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On multiplicative intersections between the Pell and Jacobsthal numbers

Abstract. We determine all integer triples (k, m, n) satisfying the Diophantine equations $P_k = J_m J_n$ and $J_k = P_m P_n$, thereby characterizing Pell numbers that are products of two Jacobsthal numbers, and vice versa. In addition, we study the intersection of the Pell and Jacobsthal sequences and show that the only common values occur when $m = 1$, corresponding to the integers 1 and 5.

1. Introduction

Numerous integer sequences have attracted significant attention in the mathematical literature because of their wide-ranging applications in both mathematics and the natural sciences. In particular, the Fibonacci and Lucas numbers, due to their remarkable appearances in nature and diverse mathematical properties, have inspired extensive research efforts. Comprehensive discussions and foundational results on these sequences can be found in the monographs by Koshy [15] and Vajda [20].

The well-known Pell and Jacobsthal numbers are defined by the recurrence relations

$$P_0 = 0, \quad P_1 = 1 \quad \text{and} \quad P_{n+1} = 2P_n + P_{n-1} \quad \text{for all } n \geq 1$$

and

$$J_0 = 0, \quad J_1 = 1 \quad \text{and} \quad J_{n+1} = J_n + 2J_{n-1} \quad \text{for all } n \geq 1,$$

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respectively. Moreover, their terms follow directly from Binet's formulas

$$P_n = \frac{\gamma^n - \delta^n}{\gamma - \delta} \quad (1)$$

and

$$J_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad (2)$$

where $\alpha = 2$ and $\beta = -1$ are the roots of $x^2 - 2x - 1 = 0$, while $\gamma = 1 + \sqrt{2}$ and $\delta = 1 - \sqrt{2}$ are the roots of $x^2 - x - 2 = 0$. Further properties of the Pell and Jacobsthal sequences can be found in [2], [3], [4] and [14], respectively.

Recently, several studies have been published on multiplicative and additive relations among classical linear recurrences such as the Fibonacci, Lucas, and Pell sequences. Daşdemir and Varol obtained complete solutions for Lucas numbers that are products of two Pell numbers [5]. Ddamulira et al. characterized Fibonacci numbers that can be written as products of two Pell numbers [6]. Subsequently, in [7], Ddamulira et al. analyzed a Pillai-type problem involving k -generalized Fibonacci numbers and powers of 2. More recently, Emin [9] investigated the Fermat-like equation $L_m^2 + L_n^2 = 2^a$, and in a subsequent work [10], considered a more general case involving Pell numbers of the form $P_m^x + P_n^x = 2^a$. In [11], Faye et al. studied the exponential Diophantine equations $P_n^x + P_{n+1}^y = P_m^x$ or $P_n^y + P_{n+1}^x = P_m^x$. Later, in [12], Gómez et al. solved an exponential Diophantine equation $F_{n+1}^x F_{n-1}^x = F_m^y$. Gueye et al. treated an exponential equation regarding the k -Fibonacci numbers in the form $(3^a \mp 1)(3^b \mp 1)$ [13]. Pongsriiam examined Fibonacci and Lucas numbers that differ by one from their products [17]. Rihane et al. focused on exponential Diophantine equations involving Pell numbers that can be written as sums of powers of two consecutive Pell numbers [18], whereas Rihane et al. determined Fibonacci and Pell numbers expressible as a ternary multiplicative form of the other sequence [19].

These works have highlighted a consistent research direction centered on multiplicative intersections and exponential relations between linear recurrence sequences, motivating the present study of analogous phenomena for Pell and Jacobsthal numbers. For this purpose, in this paper, we investigate the Diophantine equations

$$P_k = J_m J_n \quad (3)$$

and

$$J_k = P_m P_n, \quad (4)$$

where k , m , and n are non-negative integers. Owing to the commutative property of multiplication, it suffices to consider the case $m \leq n$.

These equations address several natural questions, namely: (i) which Pell numbers are also Jacobsthal numbers, and (ii) which Pell (respectively, Jacobsthal) numbers can be expressed as the product of two Jacobsthal (respectively, Pell) numbers. In particular, when $m = 1$, the first question is resolved.

