

Annales Universitatis Paedagogicae Cracoviensis
Studia Mathematica XIII (2014)*Kaliappan Vijaya, Gangadharan Murugusundaramoorthy,*
Murugesan Kasthuri*Starlike functions of complex order involving
 q -hypergeometric functions with fixed point

Abstract. Recently Kanas and Ronning introduced the classes of starlike and convex functions, which are normalized with $f(\xi) = f'(\xi) - 1 = 0$, ξ ($|\xi| = d$) is a fixed point in the open disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. In this paper we define a new subclass of starlike functions of complex order based on q -hypergeometric functions and continue to obtain coefficient estimates, extreme points, inclusion properties and neighbourhood results for the function class $\mathcal{TS}_\xi(\alpha, \beta, \gamma)$. Further, we obtain integral means inequalities for the function $f \in \mathcal{TS}_\xi(\alpha, \beta, \gamma)$.

1. Introduction

Let ξ ($|\xi| = d$) be a fixed point in the unit disc $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$. Denote by $\mathcal{A}(\xi)$ the class of functions which are regular and normalized by $f(\xi) = f'(\xi) - 1 = 0$ consisting of the functions of the form

$$f(z) = (z - \xi) + \sum_{n=2}^{\infty} a_n (z - \xi)^n, \quad (z - \xi) \in \mathbb{U}. \quad (1)$$

Also denote by $\mathcal{S}_\xi = \{f \in \mathcal{A}(\xi) : f \text{ is univalent in } \mathbb{U}\}$, the subclass of $\mathcal{A}(\xi)$. Denote by $\mathcal{T}_\xi$ the subclass of $\mathcal{S}_\xi$ consisting of the functions of the form

$$f(z) = (z - \xi) - \sum_{n=2}^{\infty} a_n (z - \xi)^n, \quad a_n \geq 0. \quad (2)$$

Note that $\mathcal{S}_0 = \mathcal{S}$ and $\mathcal{T}_0 = \mathcal{T}$ be the subclasses of $\mathcal{A} = \mathcal{A}(0)$ consisting of univalent functions in $\mathbb{U}$. By $\mathcal{S}_\xi^*(\beta)$ and $\mathcal{K}_\xi(\beta)$ respectively, we mean the classes of analytic

functions that satisfy the analytic conditions

$$\Re\left\{\frac{(z-\xi)f'(z)}{f(z)}\right\} > \beta, \quad \Re\left\{1 + \frac{(z-\xi)f''(z)}{f'(z)}\right\} > \beta \quad \text{and} \quad (z-\xi) \in \mathbb{U}$$

for $0 \leq \beta < 1$ introduced and studied by Kanas and Ronning [9]. The class $\mathcal{S}_\xi^*(0)$ is defined by geometric property that the image of any circular arc centered at ξ is starlike with respect to $f(\xi)$ and the corresponding class $\mathcal{K}_\xi^*(0)$ is defined by the property that the image of any circular arc centered at ξ is convex. We observe that the definitions are somewhat similar to the ones introduced by Goodman in [8] for uniformly starlike and convex functions, except that in this case the point ξ is fixed. In particular, $\mathcal{K} = \mathcal{K}_0(0)$ and $\mathcal{S}_0^* = \mathcal{S}^*(0)$ respectively, are the well-known standard classes of convex and starlike functions [10, 19].

We recall a generalized q -Taylors formula in fractional q -calculus and certain q -generating functions for q -hypergeometric functions studied more recently by Purohit and Raina [15] and further by Mohammed Aabed and Maslina Darus [1]. For complex parameters $a_1, \dots, a_l$ and $b_1, \dots, b_m$ ($b_j \neq 0, -1, \dots$; $j = 1, 2, \dots, m$) the q -hypergeometric function ${}_l\Psi_m(z)$ is defined by

$$\begin{aligned} {}_l\Psi_m(a_1, \dots, a_l; b_1, \dots, b_m; q, z) \\ := \sum_{n=0}^{\infty} \frac{(a_1; q)_n \dots (a_l; q)_n}{(b_1; q)_n \dots (b_m; q)_n} [(-1)^n q^{\binom{n}{2}}]^{1+m-l} z^n \end{aligned} \quad (3)$$

with $\binom{n}{2} = \frac{n(n-1)}{2}$, where $q \neq 0$ when $l > m + 1$ ($l, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$; $z \in \mathbb{U}$).

