

D.P. Patil, G. Tamone

On type sequences and Arf rings

Abstract. In this article in Section 2 we give an explicit description to compute the type sequence $t_1, \dots, t_n$ of a semigroup Γ generated by an arithmetic sequence (see 2.7); we show that the i -th term t_i is equal to 1 or to the type τ_Γ , depending on its position. In Section 3, for analytically irreducible ring R with the branch sequence $R = R_0 \subsetneq R_1 \subsetneq \dots \subsetneq R_{m-1} \subsetneq R_m = \overline{R}$, starting from a result proved in [4] we give a characterization (see 3.6) of the “Arf” property using the type sequence of R and of the rings R_j , $1 \leq j \leq m-1$. Further, we prove (see 3.9, 3.10) some relations among the integers $\ell^*(R)$ and $\ell^*(R_j)$, $1 \leq j \leq m-1$. These relations and a result of [6] allow us to obtain a new characterization (see 3.12) of semigroup rings of minimal multiplicity with $\ell^*(R) \leq \tau(R)$ in terms of the Arf property, type sequences and relations between $\ell^*(R)$ and $\ell^*(R_j)$, $1 \leq j \leq m-1$.

0. Introduction

Let $(R, \mathfrak{m}_R)$ be a noetherian local one dimensional analytically irreducible domain, i.e., the $\mathfrak{m}$ -adic completion $\hat{R}$ of R is a domain or, equivalently, the integral closure $\overline{R}$ of R in its quotient field $Q(R)$ is a discrete valuation ring and a finite R -module. We further assume that R is residually rational, i.e., R and $\overline{R}$ have the same residue field. A particular important class of rings which satisfy these assumptions are semigroup rings which are coordinate rings of algebroid monomial curves.

Let $v: Q(R) \longrightarrow \mathbb{Z} \cup \{\infty\}$ be the discrete valuation of $\overline{R}$ and let $\mathfrak{C} := \text{ann}_R(\overline{R}/R) = \{x \in R \mid x\overline{R} \subseteq R\}$ be the conductor ideal of R in $\overline{R}$. Then the value semigroup $v(R) = \{v(x) \mid x \in R, x \neq 0\}$ is a numerical semigroup, that is, $\mathbb{N} \setminus v(R)$ is finite and therefore $v(R) = \{0 = v_0, v_1, \dots, v_{n-1}\} \cup \{z \in \mathbb{N} \mid z \geq c\}$, where $0 = v_0 < v_1 < \dots < v_{n-1} < v_n := c$ are elements of $v(R)$, $n := n(R) = \ell(R/\mathfrak{C})$ and the integer $c = c(R) := \ell_{\overline{R}}(\overline{R}/\mathfrak{C})$ is also determined by $\mathfrak{C} = \{x \in Q(R) \mid v(x) \geq c\}$ or, equivalently $\mathfrak{C} = (\mathfrak{m}_{\overline{R}})^c$.

AMS (2000) Subject Classification: Primary 13A18, Secondary 20M25.

First part of this work was done while the first author was visiting the Department of Mathematics, University of Genova, Genova, Italy. The first author thanks the colleagues in Genova for their warm hospitality.

In [11] Matsuoka studied the degree of singularity $\delta = \delta(R) := \ell(\overline{R}/R) = \text{card}(\mathbb{N} \setminus v(R))$ of R by introducing the saturated chain of fractionary ideals

$$\mathfrak{C} = \mathfrak{A}_n \subsetneq \dots \subsetneq \mathfrak{A}_1 = \mathfrak{m} \subsetneq \mathfrak{A}_0 = R \subsetneq \mathfrak{A}_1^{-1} \subsetneq \dots \subsetneq \mathfrak{A}_n^{-1} = \overline{R},$$

where $\mathfrak{A}_i := \{x \in R \mid v(x) \geq v_i\}$ and $\mathfrak{A}_i^{-1} = (R : \mathfrak{A}_i)$, $i = 0, 1, \dots, n$. Moreover, each $\mathfrak{A}_i^{-1}$, $i = 0, \dots, n$ is an overring of R which satisfies the assumptions that we assume for R . The sequence $t_i = t_i(R) := \ell(\mathfrak{A}_i^{-1}/\mathfrak{A}_{i-1}^{-1})$, $i = 1, \dots, n$, is called the *type sequence* of R .

