm

$$\|f\|_q = \sup_{x \in \mathbb{R}_0} v_q(x) |f(x)|. \quad (2)$$

In this paper we will present a certain modification of the operator B_n^ν . We shall apply the modification to prove the convergence of the operators in exponential weight spaces.

2. Main results

At the beginning of this section we will recall the definition of $\overline{A}_n^\nu$ and some preliminary results from papers [2] and [3], which we shall apply to prove main theorems.

If $\nu \in \mathbb{R}_0$ and $n \in \mathbb{N}$ we considered operators of Szász-Mirakyan type as follows

$$\overline{A}_n^\nu(f; x) = \begin{cases} \sum_{k=0}^{\infty} p_{n,k}^\nu(x) f\left(\frac{2k}{n+q}\right), & x > 0; \\ f(0), & x = 0 \end{cases}$$

for $f \in E_q$.

LEMMA 1

For all $\nu \in \mathbb{R}_0$, $n \in \mathbb{N}$ and $x \in \mathbb{R}_0$ we have

$$\overline{A}_n^\nu(1; x) = 1.$$

LEMMA 2

For all $q \in \mathbb{R}_+$ and $\nu \in \mathbb{R}_0$ there exists a positive constant $M(q, \nu)$ such that for each $n \in \mathbb{N}$ we have

$$\left\| \overline{A}_n^\nu\left(\frac{1}{v_q}; \cdot\right) \right\|_q \leq M(q, \nu).$$

Applying the definition of $\overline{A}_n^\nu$ and (2) we get

LEMMA 3

For all $q \in \mathbb{R}_+$ and $\nu \in \mathbb{R}_0$ there exists a positive constant $M(q, \nu)$ such that for each $n \in \mathbb{N}$ we have

$$\left\| \overline{A}_n^\nu(f; \cdot) \right\|_q \leq M(q, \nu) \|f\|_q.$$

Notice that operators $\overline{A}_n^\nu$ are bounded and transform the space E_q into itself.

LEMMA 4

For all $q \in \mathbb{R}_+$ and $\nu \in \mathbb{R}_0$ there exists a positive constant $M(q, \nu)$ such that for each $x \in \mathbb{R}_0$ and $n \in \mathbb{N}$ we have

$$|\overline{A}_n^\nu((t-x)^2; x)| \leq M(q, \nu) \frac{x(x+1)}{n}.$$

LEMMA 5

For all $q \in \mathbb{R}_+$ and $\nu \in \mathbb{R}_0$ there exists a positive constant $M(q, \nu)$ such that for all $x \in \mathbb{R}_0$ and $n \in \mathbb{N}$ we have

$$v_q(x) \overline{A}_n^\nu\left(\frac{(t-x)^2}{v_q(t)}; x\right) \leq M(q, \nu) \frac{x(x+1)}{n}.$$

THEOREM 6

If $\nu \in \mathbb{R}_0$ and $f \in E_q$ with some $q \in \mathbb{R}_+$, then for all $x \in \mathbb{R}_0$

$$\lim_{n \rightarrow \infty} \{\bar{A}_n^\nu(f; x) - f(x)\} = 0.$$

Moreover, the above convergence is uniform on every set $[x_1, x_2]$ with $0 \leq x_1 < x_2$.

Proof. Let $f \in E_q$ with some $q \in \mathbb{R}_+$. Pick $x \in \mathbb{R}_0$ and $\varepsilon > 0$. There exists a number δ such that

$$|f(t) - f(x)| < \frac{\varepsilon}{2} \quad (3)$$

for $|t - x| < \delta$, $t \in \mathbb{R}_0$. By linearity of $\bar{A}_n^\nu$ and Lemma 1 we get

$$\begin{aligned} & |\bar{A}_n^\nu(f; x) - f(x)| \\ & \leq \bar{A}_n^\nu(|f - f(x)|; x) \\ & = \sum_{|\frac{2k}{n+q} - x| < \delta} p_{n,k}^\nu(x) \left| f\left(\frac{2k}{n+q}\right) - f(x) \right| + \sum_{|\frac{2k}{n+q} - x| \geq \delta} p_{n,k}^\nu(x) \left| f\left(\frac{2k}{n+q}\right) - f(x) \right| \\ & = I_1 + I_2. \end{aligned}$$

Hence by (3) we obtain $I_1 < \frac{\varepsilon}{2}$. Further we get

$$I_2 \leq \frac{\|f\|_q}{\delta} \bar{A}_n^\nu\left(\frac{|t - x|}{v_q(t)}; x\right) + \frac{\|f\|_q}{\delta v_q(x)} \bar{A}_n^\nu(|t - x|; x).$$