The q -shifted factorial is defined for $a, q \in \mathbb{C}$ as a product of n factors by

$$(a; q)_n = \begin{cases} 1, & n = 0, \\ (1-a)(1-aq) \dots (1-aq^{n-1}), & n \in \mathbb{N} \end{cases}$$

and in terms of basic analogue of the gamma function

$$(q^a; q)_n = \frac{\Gamma_q(a+n)(1-q)^n}{\Gamma_q(a)}, \quad n > 0. \quad (4)$$

It is interest to note that $\lim_{q \rightarrow 1^-} \frac{(q^a; q)_n}{(1-q)^n} = (a)_n = a(a+1) \dots (a+n-1)$ the familiar Pochhammer symbol.

Now for $z \in \mathbb{U}$, $0 < |q| < 1$ and $l = m + 1$, the basic q -hypergeometric function defined in (3) takes the form

$${}_l\psi_m(a_1, \dots, a_l; b_1, \dots, b_m; q, z) = \sum_{n=0}^{\infty} \frac{(a_1; q)_n \dots (a_l; q)_n}{(q; q)_n (b_1; q)_n \dots (b_m; q)_n} z^n$$

which converges absolutely in the open unit disk $\mathbb{U}$. Let

$$\mathcal{I}(a_l, b_m; q; z) = z {}_l\psi_m(a_1, \dots, a_l; b_1, \dots, b_m; q, z) = \sum_{n=0}^{\infty} \Upsilon_n^{l,m}[a_1, q] z^{n+1},$$

where for convenience,

$$\Upsilon_n^{l,m}[a_1, q] = \frac{(a_1; q)_n \cdots (a_l; q)_n}{(q; q)_n (b_1; q)_n \cdots (b_m; q)_n}.$$

The operator $\mathcal{I}(a_l, b_m; q)f(z)$ was studied recently by Aabed and Darus [1].

In this paper we define a new linear operator for $(z - \xi) \in \mathbb{U}$, $|q| < 1$ and $l = m + 1$ as follows:

$$\begin{aligned} \mathcal{I}(a_l, b_m; q, z - \xi) &= (z - \xi) {}_l\psi_m(a_1, \dots, a_l; b_1, \dots, b_m; q, z - \xi) \\ &= \sum_{n=0}^{\infty} \Upsilon_n^{l,m}[a_1, q](z - \xi)^{n+1}. \end{aligned}$$

Using the above, we let

$$\mathcal{I}(a_l, b_m; q, z - \xi) * f(z) = \mathcal{I}_m^l f(z) = (z - \xi) + \sum_{n=2}^{\infty} \Upsilon_n^{l,m}[a_1, q] a_n (z - \xi)^n, \quad (5)$$

where

$$\Upsilon_m^l(n) = \Upsilon_n^{l,m}[a_1, q] = \frac{(a_1; q)_{n-1} \cdots (a_l; q)_{n-1}}{(q; q)_{n-1} (b_1; q)_{n-1} \cdots (b_m; q)_{n-1}}$$

unless otherwise stated.

For $a_i = q^{\alpha_i}$, $b_j = q^{\beta_j}$, $\alpha_i, \beta_j \in \mathbb{C}$, and $\beta_j \neq 0, -1, -2, \dots$, ($i = 1, \dots, l$, $j = 1, \dots, m$) and $q \rightarrow 1$, we obtain the well-known Dziok-Srivastava linear operator [7, 6] (for $l = m + 1$). For $l = 1$, $m = 0$, $a_1 = q$, and further specializing the parameters, it gives many (well known and new) integral and differential operators introduced and studied in [4, 5, 10, 13, 16].

Making use of the operator $\mathcal{I}_m^l$ and motivated by the results discussed by Altintas et al. [2], (see [14] and references stated therein) and Aouf et al. [3], in this paper we introduce a new subclass $\mathcal{S}_\xi(\alpha, \beta, \gamma)$ of analytic functions of complex order associated with q -hypergeometric functions as given below.

For $-1 \leq \alpha < 1$, $\beta \geq 0$ and $\gamma \in \mathbb{C} \setminus \{0\}$, we let $\mathcal{S}_\xi(\alpha, \beta, \gamma)$ be the subclass of $\mathcal{A}(\xi)$ consisting of functions of the form (1) and satisfying the analytic criterion

$$\Re\left(1 + \frac{1}{\gamma} \left[\frac{(z - \xi)(\mathcal{I}_m^l f(z))'}{\mathcal{I}_m^l f(z)} - \alpha \right]\right) > \beta \left| 1 + \frac{1}{\gamma} \left[\frac{(z - \xi)(\mathcal{I}_m^l f(z))'}{\mathcal{I}_m^l f(z)} - 1 \right] \right|$$

for every $z \in \mathbb{U}$, where $\mathcal{I}_m^l f(z)$ is given by (5). We also let $\mathcal{TS}_\xi(\alpha, \beta, \gamma) = \mathcal{S}_\xi(\alpha, \beta, \gamma) \cap \mathcal{T}_\xi$.