Various algebraic and geometric properties of the ring R are described by some numerical invariants, for example, the degree of singularity and the type sequence. Several authors have studied these numerical invariants (see for example [1], [2], [4], [5], [16]). The first term t_1 is the Cohen–Macaulay type of R and the sum $\sum_{i=1}^n t_i$ is the degree of singularity of R . Further, the “Gorensteinness” and “almost Gorensteinness” are characterized by type sequences (see 1.2). It is worth noting here that if R is a semigroup ring, then the above properties correspond to the properties “symmetric” and “pseudo-symmetric” of numerical semigroups, respectively. These properties are of a special interest (see [7], [17]), since each numerical semigroup can be expressed as an intersection of numerical semigroups that are either symmetric or pseudo-symmetric. Furthermore, if R is analytically irreducible, then the property “Arf” can be described by its type sequence and each term t_i is related to the i -th term in the “branch sequence” of R (see § 4).

In this article we prove the following results:

- (1) If Γ is a numerical semigroup generated by an arithmetic sequence, then we explicitly compute the type sequence (see 2.7) and give (see 2.9) a characterization of almost-Gorensteinness of the semigroup ring $R = K[[\Gamma]]$. This is achieved by studying (see 2.6) the “holes” in Γ by using the explicit description (see 2.5) of the standard basis and the type of the numerical semigroup generated by arithmetic sequence given in [14] and [13], respectively.
- (2) If R is analytically irreducible, then we relate the degree of singularity of R to the branch sequence $R = R_0 \subsetneq R_1 \subsetneq \dots \subsetneq R_{m-1} \subsetneq R_m = \overline{R}$, starting from a result proved in [4] we give a characterization (see 3.6) of the “Arf” property using the type sequence (see 1.3) of R and of the rings R_j , $1 \leq j \leq m-1$. Further, we prove (see 3.9, 3.10) some relations among the integers $\ell^*(R)$ and $\ell^*(R_j)$, $1 \leq j \leq m-1$. These relations and a result of [6] allow us to obtain a new characterization (see 3.12) of semigroup rings of minimal multiplicity with $\ell^*(R) \leq \tau(R)$ in terms of the Arf property, type sequences and relations between $\ell^*(R)$ and $\ell^*(R_j)$, $1 \leq j \leq m-1$.

In Section 4, we also give some illustrative examples to describe our methods.

1. Preliminaries – Assumptions and Notation

Throughout this article we make the following assumptions and notation.

1.1. NOTATION

Let $\mathbb{N}$ and $\mathbb{Z}$ denote the set of all natural numbers and all integers, respectively. Note that we assume $0 \in \mathbb{N}$. Further, for $a, b \in \mathbb{N}$, we denote $[a, b] := \{r \in \mathbb{N} \mid a \leq r \leq b\}$ and $\mathbb{N}_a := \{n \in \mathbb{N} \mid n \geq a\}$.

Let $(R, \mathfrak{m}_R)$ be a noetherian local one dimensional analytically irreducible domain, i.e., the integral closure $\overline{R}$ of R in its quotient field $\mathbb{Q}(R)$ is a discrete valuation ring and is a finite R -module. We further assume that R is residually rational, i.e., the residue field $k_{\overline{R}}$ of $\overline{R}$ is equal to the residue field k_R of R . A particular important class of rings which satisfy these assumptions are semigroup rings which are coordinate rings of algebroid monomial curves.

1.2. MINIMAL REDUCTIONS AND REDUCTION NUMBER

If k_R is infinite, then for every non-zero ideal $\mathfrak{a}$ of R there exists $x \in \mathfrak{a}$ such that xR is a minimal reduction of $\mathfrak{a}$, i.e., $x\mathfrak{a}^m = \mathfrak{a}^{m+1}$ for some $m \in \mathbb{N}$. The natural number $r(\mathfrak{a}) := \min\{m \in \mathbb{N} \mid x\mathfrak{a}^m = \mathfrak{a}^{m+1}\}$ is called the *reduction number* of $\mathfrak{a}$ (see [12]). In particular, if $\mathfrak{a} = \mathfrak{m}$, then $r(\mathfrak{m})$ is called *reduction number* of R . By replacing R by the local ring $R[X]_{\mathfrak{m}[X]}$ of $R[X]$ at the prime ideal $\mathfrak{m}[X]$, we may assume that k_R is infinite and hence assume that a minimal reduction xR of $\mathfrak{m}$ exists.

We shall now recall the notions of *type sequences* and *almost Gorenstein rings*.