It is easy to verify that (3) and (4) admit trivial solutions corresponding to the triples $(k, m, n) = (0, 0, n)$. Throughout this paper, we focus exclusively on nontrivial solutions. We now state the main results.

THEOREM 1.1

Let k , m , and n be positive integers. Then all solutions of the Diophantine equation (3) are given by

$$(k, m, n) \in \{(1, 1, 1), (1, 1, 2), (1, 2, 2), (3, 1, 4), (3, 2, 4)\}.$$

THEOREM 1.2

Let k , m , and n be positive integers. Then all solutions of the Diophantine equation (4) are given by

$$(k, m, n) \in \{(1, 1, 1), (2, 1, 1), (4, 1, 3)\}.$$

The paper is organized as follows. Section 2 recalls several definitions and preliminary results that will be used throughout the paper. Section 3 is devoted to the proof of Theorem 1.1. In Section 4, several cases are analyzed in order to refine the bounds arising from the relevant inequalities, while the main conclusions are summarized in the last section. For brevity, only the proof of Theorem 1.1 is presented in full detail; the proof of Theorem 1.2 can be carried out in a similar manner.

2. Preliminaries

Let α be an algebraic number of degree d over $\mathbb{Q}$ with the minimal polynomial

$$a_0x^d + a_1x^{d-1} + \cdots + a_d = a_0 \prod_{i=1}^d (x - \alpha^{(i)}),$$

where $\alpha^{(i)}$ denote the conjugates of α and a_0 is the leading coefficient. In this setting, the absolute logarithmic height function $h(\cdot)$ is defined by

$$h(\alpha) = \frac{1}{d} \left(\log |a_0| + \sum_{i=1}^d \log (\max \{ |\alpha^{(i)}|, 1 \}) \right).$$

The logarithmic height function $h(\cdot)$ satisfies the following well-known properties [21, Property 3.3]:

$$h(\alpha \pm \beta) \leq h(\alpha) + h(\beta) + \log 2,$$

$$h(\alpha\beta^{\pm 1}) \leq h(\alpha) + h(\beta),$$

$$h(\alpha^s) = |s|h(\alpha).$$

In particular, if $\alpha = a/b \in \mathbb{Q}$, where $\gcd(a, b) = 1$ and $b > 0$, then $h(a/b) = \log(\max\{|a|, b\})$.

The following theorem provides a crucial lower bound for linear forms in logarithms.

THEOREM 2.1 (Matveev [16])

Let $\eta_1, \eta_2, \dots, \eta_l$ be real algebraic numbers and let $d_1, d_2, \dots, d_l$ be nonzero rational numbers. Let $d_{\mathbb{K}}$ denote the degree of the number field $\mathbb{K} = \mathbb{Q}(\eta_1, \eta_2, \dots, \eta_l)$ over $\mathbb{Q}$, and for $j = 1, 2, \dots, l$, let A_j be positive real numbers satisfying

$$A_j \geq h'(\eta_j) = \max\{d_{\mathbb{K}} h(\eta_j), |\log \eta_j|, 0.16\}.$$

Define

$$\Lambda = \eta_1^{d_1} \eta_2^{d_2} \dots \eta_l^{d_l} - 1 \quad \text{and} \quad D = \max\{|d_1|, \dots, |d_l|\}.$$

If $\Lambda \neq 0$, then

$$\log(|\Lambda|) > -1.4 \times 30^{l+3} \times l^{4.5} \times d_{\mathbb{K}}^2 \times (1 + \log(d_{\mathbb{K}})) \times (1 + \log(D)) \times A_1 \times A_2 \times \dots \times A_l.$$

In [8], Dujella and Pethő gave the following efficient reduction method to reduce huge powers.

LEMMA 2.2 (Dujella and Pethő [8])

Let M be a positive integer and let p/q be a convergent of the continued fraction expansion of an irrational number τ , with $q > 6M$. Let A , B and τ be positive real numbers such that $A > 0$ and $B > 1$. Define

$$\varepsilon = \|\mu q\| - M\|\tau q\|,$$

where $\|\cdot\|$ denotes the distance to the nearest integer. If $\varepsilon > 0$, then the inequality

$$0 < u\tau - v + \mu < AB^{-k}$$

has no integer solution (u, v, w) satisfying

$$u \leq M \quad \text{and} \quad w \geq \frac{\log(Aq/\varepsilon)}{\log B}.$$

It should be noted that this lemma is a variation of the result by Dujella and Pethő [8], and its proof was provided by Bravo et al. [1].