EXAMPLE 1

We note that $\mathcal{S}_\xi(1, 0, \gamma) \equiv \mathcal{S}_\xi^*(\gamma)$, the class of starlike functions of complex order γ ($\gamma \in \mathbb{C} \setminus \{0\}$), satisfying the following conditions

$$\frac{f(z)}{z - \xi} \neq 0 \quad \text{and} \quad \Re\left(1 + \frac{1}{\gamma} \left[\frac{(z - \xi)(\mathcal{I}_m^l f(z))'}{\mathcal{I}_m^l f(z)} - 1 \right]\right) > 0.$$

Further,

$$\mathcal{S}_\xi^*((1-\delta)\cos\lambda e^{-i\lambda}) = S_\xi^*(\delta, \lambda), \quad |\lambda| < \frac{\pi}{2}; \quad 0 \leq \delta \leq 1$$

and

$$\mathcal{S}_\xi^*(\cos\lambda e^{-i\lambda}) = \mathcal{S}_\xi^*(\lambda), \quad |\lambda| < \frac{\pi}{2},$$

where $S_\xi^*(\delta, \lambda)$ denotes the subclass of λ -spiral-like function of order δ and $S_\xi^*(\lambda)$ denotes spiral-like functions with fixed point analogous to the classes introduced and investigated by Libera [11] and Spacek [18](Also see[21]), respectively.

The main object of this paper is to study some usual properties such as the coefficient bounds, extreme points, radii of close to convexity, starlikeness and convexity for the class $\mathcal{TS}_\xi(\alpha, \beta, \gamma)$. Further, we obtain neighborhood results and integral means inequalities for aforementioned class.

2. Coefficient bounds

In this section we obtain a necessary and sufficient condition for functions $f \in \mathcal{TS}_\xi(\alpha, \beta, \gamma)$.

THEOREM 2.1

A necessary and sufficient condition for f of the form (2) to be in the class $\mathcal{TS}_\xi(\alpha, \beta, \gamma)$ is

$$\sum_{n=2}^{\infty} [(n+|\gamma|)(1-\beta) - (\alpha-\beta)](r+d)^{n-1} \Upsilon_m^l(n) a_n \leq (1-\alpha) + |\gamma|(1-\beta), \quad (6)$$

where $-1 \leq \alpha < 1$, $\beta \geq 0$ and $\gamma \in \mathbb{C} \setminus \{0\}$.

Proof. Assume that $f \in \mathcal{TS}_\xi(\alpha, \beta, \gamma)$, then

$$\Re\left(1 + \frac{1}{\gamma} \left[\frac{(z-\xi)(\mathcal{I}_m^l f(z))'}{\mathcal{I}_m^l f(z)} - \alpha \right]\right) > \beta \left| 1 + \frac{1}{\gamma} \left[\frac{(z-\xi)(\mathcal{I}_m^l f(z))'}{\mathcal{I}_m^l f(z)} - 1 \right] \right|,$$

$$\begin{aligned} \Re\left(1 + \frac{1}{\gamma} \left[\frac{(z-\xi)(1-\alpha) - \sum_{n=2}^{\infty} (n-\alpha) \Upsilon_m^l(n) a_n (z-\xi)^n}{(z-\xi) - \sum_{n=2}^{\infty} \Upsilon_m^l(n) a_n (z-\xi)^n} \right]\right) \\ > \beta \left| 1 - \frac{1}{\gamma} \left[\frac{\sum_{n=2}^{\infty} (n-1) \Upsilon_m^l(n) a_n (z-\xi)^n}{(z-\xi) - \sum_{n=2}^{\infty} \Upsilon_m^l(n) a_n (z-\xi)^n} \right] \right|. \end{aligned}$$

On choosing the values of $(z-\xi)$ on the positive real axis, where $0 < |z-\xi| \leq r+d < 1$, we have

$$\begin{aligned} \left\{ 1 + \frac{1}{|\gamma|} \left(\frac{(1-\alpha) - \sum_{n=2}^{\infty} (n-\alpha) \Upsilon_m^l(n) a_n (r+d)^{n-1}}{1 - \sum_{n=2}^{\infty} \Upsilon_m^l(n) a_n (r+d)^{n-1}} \right) \right\} \\ > \beta \left\{ 1 - \frac{1}{|\gamma|} \left(\frac{\sum_{n=2}^{\infty} (n-1) \Upsilon_m^l(n) a_n (r+d)^{n-1}}{1 - \sum_{n=2}^{\infty} \Upsilon_m^l(n) a_n (r+d)^{n-1}} \right) \right\}. \end{aligned}$$

The simple computation leads the desired inequality

$$\sum_{n=2}^{\infty} [(n + |\gamma|)(1 - \beta) - (\alpha - \beta)] \Upsilon_m^l(n) a_n (r + d)^{n-1} \leq (1 - \alpha) + |\gamma|(1 - \beta).$$