1.3. TYPE SEQUENCES — ALMOST GORENSTEIN RINGS

Let R be as in 1.1 and let $v(R)$ be its numerical semigroup, $c = c(v(R))$ be the conductor of $v(R)$, $n = n(R) = \ell(R/\mathfrak{C}) = \text{card}(v(R) \setminus \mathbb{N}_c)$ and $\delta = \delta(R) = \ell(\overline{R}/R) = \text{card}(\mathbb{N} \setminus v(R))$ be the degree of singularity of R (see [11]). Let $0 = v_0 < v_1 < \dots < v_{n-1} < v_n := c$ be elements of $v(R)$ such that $v(R) \setminus \mathbb{N}_c = \{v_0, v_1, \dots, v_{n-1}\}$. Note that (see [11]) $\delta(R)$ is the sum of n positive integers $t_i(R) := \ell(\mathfrak{A}_i^{-1}/\mathfrak{A}_{i-1}^{-1})$, $i = 1, \dots, n$, where $\mathfrak{A}_i := \{x \in R \mid v(x) \geq v_i\}$ and $\mathfrak{A}_i^{-1} := (R : \mathfrak{A}_i) := \{x \in \mathbb{Q}(R) \mid x\mathfrak{A}_i \subseteq R\}$. The first positive integer $t_1(R) = \ell(\mathfrak{m}^{-1}/R)$ is the Cohen–Macaulay type τ_R of R . The sequence $t_1(R), t_2(R), \dots, t_n(R)$ is called the *type sequence* of R . Several authors have studied the properties of type sequences (see [1], [5]). The term “type sequence” is chosen since the first term $t_1(R) = \ell(\mathfrak{m}^{-1}/R)$ is the Cohen–Macaulay type of R . Further, we have $1 \leq t_i(R) \leq \tau_R$ for every $i = 1, \dots, n$ (see [11, § 3, Proposition 2 and Proposition 3]) and hence (see also [5, Proposition 2.1]) $\ell^*(R) \leq (\tau_R - 1)(\ell(R/\mathfrak{C}) - 1)$, where $\ell^*(R) := \tau_R \cdot \ell(R/\mathfrak{C}) - \ell(\overline{R}/R)$. Moreover, the equality holds if and only if $\ell(\overline{R}/R) = \tau_R + \ell(R/\mathfrak{C}) - 1$, or equivalently $t_i(R) = 1$ for $i = 2, \dots, n$.

Type sequence of a numerical semigroup Γ can also be defined analogously: Let $c = c(\Gamma) \in \mathbb{N}$ be the conductor of Γ and let $\Gamma \setminus \mathbb{N}_c = \{0 = v_0, v_1, \dots, v_{n-1}\}$, where $0 = v_0 < v_1 < \dots < v_{n-1} < v_n := c$ are elements of Γ . Further, for $i = 0, \dots, n$, let $\Gamma_i := \{h \in \Gamma \mid h \geq v_i\}$, $\Gamma(i) := \{x \in \mathbb{Z} \mid x + \Gamma_i \subseteq \Gamma\}$ and let $t_i = \text{card}(\Gamma(i) \setminus \Gamma(i-1))$. Then $\Gamma = \Gamma(0) \subseteq \Gamma(1) \subseteq \dots \subseteq \Gamma(n-1) \subseteq \Gamma(n) = \mathbb{N}$ and the sequence $t_i, i = 1, \dots, n$ is called the *type sequence* of Γ . In particular, the cardinality t_1 of the set $T(\Gamma) := \Gamma(1) \setminus \Gamma$ is called the *Cohen–Macaulay type* of the semigroup Γ .

The type sequence of a ring R need not be the same as the type sequence of the numerical semigroup $v(R)$ of R (see for example [5]). However, if $R = K[[\Gamma]]$ is the semigroup ring of a numerical semigroup Γ over a field K , then the type sequence of R is equal to the type sequence of its semigroup $v(R) = \Gamma$.

A ring R in 1.1 is called *almost Gorenstein* if the type sequence of R is $\{\tau_R, 1, 1, \dots, 1\}$, or equivalently, $\ell^*(R)$ attains its upper bound, i.e., $\ell(\overline{R}/R) = \tau_R - 1 + \ell(R/\mathfrak{C})$. It is clear that Gorenstein rings are almost Gorenstein but not conversely (see [16, (1.2)-(1)]).

2. The type sequence of a semigroup generated by an arithmetic sequence

Let R be as in 1.1. In addition to the notations of Section 1, we also fix the following:

2.1. NOTATION

Put $\Gamma := v(R)$ and let $\Gamma_i := v(\mathfrak{A}_i)$, $\Gamma(i)$ and $t_i, i = 1, \dots, n$ be as in 1.3.