LEMMA 2.3

Let k be a positive integer and let x , y and z be positive rational numbers such that $\sqrt{x}$ and $\sqrt{z}$ are irrational. Then the number $\sqrt{x}(y + \sqrt{z})^k$ is irrational.

Proof. Set $A := \sqrt{x}(y + \sqrt{z})^k$ and assume that k is even. By the binomial theorem, we get $A = B\sqrt{x} + \sqrt{xz}C$, where

$$B := \sum_{r=0}^{\lfloor k/2 \rfloor} \binom{k}{2r} y^{k-2r} z^r \quad \text{and} \quad C := \sum_{r=0}^{\lfloor (k-1)/2 \rfloor} \binom{k}{2r+1} y^{k-2r-1} z^r.$$

Assume that A is a rational number. So, computing the square of A , we get $2BCx\sqrt{z} = A^2 - B^2x - C^2xz$, which means that $\sqrt{z} \in \mathbb{Q}$. But, this is contradiction, concluding that A is an irrational. The case of odd k can be treated similarly, completing the proof.

LEMMA 2.4

The following inequalities hold for every integer $n \geq 1$

$$\gamma^{n-2} \leq P_n \leq \gamma^{n-1}$$

and

$$\alpha^{n-2} \leq J_n \leq \alpha^{n-1},$$

where $\gamma = 1 + \sqrt{2}$ and $\alpha = 2$.

Proof. The inequalities follow readily by induction on n .

3. Obtaining effective bounds

In this section, we present explicit arguments leading to the bounds required for Theorems 1.1 and 1.2. We first focus on (4). Let k , m , and n be positive integers. For $n < 10$, we conclude that there is no solution except for the one in Theorem 1.1, and henceforth, we take $n \geq 10$. According to Lemma 2.4, the following inequalities hold:

$$\alpha^{k-2} \leq J_k = P_n P_m \leq \gamma^{m+n-2} \quad \text{and} \quad \alpha^{k-1} \geq J_k = P_n P_m \geq \gamma^{m+n-4}.$$

Hence,

$$1 + c_1(m + n - 4) \leq k \leq 2 + c_1(m + n - 2), \quad c_1 = \frac{\log \gamma}{\log \alpha} = 1.271553303 \dots,$$

which in particular implies $k < 4n$ (because of the assumption $m \leq n$).

Now, our goal is to apply Theorem 2.1. In order to do that, we need to define Λ_1 appropriately and derive the upper bound for its absolute value.

The Binet formulas in (1) and (2) together with the assumption $J_k = P_n P_m$ give

$$\left| \frac{\alpha^k}{3} - \frac{\gamma^{n+m}}{8} \right| = \left| \frac{\beta^k}{3} - \frac{\gamma^n \delta^m + \gamma^m \delta^n - \delta^{n+m}}{8} \right|.$$

Since $\beta = -1$, $\delta = -\gamma^{-1}$ and $\frac{1}{3} < \frac{3}{8}$, we obtain

$$\begin{aligned} \left| \frac{\beta^k}{3} - \frac{\gamma^n \delta^m + \gamma^m \delta^n - \delta^{n+m}}{8} \right| &< \left| \frac{\beta^k}{3} + \frac{3\gamma^{n-m}}{8} \right| \\ &< 2 \times \frac{3}{8} \max\{|\beta|^k, \gamma^{n-m}\} = \frac{3\gamma^{n-m}}{4}, \end{aligned}$$

which implies that

$$|\Lambda_1| < \frac{6}{\gamma^{2m}} \tag{5}$$

with the notation

$$\Lambda_1 = \frac{8}{3} \alpha^k \gamma^{-n-m} - 1.$$

To apply Matveev's theorem, define

$$l = 3, \quad \eta_1 = \frac{1}{3}, \quad \eta_2 = \alpha, \quad \eta_3 = \gamma, \quad d_1 = 1, \quad d_2 = k + 3 \quad \text{and} \quad d_3 = -n - m.$$