Conversely, suppose that (6) is true for $(z - \xi) \in \mathbb{U}$, then

$$\Re \left(1 + \frac{1}{\gamma} \left[\frac{(z - \xi)(\mathcal{I}_m^l f(z))'}{\mathcal{I}_m^l f(z)} - \alpha \right] \right) - \beta \left| 1 + \frac{1}{\gamma} \left[\frac{(z - \xi)(\mathcal{I}_m^l f(z))'}{\mathcal{I}_m^l f(z)} - 1 \right] \right| > 0.$$

If

$$1 + \frac{1}{|\gamma|} \left(\frac{(1 - \alpha) - \sum_{n=2}^{\infty} (n - \alpha) \Upsilon_m^l(n) a_n |z - \xi|^{n-1}}{1 - \sum_{n=2}^{\infty} \Upsilon_m^l(n) a_n |z - \xi|^{n-1}} \right) - \beta \left[1 - \frac{1}{|\gamma|} \left(\frac{\sum_{n=2}^{\infty} (n - 1) \Upsilon_m^l(n) a_n |z - \xi|^{n-1}}{1 - \sum_{n=2}^{\infty} \Upsilon_m^l(n) a_n |z - \xi|^{n-1}} \right) \right] \geq 0.$$

That is if

$$\sum_{n=2}^{\infty} [(n + |\gamma|)(1 - \beta) - (\alpha - \beta)] \Upsilon_m^l(n) a_n (r + d)^{n-1} \leq (1 - \alpha) + |\gamma|(1 - \beta),$$

which completes the proof.

COROLLARY 2.2

Let the function f defined by (2) belongs $\mathcal{TS}_{\xi}(\alpha, \beta, \gamma)$. Then

$$a_n \leq \frac{[(1 - \alpha) + |\gamma|(1 - \beta)]}{[(n + |\gamma|)(1 - \beta) - (\alpha - \beta)] \Upsilon_m^l(n) (r + d)^{n-1}},$$

$n \geq 2$, $-1 \leq \alpha < 1$, $\beta \geq 0$ and $\gamma \in \mathbb{C} \setminus \{0\}$, with equality for

$$f(z) = (z - \xi) - \frac{[(1 - \alpha) + |\gamma|(1 - \beta)]}{[(n + |\gamma|)(1 - \beta) - (\alpha - \beta)] \Upsilon_m^l(n)} (z - \xi)^n.$$

For the sake of brevity we let

$$\Theta_d(n, \alpha, \beta, \gamma) = [(n + |\gamma|)(1 - \beta) - (\alpha - \beta)] (r + d)^{n-1},$$

$$\Theta_d(2, \alpha, \beta, \gamma) = [(2 - \alpha - \beta) + |\gamma|(1 - \beta)] (r + d) \quad (7)$$

throughout our study.

In the next theorem we state extreme points for the functions of the class $\mathcal{TS}_{\xi}(\alpha, \beta, \gamma)$.

THEOREM 2.3 (EXTREME POINTS)

Let

$$\begin{aligned} f_1(z) &= (z - \xi), \\ f_n(x) &= (z - \xi) - \frac{[(1 - \alpha) + |\gamma|(1 - \beta)]}{[(n + |\gamma|)(1 - \beta) - (\alpha - \beta)] \Upsilon_m^l(n)} (z - \xi)^n, \quad n = 2, 3, \dots \end{aligned} \quad (8)$$

Then $f \in \mathcal{TS}_{\xi}(\alpha, \beta, \gamma)$ if and only if f can be expressed in the form $f(z) = \sum_{n=1}^{\infty} \omega_n f_n(z)$, where $\omega_n \geq 0$ and $\sum_{n=1}^{\infty} \omega_n = 1$.

The proof of the Theorem 2.3 follows on lines similar to the proof of the theorem on extreme points given in Silverman [19].

3. Close-to-convexity, starlikeness and convexity

In this section we obtain the radii of close-to-convexity, starlikeness and convexity for the class $\mathcal{TS}_\xi(\alpha, \beta, \gamma)$.

