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Multivalued second order differential problem

Abstract. Let K be a closed convex cone with nonempty interior in a real Banach space and let $F, G, H: K \rightarrow cc(K)$ be three given continuous additive set-valued functions. We study the existence and uniqueness of a solution of the second order differential problem

$$D^2\Phi(t, x) = \Phi(t, H(x)), \quad \Phi(0, x) = F(x), \quad D\Phi(t, x)|_{t=0} = G(x)$$

for $t \geq 0$ and $x \in K$, where $D\Phi(t, x)$ and $D^2\Phi(t, x)$ denote the Hukuhara derivative and the second Hukuhara derivative of $\Phi(t, x)$ with respect to t .

Let X be a normed linear space. By $n(X)$ we denote the set of all nonempty subsets of X and by $b(X)$ the set of all nonempty and bounded subsets of X , whereas $c(X)$ stands for the set of all compact members of $n(X)$ and $cc(X)$ stands for the set of all convex members of $c(X)$.

We introduce addition and multiplication by scalars as follows

$$A + B = \{a + b : a \in A, b \in B\}, \quad \lambda A = \{\lambda a : a \in A\}$$

for $A, B \in n(X)$ and $\lambda \in \mathbb{R}$.

A subset K of the space X is called a *cone* if $tK \subset K$ for all $t \in [0, \infty)$. We say that a cone is *convex* if it is a convex set.

Unless indicated differently, throughout the paper X denotes a normed linear space and K a convex cone in X . The Hausdorff distance d derived from the norm in X is a metric in the set $c(X)$. Concepts such as the limit of a set-valued function at a point, the continuity of a set-valued function, the integral of a set-valued function and the limit of a sequence of set-valued functions are correlated to this metric. Moreover, all linear spaces are supposed to be real.

A set-valued function $F: K \rightarrow n(X)$ is said to be *additive* if

$$F(x + y) = F(x) + F(y)$$

for all $x, y \in K$. An additive set-valued function F is *linear* if it is *homogeneous*, i.e.,

$$F(\lambda x) = \lambda F(x)$$

for all $x \in K$, $\lambda \geq 0$. An additive and continuous set-valued function with convex closed and bounded values is linear.

For two set-valued functions $F: K \rightarrow n(X)$, $G: K \rightarrow n(K)$ we define a composition $(F \circ G)(x) = F(G(x)) := \bigcup \{F(y) : y \in G(x)\}$.

Let A, B, C be sets of $cc(X)$. We say that a set C is the *Hukuhara difference* of A and B , i.e., $C = A - B$, if $B + C = A$. If this difference exists, then it is unique (see Lemma 1 in [12]).

Let $[a, b] \subset \mathbb{R}$ be a fixed interval, $F: [a, b] \rightarrow cc(X)$ and assume that the Hukuhara differences $F(t) - F(s)$ exist for all $a \leq s < t \leq b$. The *Hukuhara derivative* of F at $t \in (a, b)$ is defined by the formula

$$DF(t) = \lim_{s \rightarrow t^+} \frac{F(s) - F(t)}{s - t} = \lim_{s \rightarrow t^-} \frac{F(t) - F(s)}{t - s},$$

whenever both of these limits exist. Furthermore,

$$DF(a) = \lim_{s \rightarrow a^+} \frac{F(s) - F(a)}{s - a}, \quad DF(b) = \lim_{s \rightarrow b^-} \frac{F(b) - F(s)}{b - s}.$$

The aim of this paper is to study existence and uniqueness of a linear with respect to the second variable solution $\Phi: [0, \infty) \times K \rightarrow cc(K)$ of the following differential problem

$$D^2\Phi(t, x) = \Phi(t, H(x)), \quad \Phi(0, x) = F(x), \quad D\Phi(t, x)|_{t=0} = G(x), \quad (1)$$

where $F, G, H: K \rightarrow cc(K)$ are given continuous linear set-valued functions and $D\Phi(t, x)$ and $D^2\Phi(t, x)$ denote the Hukuhara derivative and the second Hukuhara derivative of $\Phi(t, x)$ with respect to t .

The differential problem

$$D\Phi(t, x) = \Phi(t, G(x)), \quad \Phi(0, x) = F(x),$$

where $G, F: K \rightarrow cc(K)$ are given continuous linear set-valued functions was studied in [15], while the second order differential problem

$$D^2\Phi(t, x) = \Phi(t, G(x)), \quad \Phi(0, x) = F(x), \quad D\Phi(t, x)|_{t=0} = \{0\},$$

where $G, F: K \rightarrow cc(K)$ are given continuous linear set-valued functions was investigated in [10].

Now we assume that X is a Banach space. Dinghas in [3] and Hukuhara in [4] introduced the Riemann type integral

$$\int_a^b F(t) dt$$

for set-valued functions. If there exists the integral of a function $F: [a, b] \rightarrow cc(X)$, then F is said to be *integrable*. It is known that if $F: \mathbb{R} \rightarrow cc(X)$ is continuous, then it is integrable on each interval $[a, b] \subset \mathbb{R}$ (cf. [4], p. 212).