Clearly, $\Lambda_1 \neq 0$ because $\frac{8}{3}\alpha^k \in \mathbb{Q}$, whereas $\gamma^{n+m} \notin \mathbb{Q}$. Moreover, all η_j lie in $\mathbb{L} := \mathbb{Q}(\sqrt{2})$ with $d_{\mathbb{L}} = 2$. We compute

$$h(\eta_1) = \log 3, \quad h(\eta_2) = \log \alpha \quad \text{and} \quad h(\eta_3) = \frac{\log \gamma}{2}.$$

Accordingly, choose

$$A_1 = 2 \log 3, \quad A_2 = \log \alpha, \quad A_3 = \log \gamma \quad \text{and} \quad D = 4n.$$

Applying Theorem 2.1 yields

$$\log |\Lambda_1| > -1.4 \times 30^6 \times 3^{4.5} \times 2^2 \times (1 + \log 2)(1 + \log 4n)(2 \log 3)(2 \log \alpha)(\log \gamma),$$

that is,

$$\log |\Lambda_1| > -2.61 \times 10^{12} \times (1 + \log 4n). \quad (6)$$

Combining (5) and (6) gives

$$(m - 1) \log \gamma + \log 3 < 1.4 \times 10^{12} (1 + \log 4n). \quad (7)$$

Returning to (4), we may write

$$\frac{1}{3}(\alpha^k - \beta^k) = \frac{P_m}{2\sqrt{2}}(\gamma^n - \delta^n).$$

Proceeding analogously, we obtain

$$|\Lambda_2| < \frac{2}{\gamma^n} \quad (8)$$

with $\Lambda_2 = \frac{2\sqrt{2}}{3P_m}\alpha^k\gamma^{-n} - 1$. Here we again set

$$l = 3, \quad \eta_1 = \frac{3P_m}{\sqrt{2}}, \quad \eta_2 = \alpha, \quad \eta_3 = \gamma, \quad d_1 = -1, \quad d_2 = k + 1 \quad \text{and} \quad d_3 = -n.$$

Thus,

$$h(\eta_2) = \log \alpha, \quad h(\eta_3) = \frac{1}{2} \log \gamma, \quad A_2 = 2 \log \alpha \quad \text{and} \quad A_3 = \log \gamma.$$

Since η_1 is a root of the polynomial $2x^2 - 9P_m^2$,

$$\begin{aligned} h(\eta_1) &= \frac{1}{2} \left(\log 2 + \log \left| \frac{3P_m}{\sqrt{2}} \right| + \log \left| -\frac{3P_m}{\sqrt{2}} \right| \right) \\ &= \log P_m + \log 3 \leq (m - 1) \log \gamma + \log 3, \end{aligned}$$

where the last inequality follows from Lemma 2.4. From (7), we choose

$$A_1 = 2.8 \times 10^{12}(1 + \log 4n) > 2h(\eta_1) \quad \text{and} \quad D = 4n.$$

By Lemma 2.3, $\Lambda_2 \neq 0$. Applying Matveev's theorem again gives

$$\log |\Lambda_2| > -3.32 \times 10^{24}(1 + \log 4n)^2.$$

Comparing this bound with (8), we deduce

$$n \log \gamma - \log 2 < 3.32 \times 10^{24}(1 + \log 4n)^2,$$

which implies that

$$n < 1.72 \times 10^{28}, \tag{9}$$

together with $k < 4n$.

Similarly, for (3), we find

$$n < 9 \times 10^{28}, \tag{10}$$

with $k < 3n$.

Combining (9) and (10) concludes that

$$k < 4n \quad \text{and} \quad n < 9 \times 10^{28}. \tag{11}$$

4. Reduction of the Bounds

The bounds obtained in (11) are clearly far too large for practical verification. To obtain effective results, we refine them by applying the Dujella–Pethő reduction method, dividing the argument into four cases.