THEOREM 3.1

Let $f \in \mathcal{TS}_\xi(\alpha, \beta, \gamma)$. Then f is close-to-convex of order δ ($0 \leq \delta < 1$) in the disc $|z - \xi| < R_1$, that is $\Re(f'(z)) > \delta$, where

$$R_1 = \inf_{n \geq 2} \left[\frac{(1 - \delta)\Theta_d(n, \alpha, \beta, \gamma)}{n[(1 - \alpha) + |\gamma|(1 - \beta)]} \Upsilon_m^l(n) \right]^{\frac{1}{n-1}}.$$

Proof. Given $f \in \mathcal{T}_\xi$ and f is close-to-convex of order δ , we have

$$|f'(z) - 1| < 1 - \delta. \quad (9)$$

For the left hand side of (9) we have

$$|f'(z) - 1| \leq \sum_{n=2}^{\infty} na_n R_1^{n-1}.$$

The last expression is less than $1 - \delta$ if

$$\sum_{n=2}^{\infty} \frac{n}{1 - \delta} a_n R_1^{n-1} < 1.$$

Using the fact, that $f \in \mathcal{TS}_\xi(\alpha, \beta, \gamma)$ if and only if

$$\sum_{n=2}^{\infty} \frac{\Theta_d(n, \alpha, \beta, \gamma)}{(1 - \alpha) + |\gamma|(1 - \beta)} \Upsilon_m^l(n) a_n < 1.$$

We can say (9) is true if

$$\frac{n}{1 - \delta} R_1^{n-1} \leq \frac{\Theta_d(n, \alpha, \beta, \gamma)}{(1 - \alpha) + |\gamma|(1 - \beta)} \Upsilon_m^l(n).$$

Or equivalently,

$$R_1 \leq \left[\frac{(1 - \delta)\Theta_d(n, \alpha, \beta, \gamma)}{n[(1 - \alpha) + |\gamma|(1 - \beta)]} \Upsilon_m^l(n) \right]^{\frac{1}{n-1}}.$$

Which completes the proof.

THEOREM 3.2

Let $f \in \mathcal{TS}_\xi(\alpha, \beta, \gamma)$. Then

1. f is starlike of order δ ($0 \leq \delta < 1$) in the disc $|z - \xi| < R_2$; that is, $\Re\left(\frac{(z - \xi)f'(z)}{f(z)}\right) > \delta$, where

$$R_2 = \inf_{n \geq 2} \left\{ \frac{(1 - \delta)}{(n - \delta)} \frac{\Theta_d(n, \alpha, \beta, \gamma)}{[(1 - \alpha) + |\gamma|(1 - \beta)]} \Upsilon_m^l(n) \right\}^{\frac{1}{n-1}},$$

2. f is convex of order δ ($0 \leq \delta < 1$) in the unit disc $|z - \xi| < R_3$, that is $\Re(1 + \frac{(z-\xi)f''(z)}{f'(z)}) > \delta$, where

$$R_3 = \inf_{n \geq 2} \left\{ \frac{(1-\delta)}{n(n-\delta)} \frac{\Theta_d(n, \alpha, \beta, \gamma)}{[(1-\alpha) + |\gamma|(1-\beta)]} \Upsilon_m^l(n) \right\}^{\frac{1}{n-1}}.$$

These results are sharp for the extremal function f given by (8).

Proof. For the case 1, notice that for given $f \in \mathcal{T}_\xi$ and f is starlike of order δ , we have

$$\left| \frac{(z-\xi)f'(z)}{f(z)} - 1 \right| < 1 - \delta. \quad (10)$$

For the left hand side of (10) we obtain

$$\left| \frac{(z-\xi)f'(z)}{f(z)} - 1 \right| \leq \frac{\sum_{n=2}^{\infty} (n-1)a_n |z-\xi|^{n-1}}{1 - \sum_{n=2}^{\infty} a_n |z-\xi|^{n-1}}.$$

The last expression is less than $1 - \delta$ if

$$\sum_{n=2}^{\infty} \frac{n-\delta}{1-\delta} a_n |z-\xi|^{n-1} < 1.$$

Using the fact, that $f \in \mathcal{TS}_\xi(\alpha, \beta, \gamma)$ if and only if

$$\sum_{n=2}^{\infty} \frac{\Theta_d(n, \alpha, \beta, \gamma)}{(1-\alpha) + |\gamma|(1-\beta)} \Upsilon_m^l(n) a_n < 1.$$

We can say (10) is true if

$$\frac{n-\delta}{1-\delta} |z-\xi|^{n-1} < \frac{\Theta_d(n, \alpha, \beta, \gamma)}{(1-\alpha) + |\gamma|(1-\beta)} \Upsilon_m^l(n).$$

Or equivalently,

$$R_3^{n-1} < \frac{(1-\delta)\Theta_d(n, \alpha, \beta, \gamma)}{(n-\delta)[(1-\alpha) + |\gamma|(1-\beta)]} \Upsilon_m^l(n)$$

which yields the starlikeness of the family.

Notice that we can prove case 2, on lines similar the proof of case 1, it is sufficient to use the fact that f is convex if and only if $(z-\xi)f'$ is starlike.