Following lemmas introduce some important properties of this integral.

LEMMA 1 ([4, P. 212])

If $F: [a, b] \rightarrow cc(X)$ is continuous and $a < c < b$, then

$$\int_a^b F(t) dt = \int_a^c F(t) dt + \int_c^b F(t) dt.$$

LEMMA 2 ([4, P. 211])

If $F, G: [a, b] \rightarrow cc(X)$ are continuous, then

$$d\left(\int_a^b F(t) dt, \int_a^b G(t) dt\right) \leq \int_a^b d(F(t), G(t)) dt.$$

LEMMA 3 ([4, P. 211])

If $F: [a, b] \rightarrow cc(X)$ is continuous, then

$$\left\| \int_a^b F(t) dt \right\| \leq \int_a^b \|F(t)\| dt.$$

LEMMA 4 ([9, LEMMA 10])

If $F: [a, b] \rightarrow cc(X)$ is continuous, then the set-valued function

$$H(t) = \int_a^t F(u) du \quad \text{for } t \in [a, b]$$

is continuous.

LEMMA 5 ([15, LEMMA 4])

If $F: [a, b] \rightarrow cc(X)$ is continuous and $H(t) = \int_a^t F(u) du$, then $DH(t) = F(t)$ for $t \in [a, b]$.

LEMMA 6 ([15, LEMMA 5])

If $F, G: [a, b] \rightarrow cc(X)$ are two differentiable set-valued functions such that $DF(t) = DG(t)$ for $t \in [a, b]$ and $F(a) = G(a)$, then

$$F(t) = G(t) \quad \text{for } t \in [a, b].$$

DEFINITION 1

Let X be a Banach space and let set-valued functions $F, G, H: K \rightarrow cc(K)$ be continuous and additive. A map $\Phi: [0, \infty) \times K \rightarrow cc(K)$ is said to be a solution of problem (1) if it is continuous, twice differentiable with respect to t and it satisfies the differential equation from (1) in $[0, \infty) \times K$ and the initial conditions in K .

To the problem (1) we associate the following integral equation

$$\Phi(t, x) = F(x) + tG(x) + \int_0^t \left(\int_0^s \Phi(u, H(x)) du \right) ds \quad (2)$$

for $(t, x) \in [0, \infty) \times K$, where $F, G, H: K \rightarrow cc(K)$ are given continuous linear set-valued maps.

DEFINITION 2

Let X be a Banach space and let set-valued functions $F, G, H: K \rightarrow cc(K)$ be continuous and additive. A map $\Phi: [0, \infty) \times K \rightarrow cc(K)$ is said to be a solution of (2) if it is continuous and satisfies (2) in $[0, \infty) \times K$.

The proofs of the next two theorems are based on ideas from the proofs of Proposition and Theorem 1 in [15] and Theorems 1, 2 in [10]. We repeat them with inevitable changes for the reader's convenience.

THEOREM 1

Let X be a Banach space and let set-valued functions $F, G, H: K \rightarrow cc(K)$ be continuous and additive. Set-valued function $\Phi: [0, \infty) \times K \rightarrow cc(K)$ is a solution of problem (1) if and only if it is a solution of (2).

Proof. 1° Suppose that a set-valued function $\Phi: [0, \infty) \times K \rightarrow cc(K)$ is a solution of (2). Then Φ is continuous in $[0, \infty) \times K$. Hence, since H is continuous in K , from Theorems 1 and 1' in [1, Chap. VI, p. 113] we get continuity of a map $(u, x) \mapsto \Phi(u, H(x))$ in $[0, \infty) \times K$. In particular, for every $x \in K$ a set-valued function

$$u \mapsto \Phi(u, H(x))$$

is continuous in $[0, \infty)$. Thus by Lemmas 4 and 5 the set-valued function

$$\Psi(t, x) = F(x) + tG(x) + \int_0^t \left(\int_0^s \Phi(u, H(x)) du \right) ds \quad (3)$$

is twice differentiable with respect to t ,

$$D\Psi(t, x) = G(x) + D \int_0^t \left(\int_0^s \Phi(u, H(x)) du \right) ds = G(x) + \int_0^t \Phi(s, H(x)) ds,$$

and

$$D^2\Psi(t, x) = D \int_0^t \Phi(s, H(x)) ds = \Phi(t, H(x)).$$

By (2) we have $\Phi(t, x) = \Psi(t, x)$ for all $(t, x) \in [0, \infty) \times K$, therefore

$$D^2\Phi(t, x) = \Phi(t, H(x)), \quad \Phi(0, x) = F(x) \quad \text{and} \quad D\Phi(t, x)|_{t=0} = G(x).$$

Hence Φ satisfies (1).