In order to compute type sequences explicitly, we need to study the “holes” of Γ , i.e. elements of $\mathbb{N} \setminus \Gamma$. The positions of the holes will therefore determine the type sequence of Γ . To make these things more precise first let us make the following:

2.2. DEFINITION

An element $z \in \mathbb{Z} \setminus \Gamma$ is called a *hole of first type* (respectively, *hole of second type*) of Γ if $c - 1 - z \in \Gamma$ (respectively, if $c - 1 - z \notin \Gamma$). Then $\Gamma' := \{z \in \mathbb{Z} \setminus \Gamma \mid c - 1 - z \in \Gamma\} = \{c - 1 - h \mid h \in \Gamma\}$ is the set of holes of first type of Γ and $\Gamma'' := \{z \in \mathbb{Z} \setminus \Gamma \mid c - 1 - z \notin \Gamma\}$ is the set of holes of second type of Γ . Therefore $\mathbb{Z} = \Gamma \cup \Gamma' \cup \Gamma''$. Further, it is easy to see that:

$$(2.2.a) \quad \begin{cases} \Gamma' \cap \mathbb{N} = \{c - 1 - v_i \mid i \in [0, n - 1]\}; & |\Gamma' \cap \mathbb{N}| = n = c - \delta, \\ \Gamma'' \subseteq \mathbb{N} \setminus \Gamma, & c - 1 \notin \Gamma'' \quad \text{and} \quad T(\Gamma) \subseteq \{c - 1\} \cup \Gamma''. \end{cases}$$

In particular, Γ is symmetric if and only if $\Gamma'' = \emptyset$. For this reason the cardinality of Γ'' is called the *symmetry-defect* of Γ .

The following lemma describes the holes of first type of Γ .

2.3. LEMMA

$(\Gamma(i) \setminus \Gamma(i-1)) \cap \Gamma' = \{c-1-v_{i-1}\}$ for each $i = 1, \dots, n$.

Proof. Easy to verify (this essentially follows from [11, Proposition 2]).

In order to describe the holes of second type, we assume that Γ is generated by an arithmetic sequence (the description of the holes of second type in the general case is given in § 2 and § 3 of [15]). For this in addition to the notation in 2.1 and 2.2, we further fix the following notation:

2.4. NOTATION

Let $m, d \in \mathbb{N}$, $m \geq 2$, $d \geq 1$ be such that $\gcd(m, d) = 1$ and let p be an integer $p \geq 1$, $m_i := m + id$ for $i = 0, 1, \dots, p+1$. Let $\Gamma := \sum_{i=0}^{p+1} \mathbb{N}m_i$ be the semigroup generated by the arithmetic sequence $m_0, m_1, \dots, m_{p+1}$.

For any positive natural number $k \in \mathbb{N}^+$, let $q_k \in \mathbb{N}$ and $r_k \in [1, p+1]$ be the unique integers defined by the equation $k = q_k(p+1) + r_k$. We put $q := q_{m-1}$ and $r := r_{m-1} - 1$. Therefore $q \in \mathbb{N}$, $r \in [0, p]$ and $m-2 = q(p+1) + r$.

Put $s_0 = 0$ and $s_k := m_{r_k} + q_k m_{p+1} = (1 + q_k)m + (r_k + q_k(p+1))d$ for $k \in [1, m-1]$. Further, we put $S_1 := \{m_i + jm_{p+1} \mid i \in [1, p+1] \text{ and } j \in [0, q-1]\}$ and $S_2 := \{m_i + qm_{p+1} \mid i \in [1, r+1]\}$. Note that $S_1 = \emptyset$, if $q = 0$.

Let $0 = v_0 < v_1 < \dots < v_{n-1} < v_n := c$ be elements of Γ such that $\Gamma \setminus \mathbb{N}_c = \{0 = v_0, v_1, \dots, v_{n-1}\}$. For $i \in [0, n]$, the element $v_i \in \Gamma$ is called the i -th element of Γ .