4. Modified Hadamard products

For functions of the form

$$f_j(z) = (z-\xi) - \sum_{n=2}^{\infty} a_{n,j} (z-\xi)^n, \quad j = 1, 2$$

we define the modified Hadamard product as

$$(f_1 * f_2)(z) = (z-\xi) - \sum_{n=2}^{\infty} a_{n,1} a_{n,2} (z-\xi)^n.$$

THEOREM 4.1

If $f_j \in \mathcal{TS}_\xi(\alpha, \beta, \gamma)$, $j = 1, 2$, then $(f_1 * f_2)(z) \in \mathcal{TS}_\xi(\alpha, \beta, \gamma)$, where

$$\xi = \frac{(2 - \beta)\Theta_d(2, \alpha, \beta, \gamma)\Upsilon_m^l(2) - 2(1 - \beta)[(1 - \alpha) + |\gamma|(1 - \beta)]}{(2 - \beta)\Theta_d(2, \alpha, \beta, \gamma)\Upsilon_m^l(2) - (1 - \beta)[(1 - \alpha) + |\gamma|(1 - \beta)]},$$

with $\Upsilon_m^l(2)$ be defined as in (7).

Proof. Since $f_j \in \mathcal{TS}_\xi(\alpha, \beta, \gamma)$, $j = 1, 2$, we have

$$\sum_{n=2}^{\infty} \Theta_d(n, \alpha, \beta, \gamma)\Upsilon_m^l(n)a_{n,j} \leq (1 - \alpha) + |\gamma|(1 - \beta), \quad j = 1, 2.$$

The Cauchy-Schwartz inequality leads to

$$\sum_{n=2}^{\infty} \frac{\Theta_d(n, \alpha, \beta, \gamma)\Upsilon_m^l(n)}{(1 - \alpha) + |\gamma|(1 - \beta)} \sqrt{a_{n,1}a_{n,2}} \leq 1. \quad (11)$$

Note that we need to find the largest ρ such that

$$\sum_{n=2}^{\infty} \frac{\Theta_d(n, \alpha, \rho, \gamma)\Upsilon_m^l(n)}{(1 - \alpha) + |\gamma|(1 - \rho)} a_{n,1}a_{n,2} \leq 1. \quad (12)$$

Therefore, in view of (11) and (12), whenever

$$\frac{n - \xi}{1 - \xi} \sqrt{a_{n,1}a_{n,2}} \leq \frac{n - \beta}{1 - \beta}, \quad n \geq 2$$

holds, then (12) is satisfied. We have, from (11),

$$\sqrt{a_{n,1}a_{n,2}} \leq \frac{(1 - \alpha) + |\gamma|(1 - \beta)}{\Theta_d(n, \alpha, \beta, \gamma)\Upsilon_m^l(n)}, \quad n \geq 2. \quad (13)$$

Thus, if

$$\left(\frac{n - \xi}{1 - \xi}\right) \left[\frac{(1 - \alpha) + |\gamma|(1 - \beta)}{\Theta_d(n, \alpha, \beta, \gamma)\Upsilon_m^l(n)}\right] \leq \frac{n - \beta}{1 - \beta}, \quad n \geq 2,$$

or, if

$$\xi = \frac{(n - \beta)\Theta_d(n, \alpha, \beta, \gamma)\Upsilon_m^l(n) - n(1 - \beta)[(1 - \alpha) + |\gamma|(1 - \beta)]}{(n - \beta)\Theta_d(n, \alpha, \beta, \gamma)\Upsilon_m^l(n) - (1 - \beta)[(1 - \alpha) + |\gamma|(1 - \beta)]}, \quad n \geq 2,$$

then (11) is satisfied. Note that the right hand side of the above expression is an increasing function on n . Hence, setting $n = 2$ in the above inequality gives the required result. Finally, by taking the function

$$f(z) = (z - \xi) - \frac{(1 - \alpha) + |\gamma|(1 - \beta)}{(2 - \beta)[\Theta_d(2, \alpha, \beta, \gamma)]\Upsilon_m^l(2)}(z - \xi)^2,$$

we see that the result is sharp.

5. Integral means

In order to find the integral means inequality and to verify the Silverman Conjecture [20] for $f \in \mathcal{TS}_\xi(\alpha, \beta, \gamma)$ we need the following subordination result due to Littlewood [12].

LEMMA 5.1 ([12])

If the functions f and g are analytic in $\mathbb{U}$ with $g \prec f$, then

$$\int_0^{2\pi} |g(re^{i\theta})|^\eta d\theta \leq \int_0^{2\pi} |f(re^{i\theta})|^\eta d\theta, \quad \eta > 0, \quad z = re^{i\theta} \text{ and } 0 < r < 1.$$

Applying Theorem 2.1 with extremal function given by (8) and Lemma 5.1, we prove the following theorem.