2° Now assume that $\Phi: [0, \infty) \times K \rightarrow cc(K)$ is a solution of (1) and let Ψ be defined by equation (3) for $(t, x) \in [0, \infty) \times K$. By Lemmas 4 and 5 we get

$$D\Psi(t, x) = G(x) + \int_0^t \Phi(u, H(x)) du$$

and

$$D^2\Psi(t, x) = \Phi(t, H(x)).$$

Since $D^2\Psi(t, x) = D^2\Phi(t, x)$ and $D\Psi(t, x)|_{t=0} = G(x) = D\Phi(t, x)|_{t=0}$, by Lemma 6 we obtain

$$D\Psi(t, x) = D\Phi(t, x) \quad \text{for } (t, x) \in [0, \infty) \times K.$$

Thus, since $\Psi(0, x) = F(x) = \Phi(0, x)$, similarly we obtain

$$\Psi(t, x) = \Phi(t, x) \quad \text{for } (t, x) \in [0, \infty) \times K.$$

Therefore Φ satisfies (2).

Let K be a closed convex cone in X and Y be a normed linear space. The functional

$$F \mapsto \|F\| := \sup_{x \in K, x \neq 0} \frac{\|F(x)\|}{\|x\|}$$

is finite for every continuous linear set-valued function $F: K \rightarrow c(Y)$. This functional will be called a *norm* (cf. [13]).

Next lemmas will be used in the proof of Theorem 2.

LEMMA 7 ([16, THEOREM 3], [13, LEMMA 4])

Let Y be a normed linear space. Suppose that $\{F_i : i \in I\}$ is a family of continuous linear set-valued functions $F_i: K \rightarrow c(Y)$. If K is of the second category in K and $\bigcup_{i \in I} F_i(x) \in b(Y)$ for all $x \in K$, then there exists a positive constant M such that

$$\sup_{i \in I} \|F_i(x)\| \leq M\|x\| \quad \text{for } x \in K.$$

LEMMA 8 ([13, LEMMA 5])

Let Y be a normed linear space and let d be the Hausdorff distance derived from the norm in Y . Suppose that K is a convex cone with nonempty interior in X . Then there exists a positive constant M_0 such that for every linear continuous set-valued function $F: K \rightarrow c(Y)$ the inequality

$$d(F(x), F(y)) \leq M_0\|F\|\|x - y\|$$

holds for all $x, y \in K$.

Assume that X is a Banach space and $\text{int } K \neq \emptyset$. Let T be a positive real number and let $\mathcal{E}$ be the set of all continuous set-valued functions $\Phi: [0, T] \times K \rightarrow cc(K)$, which are linear with respect to the second variable. Define a functional ρ in $\mathcal{E} \times \mathcal{E}$ by

$$\rho(\Phi, \Psi) = \sup\{d(\Phi(t, A), \Psi(t, A)) : t \in [0, T], A \in cc(K), \|A\| \leq 1\}$$

for $\Phi, \Psi \in \mathcal{E}$ (see proof of Theorem 1 in [15] and proof of Theorem 2 in [10]). Sets

$$\Phi([0, T], x) = \bigcup_{t \in [0, T]} \Phi(t, x)$$

are compact for $\Phi \in \mathcal{E}$ and $x \in K$ by Theorem 3 in [1, Chap. VI, p. 110], thus they are bounded. Therefore by Lemma 7, for every Φ there exists a positive constant M_Φ such that

$$\|\Phi(t, x)\| \leq M_\Phi \|x\|$$

for $t \in [0, T]$ and $x \in K$. Hence

$$\begin{aligned} d(\Phi(t, A), \Psi(t, A)) &\leq d(\Phi(t, A), \{0\}) + d(\{0\}, \Psi(t, A)) = \|\Phi(t, A)\| + \|\Psi(t, A)\| \\ &\leq M_\Phi + M_\Psi \end{aligned}$$

for $t \in [0, T]$ and $A \in cc(K)$ with $\|A\| \leq 1$. Thus

$$\rho(\Phi, \Psi) \leq M_\Phi + M_\Psi < \infty,$$

so the functional ρ is finite. It is easy to verify that ρ is a metric in $\mathcal{E}$.

Since the space $(cc(K), d)$ is complete (see [2]), $(\mathcal{E}, \rho)$ is a complete metric space.

THEOREM 2

Let K be a closed convex cone with nonempty interior in a Banach space and let set-valued functions $F, G, H: K \rightarrow cc(K)$ be continuous and additive. Then there exists exactly one solution of problem (1). Moreover, this solution is linear with respect to the second variable.

Proof. Fix $T > 0$ arbitrarily. On $\mathcal{E}$ we introduce a map Γ which values are set-valued functions defined by

$$(\Gamma\Phi)(t, x) := F(x) + tG(x) + \int_0^t \left(\int_0^s \Phi(u, H(x)) du \right) ds$$

for $(t, x) \in [0, T] \times K$. It is easy to see that every set $(\Gamma\Phi)(t, x)$ belongs to $cc(K)$.