2.5. PROPOSITION

With the notations as in 2.4 we have:

- (1) The standard basis $S := S_m(\Gamma)$ with respect to the multiplicity $m = m_0$ of Γ is

$$S = \{s_k \mid k \in [0, m-1]\} = \{0\} \cup S_1 \cup S_2.$$

- (2) The conductor $c := c(\Gamma)$ and the degree of singularity $\delta := \delta(\Gamma)$ of Γ are

$$c = (m-1)(d+q) + q + 1 \quad \text{and} \quad \delta = ((m-1)(d+q) + (r+1)(q+1)) / 2.$$

- (3) The set $T := T(\Gamma) = \Gamma(1) \setminus \Gamma = \{m_i + qm_{p+1} - m_0 \mid i \in [1, r+1]\} = \{c-1-(r-i+1)d \mid i \in [1, r+1]\}$. In particular, the Cohen-Macaulay type of Γ is $\tau = \tau_\Gamma = r+1$.

Proof. (1) and (3) are special cases of the general results proved in [14, (3.5)] and [13, § 5]. (2) is proved in [18, § 3, Supplement 6].

Now we give an explicit description of the positions of the holes of second type of Γ .

2.6. LEMMA

With the notations as in 2.1, 2.2 and 2.4, we have:

- (1) $\text{card}(\Gamma'') = (q+1)r$.
- (2) $\Gamma'' = \{x - jm_{p+1} \mid x \in \Gamma(1) \setminus \Gamma, x \neq c-1 \text{ and } j \in [0, q]\}$.
- (3) For each $j \in [0, q]$, there exists a unique integer $i(j) \in [0, n-1]$ such that $jm_{p+1} = v_{i(j)}$ is the $i(j)$ -th element of Γ . Moreover,

$$\Gamma(i(j)+1) \setminus \Gamma(i(j)) = \{x - jm_{p+1} \mid x \in \Gamma(1) \setminus \Gamma\}.$$

In particular, $\text{card}(\Gamma(i(j)+1) \setminus \Gamma(i(j))) = \tau_\Gamma = r+1$.

Proof. (1) Immediate from 2.5-(2). (2) Easy to verify using 2.5-(3). For the proof of (3) see [15, § 2 and § 3].

Now we give an explicit description of the type sequence of a semigroup generated by an arithmetic sequence.

2.7. THEOREM

Let $m, d \in \mathbb{N}$, $m \geq 3$, $d \geq 1$ be such that $\gcd(m, d) = 1$ and let p be an integer with $1 \leq p \leq m-2$. Let $\Gamma := \sum_{k=0}^{p+1} \mathbb{N}m_k$ be the semigroup generated by the arithmetic sequence $m_k := m + kd$, $k = 0, 1, \dots, p+1$. Let $q \in \mathbb{N}$ and $r \in [0, p]$ be the unique integers defined by the equation $m-2 = q(p+1) + r$. Further, let $c \in \Gamma$ be the conductor of Γ , $\mathbb{N}_c = \{z \in \mathbb{N} \mid z \geq c\}$ and let $\Gamma \setminus \mathbb{N}_c = \{0 = v_0, v_1, \dots, v_{n-1}\}$ with $v_0 < v_1 < \dots < v_{n-1} < v_n := c$. Then the i -th term $t_i = t_i(\Gamma)$ of the type sequence $(t_1, t_2, \dots, t_n)$ of Γ is

$$t_i = \begin{cases} 1, & \text{if } v_{i-1} \neq jm_{p+1} \text{ for every } j \in [0, q], \\ r+1, & \text{if } v_{i-1} = jm_{p+1} \text{ for some } j \in [0, q]. \end{cases}$$

Proof. If $v_{i-1} \neq jm_{p+1}$ for every $j \in [0, q]$, then $\Gamma(i) \setminus \Gamma(i-1) = \{c-1 - v_{i-1}\}$ by 2.6-(1), (2), (3) and hence $\text{card}(\Gamma(i) \setminus \Gamma(i-1)) = 1$. If $v_{i-1} = jm_{p+1}$ for some $j \in [0, q]$, then $\text{card}(\Gamma(i) \setminus \Gamma(i-1)) = r+1$ by 2.6-(3).

2.8. COROLLARY

In addition to the notations and assumptions as in 2.7, further assume that $d = 1$. Then the i -th term t_i of the type sequence $(t_1, t_2, \dots, t_n)$ of Γ is

$$t_i = \begin{cases} r+1, & \text{if } i = \binom{j+1}{2}(p+1) + j + 1 \text{ for some } j \in [0, q], \\ 1, & \text{otherwise.} \end{cases}$$

Proof. It is easy to check that for every $j \in [0, q]$, we have

$$i(j) = \text{card}\left(\biguplus_{t=0}^j \Gamma^{(t)}\right) = \sum_{t=0}^j (t(p+1) + 1) = \binom{j+1}{2}(p+1) + j + 1$$

and jm_{p+1} is the $(i(j) - 1)$ -th element $v_{i(j)-1}$ in Γ . Now the assertion is clear from 2.7.