THEOREM 5.2

Let $\eta > 0$. If $f \in \mathcal{TS}_\xi(\alpha, \beta, \gamma)$ and $\{\Phi(\alpha, \beta, \gamma, n)\}_{n=2}^\infty$ is non-decreasing sequence, then for $(z - \xi) = re^{i\theta}$ and $0 < r + d < 1$ we have

$$\int_0^{2\pi} |f(re^{i\theta})|^\eta d\theta \leq \int_0^{2\pi} |f_2(re^{i\theta})|^\eta d\theta,$$

where

$$f_2(z) = (z - \xi) - \frac{(1 - \alpha) + |\gamma|(1 - \beta)}{\Theta_d(2, \alpha, \beta, \gamma)\Upsilon_m^l(2)}(z - \xi)^2.$$

Proof. Let $f(z)$ of the form (2) and

$$f_2(z) = (z - \xi) - \frac{(1 - \alpha) + |\gamma|(1 - \beta)}{\Theta_d(2, \alpha, \beta, \gamma)\Upsilon_m^l(2)}(z - \xi)^2,$$

then we must show that

$$\int_0^{2\pi} \left| 1 - \sum_{n=2}^{\infty} a_n(z - \xi)^{n-1} \right|^\eta d\theta \leq \int_0^{2\pi} \left| 1 - \frac{(1 - \alpha) + |\gamma|(1 - \beta)}{\Theta_d(2, \alpha, \beta, \gamma)\Upsilon_m^l(2)}(z - \xi) \right|^\eta d\theta.$$

By Lemma 5.1, it suffices to show that

$$1 - \sum_{n=2}^{\infty} a_n(z - \xi)^{n-1} \prec 1 - \frac{(1 - \alpha) + |\gamma|(1 - \beta)}{\Theta_d(2, \alpha, \beta, \gamma)\Upsilon_m^l(2)}(z - \xi).$$

Setting

$$1 - \sum_{n=2}^{\infty} a_n(z - \xi)^{n-1} = 1 - \frac{(1 - \alpha) + |\gamma|(1 - \beta)}{\Theta_d(2, \alpha, \beta, \gamma)\Upsilon_m^l(2)}w(z). \quad (14)$$

From (14) and (6) we obtain

$$\begin{aligned}
 |w(z)| &= \left| \sum_{n=2}^{\infty} \frac{\Theta_d(n, \alpha, \beta, \gamma) \Upsilon_m^l(n)}{(1-\alpha) + |\gamma|(1-\beta)} a_n (z-\xi)^{n-1} \right| \\
 &\leq |z-\xi| \sum_{n=2}^{\infty} \frac{\Theta_d(n, \alpha, \beta, \gamma) \Upsilon_m^l(n)}{(1-\alpha) + |\gamma|(1-\beta)} a_n \\
 &\leq |z-\xi| \\
 &< 1.
 \end{aligned}$$

This completes the proof of the Theorem 5.2.

6. Inclusion relations involving $N_\delta(e)$

In this section following [14, 17], we define the n, δ neighborhood of function $f \in \mathcal{T}_\xi$ and discuss the inclusion relations involving $N_\delta(e)$.

$$N_\delta(f) = \left\{ g \in \mathcal{T}_\xi : g(z) = (z-\xi) - \sum_{n=2}^{\infty} b_n (z-\xi)^n \text{ and } \sum_{n=2}^{\infty} n|a_n - b_n| \leq \delta \right\}.$$

In particular, for the identity function $e(z) = z$ we have

$$N_\delta(e) = \left\{ g \in \mathcal{T}_\xi : g(z) = (z-\xi) - \sum_{n=2}^{\infty} b_n z^n \text{ and } \sum_{n=2}^{\infty} n|b_n| \leq \delta \right\}.$$

THEOREM 6.1

Let

$$\delta = \frac{2[(1-\alpha) + |\gamma|(1-\beta)]}{\Theta_d(2, \alpha, \beta, \gamma) \Upsilon_m^l(2)},$$

where $-1 \leq \alpha < 1$, $\beta \geq 0$ and $\gamma \in \mathbb{C} \setminus \{0\}$. Then $\mathcal{TS}_\xi(\alpha, \beta, \gamma) \subset N_\delta(e)$.