Let $\Phi \in \mathcal{E}$. We shall prove that $\Gamma\Phi$ is continuous. Fix $x, y \in K$. As above, by Lemma 7 there exists a positive constant M_Φ such that

$$\|\Phi(u, a)\| \leq M_\Phi \|a\| \quad (4)$$

for $u \in [0, T]$ and $a \in K$. Hence

$$\|\Phi(u, H(x))\| \leq M_\Phi \|H(x)\|$$

for $u \in [0, T]$. Let $0 \leq t_1 \leq t_2 \leq T$. By Lemma 3

$$\begin{aligned} &\left\| \int_{t_1}^{t_2} \left(\int_0^s \Phi(u, H(x)) du \right) ds \right\| \\ &\leq \int_{t_1}^{t_2} \left(\int_0^s \|\Phi(u, H(x))\| du \right) ds \leq \int_{t_1}^{t_2} \left(\int_0^s M_\Phi \|H(x)\| du \right) ds \quad (5) \\ &= \int_{t_1}^{t_2} s M_\Phi \|H(x)\| ds = M_\Phi \|H(x)\| \frac{t_2^2 - t_1^2}{2} \\ &\leq (t_2 - t_1) T M_\Phi \|H(x)\|. \end{aligned}$$

From Lemma 8 and (4) there exists a positive constant M_0 such that

$$d(\Phi(u, a), \Phi(u, b)) \leq M_0 \|\Phi(u, \cdot)\| \|a - b\| \leq M_0 M_\Phi \|a - b\|$$

for $u \in [0, T]$ and $a, b \in K$. This implies that

$$\Phi(u, a) \subset \Phi(u, b) + M_0 M_\Phi \|a - b\| S$$

for $u \in [0, T]$ and $a, b \in K$, where S is the closed unit ball centered at zero in X .

Let $a \in H(x)$. There exists $b \in H(y)$ for which

$$\|a - b\| = \inf\{\|a - u\| : u \in H(y)\}.$$

Consequently, for every $a \in H(x)$ there exists $b \in H(y)$ such that

$$\begin{aligned} \Phi(u, a) &\subset \Phi(u, b) + M_0 M_\Phi d(H(x), H(y)) S \\ &\subset \Phi(u, H(y)) + M_0 M_\Phi d(H(x), H(y)) S, \end{aligned}$$

whence

$$\Phi(u, H(x)) \subset \Phi(u, H(y)) + M_0 M_\Phi d(H(x), H(y)) S$$

for every $u \in [0, T]$. Since $x, y \in K$ are arbitrary, we obtain

$$d(\Phi(u, H(x)), \Phi(u, H(y))) \leq M_0 M_\Phi d(H(x), H(y))$$

for every $u \in [0, T]$. Therefore by Lemma 2

$$\begin{aligned} &d\left(\int_0^{t_1} \left(\int_0^s \Phi(u, H(x)) du\right) ds, \int_0^{t_1} \left(\int_0^s \Phi(u, H(y)) du\right) ds\right) \\ &\leq \int_0^{t_1} \left(\int_0^s d(\Phi(u, H(x)), \Phi(u, H(y))) du\right) ds \\ &\leq \int_0^{t_1} \left(\int_0^s M_0 M_\Phi d(H(x), H(y)) du\right) ds \\ &= \frac{t_1^2}{2} M_0 M_\Phi d(H(x), H(y)). \end{aligned} \tag{6}$$

Using Lemma 1 and properties of the Hausdorff distance we get

$$\begin{aligned} &d((\Gamma\Phi)(t_1, x), (\Gamma\Phi)(t_2, y)) \\ &\leq d(F(x), F(y)) + d(t_1 G(x), t_2 G(y)) \\ &\quad + d\left(\int_0^{t_1} \left(\int_0^s \Phi(u, H(x)) du\right) ds, \int_0^{t_2} \left(\int_0^s \Phi(u, H(y)) du\right) ds\right) \\ &\leq d(F(x), F(y)) + t_1 d(G(x), G(y)) + (t_2 - t_1) \|G(y)\| \end{aligned}$$

$$\begin{aligned}
& + d\left(\int_0^{t_1} \left(\int_0^s \Phi(u, H(x)) du\right) ds, \int_0^{t_1} \left(\int_0^s \Phi(u, H(y)) du\right) ds\right) \\
& + \left\| \int_{t_1}^{t_2} \left(\int_0^s \Phi(u, H(y)) du\right) ds \right\|,
\end{aligned}$$

hence from inequalities (5) and (6)

$$\begin{aligned}
& d((\Gamma\Phi)(t_1, x), (\Gamma\Phi)(t_2, y)) \\
& \leq d(F(x), F(y)) + t_1 d(G(x), G(y)) + (t_2 - t_1) \|G(y)\| \\
& \quad + \frac{t_1^2}{2} M_0 M_\Phi d(H(x), H(y)) + (t_2 - t_1) T M_\Phi \|H(y)\|.
\end{aligned}$$

Since F , G and H are continuous, this shows that $\Gamma\Phi$ is a continuous set-valued function. It is easily seen that $x \mapsto (\Gamma\Phi)(t, x)$ are linear for all $t \in [0, T]$. This implies that $\Gamma(\mathcal{E}) \subset \mathcal{E}$.