Case I. Refinement of (5) for $m \geq 2$. Define

$$\Gamma_1 := k \log \alpha - (n + m) \log \gamma + \log \frac{8}{3}.$$

From (5), we have

$$|e^{\Gamma_1} - 1| = |\Lambda_1| < \frac{6}{\gamma^{2m}} < \frac{1}{5}. \tag{12}$$

By removing the absolute value in (12) and taking the logarithm of the resulting expression, we obtain $|\Gamma_1| < 1/4$. Since $|x| < \frac{1}{4}$ implies that $|x| < \frac{3}{2}|e^x - 1|$, it follows that

$$|\Gamma_1| < \frac{9}{\gamma^{2m}}.$$

As $\Lambda_1 \neq 0$, the sign of Γ_1 is either positive or negative. Suppose first that $\Gamma_1 > 0$. Then

$$0 < k \left(\frac{\log \alpha}{\log \gamma} \right) - (n + m) + \left(\frac{\log(8/3)}{\log \gamma} \right) < \frac{9}{\gamma^{2m} \log \gamma} < \frac{11}{\gamma^{2m}}.$$

We now invoke Dujella and Pethő reduction lemma. From (11), take $M = 3.6 \times 10^{29}$ and set

$$\tau = \frac{\log \alpha}{\log \gamma}, \quad \mu = \frac{\log(8/3)}{\log \gamma}, \quad A = 11 \quad \text{and} \quad B = \gamma^2. \quad (13)$$

Using the continued fraction expansion of τ , we compute

$$\frac{p_{60}}{q_{60}} = \frac{6332847229674209482244367144203}{8052552813322770308759378039685},$$

for which $q_{60} > 6M$. Consequently, we compute

$$\varepsilon = \|\mu q\| - M\|\tau q\| > 0.41 > 0.$$

Applying Lemma 2.2 thus gives $m \leq 42$. The case $\Gamma_1 < 0$ yields the same conclusion.

Case II. Refinement for (3). Repeating the above reasoning for (3) under identical assumptions gives $m \leq 108$.

Case III. Refinement for (8). Let $n > 1$ and consider (8). Since $|x| < \frac{3}{5}$ implies $|x| < \frac{3}{2}|e^x - 1|$, we deduce

$$|\Gamma_2| < \frac{3}{\gamma^n}$$

with

$$\Gamma_2 = k \log \alpha - n \log \gamma + \log \frac{2\sqrt{2}}{3P_m}.$$

Hence,

$$0 < k \frac{\log \alpha}{\log \gamma} - n + \frac{\log(2\sqrt{2}/(3P_m))}{\log \gamma} < \frac{3}{(\log \gamma)\gamma^n} < \frac{4}{\gamma^n}.$$

If $\Gamma_2 > 0$, the values of M , τ , and q remain as in (13). For $m \leq 108$, we set

$$\mu'' = \frac{\log(2\sqrt{2}/(3P_m))}{\log \gamma}, \quad A'' = 4 \quad \text{and} \quad B'' = \gamma.$$

We compute the minimum and maximum values of ε'' over the interval $1 \leq m \leq 108$. For each ε'' , the largest of the upper bound of n is determined, i.e. $n \leq 87$. The case gives the same bound.

Case IV. Refinement for (3). Applying the same argument to (3), leads to $n \leq 112$.

After these reductions, we implemented a complete search using MATHEMATICA over the ranges $n \leq 112$ and $m \leq 108$. The computation confirms that there are no solutions other than those given in Theorems 1.1 and 1.2, thereby completing the proofs of our theorems. Furthermore, the solution set of each of (3) and (4) is $\{1, 5\}$, i.e. $J_k; P_k \in \{1, 5\}$.

Conclusion

In this work, we have determined all integer solutions to the Diophantine equations $P_k = J_m J_n$ and $J_k = P_m P_n$, which describe multiplicative relations between the Pell and Jacobsthal sequences.

By combining explicit estimates derived from the Binet formulas with Matveev's lower bound for linear forms in logarithms, we established effective upper bounds for the indices involved. These bounds were then sharpened using the Dujella–Pethő reduction lemma, leading to a complete classification of all admissible triples (k, m, n) .

Our analysis shows that the only common elements between the two sequences occur in the trivial case $m = 1$, yielding the shared values 1 and 5. Beyond these, no further coincidences exist.

The methods employed here may be adapted to study analogous multiplicative intersections among other linear recurrence sequences, particularly those of the Fibonacci–Lucas type, offering a potential direction for future work.

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