Proof. For $f \in \mathcal{TS}_\xi(\alpha, \beta, \gamma)$ Theorem 2.1 yields

$$\Theta_d(2, \alpha, \beta, \gamma) \Upsilon_m^l(2) \sum_{n=2}^{\infty} a_n \leq (1-\alpha) + |\gamma|(1-\beta)$$

so that

$$\sum_{n=2}^{\infty} a_n \leq \frac{(1-\alpha) + |\gamma|(1-\beta)}{[\Theta_d(2, \alpha, \beta, \gamma) \Upsilon_m^l(2)]}. \quad (15)$$

On the other hand, from (6) and (15) we have

$$\begin{aligned}
& (1 - \beta)(r + d)\Upsilon_m^l(2) \sum_{n=2}^{\infty} na_n \\
& \leq (1 - \alpha) + |\gamma|(1 - \beta) + [(\alpha - \beta) - |\gamma|(1 - \beta)](r + d)\Upsilon_m^l(2) \sum_{n=2}^{\infty} a_n \\
& \leq (1 - \alpha) + |\gamma|(1 - \beta) + [(\alpha - \beta) - |\gamma|(1 - \beta)](r + d)\Upsilon_m^l(2) \\
& \quad \times \frac{(1 - \alpha) + |\gamma|(1 - \beta)}{[(2 - \alpha + \beta) + |\gamma|(1 - \beta)](r + d)\Upsilon_m^l(2)} \\
& \leq \frac{[(1 - \alpha) + |\gamma|(1 - \beta)]2(1 - \beta)}{(2 - \alpha + \beta) + |\gamma|(1 - \beta)}.
\end{aligned}$$

Hence

$$\sum_{n=2}^{\infty} na_n \leq \frac{2[(1 - \alpha) + |\gamma|(1 - \beta)]}{[(2 - \alpha + \beta) + |\gamma|(1 - \beta)](r + d)\Upsilon_m^l(2)}$$

and

$$\sum_{n=2}^{\infty} na_n \leq \frac{2[(1 - \alpha) + |\gamma|(1 - \beta)]}{\Theta_d(2, \alpha, \beta, \gamma)\Upsilon_m^l(2)} = \delta. \quad (16)$$

Now we determine the neighborhood for each of the function class $\mathcal{TS}_{\xi}(\alpha, \beta, \gamma)$ which we define as follows:

A function $f \in \mathcal{T}_{\xi}$ is said to be in the class $\mathcal{TS}_{\xi}(\alpha, \beta, \gamma, \eta)$ if there exists a function $g \in \mathcal{TS}_{\xi}(\alpha, \beta, \gamma)$ such that

$$\left| \frac{f(z)}{g(z)} - 1 \right| < 1 - \eta, \quad (z - \xi) \in \mathbb{U}, \quad 0 \leq \eta < 1. \quad (17)$$

THEOREM 6.2

If $g \in \mathcal{TS}_{\xi}(\alpha, \beta, \gamma)$ and

$$\eta = 1 - \frac{\delta\Theta_d(2, \alpha, \beta, \gamma)\Upsilon_m^l(2)}{\Theta_d(2, \alpha, \beta, \gamma)\Upsilon_m^l(2) - 2[(1 - \alpha) + |\gamma|(1 - \beta)]}. \quad (18)$$

Then $N_{\delta}(g) \subset \mathcal{TS}_{\xi}(\alpha, \beta, \gamma, \eta)$.

Proof. Suppose that $f \in N_{\delta}(g)$, then we find from (16) that

$$\sum_{n=2}^{\infty} n|a_n - b_n| \leq \delta,$$

which implies the coefficient inequality

$$\sum_{n=2}^{\infty} |a_n - b_n| \leq \frac{\delta}{2}.$$

Next, since $g \in \mathcal{TS}_{\xi}(\alpha, \beta, \gamma)$, we have

$$\sum_{n=2}^{\infty} b_n \leq \frac{2[(1 - \alpha) + |\gamma|(1 - \beta)]}{\Theta_d(2, \alpha, \beta, \gamma)\Upsilon_m^l(2)}.$$

So that

$$\begin{aligned} \left| \frac{f(z)}{g(z)} - 1 \right| &< \frac{\sum_{n=2}^{\infty} |a_n - b_n|}{1 - \sum_{n=2}^{\infty} b_n} \\ &\leq \frac{\delta}{2} \times \frac{\Theta_d(2, \alpha, \beta, \gamma) \Upsilon_m^l(2)}{\Theta_d(2, \alpha, \beta, \gamma) \Upsilon_m^l(2) - 2[(1 - \alpha) + |\gamma|(1 - \beta)]} \\ &\leq 1 - \eta, \end{aligned}$$

provided that η is given precisely by (18). Thus by definition, $f \in \mathcal{TS}_\xi(\alpha, \beta, \gamma, \eta)$ for η given by (18), which completes the proof.

Concluding Remarks: By suitably specializing the various parameters involved in Theorem 6 to Theorem 6.2 we can state the corresponding results for the new subclasses defined in Example 1 and also for many relatively more familiar function classes.

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*School of Advanced Sciences
VIT University
Vellore - 632014
India
E-mail: kvijaya@vit.ac.in
gmsmoorthy@yahoo.com*

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