Next we shall prove that Γ has exactly one fixed point. Fix $\Phi, \Psi \in \mathcal{E}$ arbitrarily. By Lemma 2 we have

$$\begin{aligned}
& d((\Gamma\Phi)(t, x), (\Gamma\Psi)(t, x)) \\
& = d\left(\int_0^t \left(\int_0^s \Phi(u, H(x)) du\right) ds, \int_0^t \left(\int_0^s \Psi(u, H(x)) du\right) ds\right) \quad (7) \\
& \leq \int_0^t \left(\int_0^s d(\Phi(u, H(x)), \Psi(u, H(x))) du\right) ds
\end{aligned}$$

for $t \in [0, T]$ and $x \in K$, which implies that

$$d((\Gamma\Phi)(t, x), (\Gamma\Psi)(t, x)) \leq \frac{t^2}{2} \|H(x)\| \rho(\Phi, \Psi) \quad (8)$$

for $t \in [0, T]$ and $x \in K$ and consequently

$$\rho(\Gamma\Phi, \Gamma\Psi) \leq \frac{T^2}{2} \|H\| \rho(\Phi, \Psi).$$

Let

$$\Phi_1(t, x) := (\Gamma\Phi)(t, x), \quad \Psi_1(t, x) := (\Gamma\Psi)(t, x).$$

From (7) we have

$$\begin{aligned}
& d((\Gamma^2\Phi)(t, x), (\Gamma^2\Psi)(t, x)) = d((\Gamma\Phi_1)(t, x), (\Gamma\Psi_1)(t, x)) \\
& \leq \int_0^t \left(\int_0^s d(\Phi_1(u, H(x)), \Psi_1(u, H(x))) du\right) ds
\end{aligned}$$

for $t \in [0, T]$ and $x \in K$, whereas from (8)

$$d(\Phi_1(u, y), \Psi_1(u, y)) \leq \frac{u^2}{2} \|H(y)\| \rho(\Phi, \Psi)$$

for all $y \in H(x)$, thus

$$d(\Phi_1(u, y), \Psi_1(u, y)) \leq \frac{u^2}{2} \|H(H(x))\| \rho(\Phi, \Psi).$$

It is easy to verify that this implies that

$$d(\Phi_1(u, H(x)), \Psi_1(u, H(x))) \leq \frac{u^2}{2} \|H^2(x)\| \rho(\Phi, \Psi),$$

and therefore we get

$$\begin{aligned} d((\Gamma^2 \Phi)(t, x), (\Gamma^2 \Psi)(t, x)) &\leq \int_0^t \left(\int_0^s \frac{u^2}{2} \|H^2(x)\| \rho(\Phi, \Psi) du \right) ds \\ &= \frac{t^4}{4!} \|H^2(x)\| \rho(\Phi, \Psi) \end{aligned}$$

for $t \in [0, T]$ and $x \in K$, thus

$$\rho(\Gamma^2 \Phi, \Gamma^2 \Psi) \leq \frac{T^4}{4!} \|H^2\| \rho(\Phi, \Psi).$$

By induction we can prove that

$$\rho(\Gamma^n \Phi, \Gamma^n \Psi) \leq \frac{T^{2n}}{(2n)!} \|H\|^n \rho(\Phi, \Psi)$$

for every positive integer n . Since T is a positive constant, there is $n \in \mathbb{N}$ such that $\frac{T^{2n}}{(2n)!} \|H\|^n < 1$. From the Banach's fixed point Theorem Γ^n has exactly one fixed point Φ . But

$$\Gamma^n(\Gamma \Phi) = \Gamma(\Gamma^n \Phi) = \Gamma \Phi.$$

Since Φ is a unique fixed point of Γ^n , we get $\Gamma \Phi = \Phi$. If Φ was not unique, Γ^n would also have more than one fixed point. Therefore we obtain existence and uniqueness of $\Phi \in \mathcal{E}$ satisfying the differential equation from (1) in $[0, T] \times K$ and the initial conditions in K . Since T was arbitrary, this finishes the proof.

Let $\{F_t : t \geq 0\}$ be a family of set-valued functions $F_t: K \rightarrow n(X)$, $t \geq 0$. A family $\{E_t : t \geq 0\}$ of set-valued functions $E_t: K \rightarrow n(K)$, $t \geq 0$, is called a *sine family associated with family* $\{F_t : t \geq 0\}$, if

$$E_{t+s}(x) = E_{t-s}(x) + 2F_t(E_s(x)) \quad (9)$$

for $0 \leq s \leq t$ and $x \in K$.