Using the Hölder inequality and Lemmas 2–5 we have

$$\begin{aligned} I_2 & \leq \frac{\|f\|_q}{\delta} \left[\bar{A}_n^\nu\left(\frac{(t - x)^2}{v_q(t)}; x\right) \right]^{\frac{1}{2}} \left[\bar{A}_n^\nu\left(\frac{1}{v_q(t)}; x\right) \right]^{\frac{1}{2}} + \frac{\|f\|_q}{\delta v_q(x)} \left[\bar{A}_n^\nu((t - x)^2; x) \right]^{\frac{1}{2}} \\ & \leq \frac{\|f\|_q}{\delta} \left[M(q, \nu) \frac{x(x+1)}{nv_q(x)} \right]^{\frac{1}{2}} \left[M(q, \nu) \frac{1}{v_q(x)} \right]^{\frac{1}{2}} + \frac{\|f\|_q}{\delta v_q(x)} \left[M(q, \nu) \frac{x(x+1)}{n} \right]^{\frac{1}{2}} \\ & < \frac{\varepsilon}{2}. \end{aligned}$$

The above estimations imply the convergence in Theorem 6.

In the space E_q we define the following class of partial operators

$$\bar{B}_n^\nu(f, a_n; x) = \begin{cases} \sum_{k=0}^{[(n+q)(x+a_n)]} p_{n,k}^\nu(x) f\left(\frac{2k}{n+q}\right), & x > 0; \\ f(0), & x = 0, \end{cases}$$

where we replace the infinite summing in $\bar{A}_n^\nu$ by the finite one and we still have the assumption (1).

THEOREM 7

If $\nu \in \mathbb{R}_0$ and $f \in E_q$ with some $q \in \mathbb{R}_+$ then for all $x \in \mathbb{R}_0$

$$\lim_{n \rightarrow \infty} \{\bar{B}_n^\nu(f, a_n; x) - f(x)\} = 0.$$

Moreover, the above convergence is uniform on every interval $[x_1, x_2]$, $x_2 > x_1 \geq 0$.

Proof. Let $f \in E_q$ with some $q \in \mathbb{R}_+$. From the definitions of the operators $\bar{A}_n^\nu$ and $\bar{B}_n^\nu$ we get

$$\begin{aligned} & \bar{B}_n^\nu(f, a_n; x) - f(x) \\ &= \sum_{k=0}^{[(n+q)(x+a_n)]} p_{n,k}^\nu(x) f\left(\frac{2k}{n+q}\right) - f(x) \\ &= \sum_{k=0}^{\infty} p_{n,k}^\nu(x) f\left(\frac{2k}{n+q}\right) - f(x) - \sum_{k=[(n+q)(x+a_n)]+1}^{\infty} p_{n,k}^\nu(x) f\left(\frac{2k}{n+q}\right) \\ &= \bar{A}_n^\nu(f; x) - f(x) - R_n^\nu(f, a_n; x) \end{aligned}$$

for $x \in \mathbb{R}_0$ and $n \in \mathbb{N}$. Observe that

$$\left| \frac{2k}{n+q} - x \right| \geq a_n \quad \text{if} \quad k \geq [(n+q)(x+a_n)] + 1.$$

Analogously as in the previous proof we can write the following estimation

$$\begin{aligned} |R_n^\nu(f, a_n; x)| &\leq \sum_{k=[(n+q)(x+a_n)]+1}^{\infty} p_{n,k}^\nu(x) \left| f\left(\frac{2k}{n+q}\right) \right| \\ &\leq \sum_{\left| \frac{2k}{n+q} - x \right| \geq a_n} p_{n,k}^\nu(x) \left| f\left(\frac{2k}{n+q}\right) \right| \leq \frac{\|f\|_q}{a_n} \sum_{k=0}^{\infty} p_{n,k}^\nu(x) \frac{\left| \frac{2k}{n+q} - x \right|}{v_q\left(\frac{2k}{n+q}\right)} \\ &= \frac{\|f\|_q}{a_n} \bar{A}_n^\nu\left(\frac{|t-x|}{v_q(t)}; x\right). \end{aligned}$$

The Hölder inequality, Lemmas 2 and 5 imply

$$\begin{aligned} |R_n^\nu(f, a_n; x)| &\leq \frac{\|f\|_q}{a_n} \left[\bar{A}_n^\nu\left(\frac{(t-x)^2}{v_q(t)}; x\right) \right]^{\frac{1}{2}} \left[\bar{A}_n^\nu\left(\frac{1}{v_q(t)}; x\right) \right]^{\frac{1}{2}} \\ &\leq M(q, \nu) \frac{\|f\|_q}{a_n} \frac{\sqrt{x(x+1)}}{\sqrt{n}} \frac{1}{v_q(x)}. \end{aligned}$$

In view of (1) we obtain the required result.

Notice that the same simplified method we can use in polynomial weight spaces C_p to estimate the reminder of the series A_n^ν , which was considered in [4].

References

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