A sine family $\{E_t : t \geq 0\}$ of set-valued functions with compact values is called *regular* if $\lim_{t \rightarrow 0^+} \frac{E_t(x)}{t} = \{x\}$ (cf. [5]).

LEMMA 9 ([5, PROPOSITION 1])

Assume that $\{F_t : t \geq 0\}$ and $\{E_t : t \geq 0\}$ are families of set-valued functions $F_t : K \rightarrow n(X)$, $E_t : K \rightarrow n(X)$ such that F_0 is continuous linear, $F_0(x) \in c(K)$, $E_0(x) \in cc(K)$, $x \in F_0(x)$ for $x \in K$. If $\{E_t : t \geq 0\}$ is a sine family associated with the family $\{F_t : t \geq 0\}$, then $E_0(x) = \{0\}$ for $x \in K$.

LEMMA 10 ([5, THEOREM 3], [6, THEOREM 3])

Let X be a Banach space, K a closed convex cone with nonempty interior in X and let $\{F_t : t \geq 0\}$ and $\{E_t : t \geq 0\}$ be families of continuous additive set-valued functions $F_t : K \rightarrow cc(K)$, $E_t : K \rightarrow cc(K)$, $F_0(x) = \{x\}$ for $x \in K$ and $x \in F_t(x)$ for $x \in K$ and $t > 0$. Assume that $\{E_t : t \geq 0\}$ is a regular sine family associated with $\{F_t : t \geq 0\}$. Then the set-valued function $u \mapsto F_u(x)$ is continuous for every $x \in K$ and

$$E_t(x) = \int_0^t F_u(x) du, \quad t \geq 0, x \in K.$$

A family $\{F_t : t \geq 0\}$ of set-valued functions $F_t : K \rightarrow n(K)$ is called a *cosine family*, if

$$F_0(x) = \{x\} \quad (10)$$

for all $x \in K$ and

$$F_{t+s}(x) + F_{t-s}(x) = 2F_t(F_s(x)), \quad (11)$$

whenever $0 \leq s \leq t$ and $x \in K$.

A cosine family $\{F_t : t \geq 0\}$ of set-valued functions with compact values is called *regular* if $\lim_{t \rightarrow 0^+} F_t(x) = \{x\}$ (cf. [13]).

LEMMA 11 ([8, THEOREM])

Let K be a closed convex cone with nonempty interior in a Banach space X . Suppose that $\{F_t : t \geq 0\}$ is a regular cosine family of continuous linear set-valued functions $F_t : K \rightarrow cc(K)$, $x \in F_t(x)$ for all $x \in K$, $t > 0$ and $F_t \circ F_s = F_s \circ F_t$ for all $s, t > 0$. Then this cosine family is twice differentiable and

$$D^2F_t(x) = F_t(H(x)) \quad \text{and} \quad DF_t(x)|_{t=0} = \{0\} \quad (12)$$

for $x \in K$, $t \geq 0$, where $DF_t(x)$ and $D^2F_t(x)$ denote the Hukuhara derivative and the second Hukuhara derivative of $F_t(x)$ with respect to t , respectively, and $H(x) = D^2F_t(x)|_{t=0}$.

We shall need some further properties of the Hukuhara derivative and of the Riemann integral.

LEMMA 12 ([14, LEMMA 3])

Let K be a closed convex cone in a linear space X . Assume that $F : K \rightarrow cc(K)$ is a continuous additive set-valued function and $A, B \in cc(K)$. If there exists the difference $A - B$, then there exists $F(A) - F(B)$ and $F(A) - F(B) = F(A - B)$.

LEMMA 13

Let K be a closed convex cone in X and $[a, b] \subset \mathbb{R}$ be a given interval. Let $F: K \rightarrow cc(K)$ be a continuous additive set-valued function and $G: [a, b] \rightarrow cc(K)$ be a differentiable set-valued function. Then $D(F \circ G(t))$ exists and $D(F \circ G(t)) = F \circ DG(t)$.

Proof. By the definition of the Hukuhara derivative and Lemma 12

$$D(F \circ G)(t) = \lim_{s \rightarrow t^+} \frac{F \circ G(s) - F \circ G(t)}{s - t} = \lim_{s \rightarrow t^+} \frac{F[G(s) - G(t)]}{s - t}.$$

Since F is linear and continuous we have

$$\lim_{s \rightarrow t^+} \frac{F[G(s) - G(t)]}{s - t} = F\left(\lim_{s \rightarrow t^+} \frac{G(s) - G(t)}{s - t}\right) = F \circ DG(t)$$

(see Lemma 8 and [13, Lemma 6]). We use the same reasoning when s converges to t from the left.

LEMMA 14

Let K be a convex cone with nonempty interior in a Banach space and let $\{F_t : t \geq 0\}$ be a regular cosine family of continuous additive set-valued functions $F_t: K \rightarrow cc(K)$ such that $F_t \circ F_s = F_s \circ F_t$ for all $s, t \geq 0$. Assume that $\Phi: K \rightarrow cc(K)$ is continuous additive and $F_t \circ \Phi = \Phi \circ F_t$ for all $t \geq 0$. Then

$$\left(\int_0^s F_u(\cdot) du \right) (\Phi(x)) = \int_0^s F_u(\Phi(x)) du.$$

Proof. First we shall prove that

$$(F_{\frac{s}{4}} + F_{\frac{3s}{4}})(\Phi(x)) = F_{\frac{s}{4}}(\Phi(x)) + F_{\frac{3s}{4}}(\Phi(x)).$$

Indeed, since $\{F_t : t \geq 0\}$ is a cosine family, by (11) and the commutativity of Φ and all F_t we have

$$\begin{aligned} (F_{\frac{s}{4}} + F_{\frac{3s}{4}})(\Phi(x)) &= (F_{\frac{s}{2} - \frac{s}{4}} + F_{\frac{s}{2} + \frac{s}{4}})(\Phi(x)) = 2(F_{\frac{s}{2}} \circ F_{\frac{s}{4}})(\Phi(x)) \\ &= \Phi(2F_{\frac{s}{2}} \circ F_{\frac{s}{4}}(x)) = \Phi(F_{\frac{s}{4}}(x) + F_{\frac{3s}{4}}(x)) \\ &= \Phi(F_{\frac{s}{4}}(x)) + \Phi(F_{\frac{3s}{4}}(x)) \\ &= F_{\frac{s}{4}}(\Phi(x)) + F_{\frac{3s}{4}}(\Phi(x)). \end{aligned}$$

Let us now assume that for some positive integer n and for all $x \in K$ we have

$$\left(\sum_{i=0}^{2^n-1} F_{(\frac{1}{2^{n+1}} + \frac{i}{2^n})s} \right) (\Phi(x)) = \sum_{i=0}^{2^n-1} F_{(\frac{1}{2^{n+1}} + \frac{i}{2^n})s} (\Phi(x)). \quad (13)$$

Then, by (11) we obtain

$$\begin{aligned}
 & \left(\sum_{i=0}^{2^{n+1}-1} F_{(\frac{1}{2^{n+2}} + \frac{i}{2^{n+1}})_s} \right) (\Phi(x)) \\
 &= \left(\sum_{i=0}^{2^n-1} \left[F_{(\frac{1}{2^{n+2}} + \frac{2i+1}{2^{n+1}})_s} + F_{(\frac{1}{2^{n+2}} + \frac{2i}{2^{n+1}})_s} \right] \right) (\Phi(x)) \\
 &= \left(\sum_{i=0}^{2^n-1} \left[F_{(\frac{1}{2^{n+1}} + \frac{i}{2^n} + \frac{1}{2^{n+2}})_s} + F_{(\frac{1}{2^{n+1}} + \frac{i}{2^n} - \frac{1}{2^{n+2}})_s} \right] \right) (\Phi(x)) \\
 &= \left(\sum_{i=0}^{2^n-1} 2 \left[F_{(\frac{1}{2^{n+1}} + \frac{i}{2^n})_s} \circ F_{\frac{s}{2^{n+2}}} \right] \right) (\Phi(x)).
 \end{aligned}$$

From the commutativity of F_t , F_s and Φ and (13), on account of the fact that Φ and F_t are additive we have

$$\begin{aligned}
 & \left(\sum_{i=0}^{2^n-1} 2 \left[F_{(\frac{1}{2^{n+1}} + \frac{i}{2^n})_s} \circ F_{\frac{s}{2^{n+2}}} \right] \right) (\Phi(x)) \\
 &= F_{\frac{s}{2^{n+2}}} \left[\left(\sum_{i=0}^{2^n-1} 2 F_{(\frac{1}{2^{n+1}} + \frac{i}{2^n})_s} \right) (\Phi(x)) \right] = F_{\frac{s}{2^{n+2}}} \left[\sum_{i=0}^{2^n-1} 2 F_{(\frac{1}{2^{n+1}} + \frac{i}{2^n})_s} (\Phi(x)) \right] \\
 &= \Phi \left[\sum_{i=0}^{2^n-1} 2 F_{(\frac{1}{2^{n+1}} + \frac{i}{2^n})_s} \circ F_{\frac{s}{2^{n+2}}} (x) \right].
 \end{aligned}$$

Again using (11) and the commutativity we get

$$\begin{aligned}
 & \Phi \left[\sum_{i=0}^{2^n-1} 2 F_{(\frac{1}{2^{n+1}} + \frac{i}{2^n})_s} \circ F_{\frac{s}{2^{n+2}}} (x) \right] \\
 &= \Phi \left[\sum_{i=0}^{2^n-1} F_{(\frac{1}{2^{n+1}} + \frac{i}{2^n} + \frac{1}{2^{n+2}})_s} (x) + F_{(\frac{1}{2^{n+1}} + \frac{i}{2^n} - \frac{1}{2^{n+2}})_s} (x) \right] \\
 &= \sum_{i=0}^{2^n-1} \left[F_{(\frac{1}{2^{n+2}} + \frac{2i+1}{2^{n+1}})_s} (\Phi(x)) + F_{(\frac{1}{2^{n+2}} + \frac{2i}{2^{n+1}})_s} (\Phi(x)) \right] \\
 &= \sum_{i=0}^{2^{n+1}-1} F_{(\frac{1}{2^{n+2}} + \frac{i}{2^{n+1}})_s} (\Phi(x)).
 \end{aligned}$$

Hence we have proved equality (13) for all positive integers n . Multiplying both sides of it by $\frac{s}{2^n}$ we get

$$\left(\sum_{i=0}^{2^n-1} \frac{s}{2^n} F_{(\frac{1}{2^{n+1}} + \frac{i}{2^n})_s} \right) (\Phi(x)) = \sum_{i=0}^{2^n-1} \frac{s}{2^n} F_{(\frac{1}{2^{n+1}} + \frac{i}{2^n})_s} (\Phi(x))$$

Since $\{F_t: t \geq 0\}$ is regular, it is continuous (cf. Theorem 2 in [13]) and therefore integrable. Hence letting n tend to infinity in the above equality, by Lemma 5 in [11] we obtain the result.

The following theorem gives the solution of the problem (1) in two special cases.

THEOREM 3

Let K be a closed convex cone with nonempty interior in a Banach space. Let $\{F_t : t \geq 0\}$ be a regular cosine family of continuous additive set-valued functions $F_t : K \rightarrow cc(K)$ such that $x \in F_t(x)$ for all $x \in K$, $t \geq 0$, $F_t \circ F_s = F_s \circ F_t$ for all $s, t \geq 0$ and $H(x)$ is the second Hukuhara derivative of $F_t(x)$ at $t = 0$.

- (a) Assume that there is $G(x) = \{0\}$ in problem (1). Then $\Phi(t, x) = F \circ F_t(x)$, $(t, x) \in [0, \infty) \times K$ is the unique solution of this problem.
- (b) Let $\{E_t : t \geq 0\}$ be a regular sine family of continuous additive set-valued functions $E_t : K \rightarrow cc(K)$ associated with $\{F_t : t \geq 0\}$. Assume that $F_t \circ H = H \circ F_t$ for all $t \geq 0$ and there is $F(x) = \{0\}$ in problem (1). Then $\Phi(t, x) = G \circ E_t(x)$, $(t, x) \in [0, \infty) \times K$ is the unique solution of this problem.

Proof. (a) From Lemmas 11 and 13 the set-valued function Φ fulfills equality (1). The initial conditions

$$\Phi(0, x) = F(x), \quad D\Phi(t, x)|_{t=0} = \{0\}$$

are satisfied on account of (10) and (12). By Theorem 2 this solution is unique.

(b) First we shall prove that the set-valued function $(t, x) \mapsto E_t(x)$ satisfies (1). From Lemma 10 we have $DE_t(x) = F_t(x)$ and therefore

$$D^2E_t(x) = DF_t(x) =: G_t(x), \quad H_t(x) := D^2F_t(x) = DG_t(x).$$

Since $G_0(x) = \{0\}$ and $H_t(x) = F_t(H(x))$ (cf. Lemma 11), from Lemma 14 we obtain

$$\begin{aligned} D^2E_t(x) = G_t(x) &= \int_0^t H_u(x) du = \int_0^t F_t(H(x)) du = \left(\int_0^t F_t(\cdot) du \right) (H(x)) \\ &= E_t(H(x)). \end{aligned}$$

By Lemma 13

$$D^2\Phi(t, x) = D^2(G \circ E_t(x)) = G \circ D^2E_t(x) = G \circ E_t(H(x)) = \Phi(t, H(x)).$$

Of course

$$\Phi(0, x) = G \circ E_0(x) = \{0\}$$

and

$$D\Phi(t, x)|_{t=0} = G \circ DE_t(x)|_{t=0} = G(F_0(x)) = G(x).$$

A simple corollary of Theorem 3 is the following

COROLLARY 1

Under assumptions of Theorem 3, if $F_t \circ H = H \circ F_t$ for all $t \geq 0$ and the map $H(x)$ is single-valued, then the set-valued function $\Phi(t, x) = F \circ F_t(x) + G \circ E_t(x)$ is the unique solution of problem (1).

Proof. By Theorem 3 we have

$$D^2\Phi(t, x) = D^2(F \circ F_t(x)) + D^2(G \circ E_t(x)) = F \circ F_t(H(x)) + G \circ E_t(H(x)).$$

Therefore, since H is single-valued, we obtain

$$D^2\Phi(t, x) = (F \circ F_t + G \circ E_t)(H(x)) = \Phi(t, H(x)).$$

It is easy to see that Φ fulfills also the initial conditions.

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