2.9. COROLLARY

Let m, d, p, q, r and Γ be as in 2.7 and let $R := K[[\Gamma]]$ be the semigroup ring of Γ over a field K . Then

- (1) R is Gorenstein if and only if $r = 0$.
- (2) Assume that R is not Gorenstein. Then R is almost Gorenstein if and only if $m = p + 2$. Moreover, in this case we have $\tau_R = m - 1$.

Proof. (1) Note that $\tau_R = r + 1$ by 2.5-(3). Therefore R is Gorenstein if and only if $r + 1 = \tau_R = 1$, i.e., $r = 0$.

(2) R is almost Gorenstein if and only if the type sequence of R is $\tau_R = r + 1, 1, \dots, 1$ or equivalently (by 2.7) $q = 0$, i.e. $m - 2 = r$. Now, since $m \geq p + 2$ and $r \leq p$, we have $m - 2 = r$ if and only if $m - 2 = p$.

3. Numerical invariants of analytically irreducible Arf rings

In this section we first recall some definitions and results proved in [9] on blowing-up and Arf rings. These results hold more generally, for semi-local 1-dimensional Cohen–Macaulay rings.

Let R be a semi-local Cohen–Macaulay ring of dimension 1 and let $\mathfrak{m}$ be the (Jacobson) radical of R . Let $\overline{R}$ be the integral closure of R in its total quotient ring $Q(R)$. An ideal $\mathfrak{a}$ in R is called *open* if it is open in the $\mathfrak{m}$ -adic topology on R , or, equivalently, $\mathfrak{m}^n \subseteq \mathfrak{a}$ for some $n \geq 1$, or, equivalently, the length $\ell(R/\mathfrak{a})$ is finite. For any two R -submodules M, N of $\overline{R}$, we put $(M : N) := \{y \in \overline{R} \mid yN \subseteq M\}$.

For an open ideal $\mathfrak{a}$ in R , let $B(\mathfrak{a}) := \cup_{n \in \mathbb{N}} (\mathfrak{a}^n : \mathfrak{a}^n)$. The ring $B(\mathfrak{a})$ is called the *blowing-up* of R along $\mathfrak{a}$ or the *first neighbourhood ring* of $\mathfrak{a}$.

3.1. PROPOSITION ([9, Proposition 1.1])

For an open ideal $\mathfrak{a}$ in R , the ring $B(\mathfrak{a})$ is a finitely generated R -module and $R \subseteq B(\mathfrak{a}) \subseteq \overline{R}$. Moreover, if R is local and if $\mathfrak{a}$ is a $\mathfrak{m}$ -primary ideal which is not principal, then $R \subsetneq B(\mathfrak{a})$. In particular, if R is local and if R is not a discrete valuation ring, then $R \subsetneq B(\mathfrak{m})$. Furthermore, there exists a non-zero divisor $x \in \mathfrak{a}$ such that $B(\mathfrak{a}) = R[\frac{z_1}{x}, \dots, \frac{z_r}{x}]$, where $z_1, \dots, z_r$ is a generating set for the ideal $\mathfrak{a}$. In particular, $\mathfrak{a}B(\mathfrak{a}) = xB(\mathfrak{a})$.

An open ideal $\mathfrak{a}$ in R is called *stable* in R if $B(\mathfrak{a}) = (\mathfrak{a} : \mathfrak{a})$, or, equivalently, $\mathfrak{a}B(\mathfrak{a}) = \mathfrak{a}$. It is clear that if $\mathfrak{a}$ is an open ideal in R , then $\mathfrak{a}^n$ is stable for some $n > 0$ and if $\mathfrak{a}^n$ is stable, then $\mathfrak{a}^m$ is stable for every $m \geq n$.

Recall that an ideal $\mathfrak{a}$ of R is said to be *integrally closed* in R if $\mathfrak{a} = \overline{\mathfrak{a}} := \{z \in R \mid z^n + a_1 z^{n-1} + \cdots + a_n = 0 \text{ with } a_j \in \mathfrak{a}^j \text{ for every } j = 1, \dots, n\}$.

Now we recall the definition of an *Arf ring* studied by Lipman in [9].

3.2. BRANCH SEQUENCE AND ARF RINGS

Let R be a ring as above. Since $\overline{R}$ is a finite R -module, there exists a finite sequence

$$(3.2.1) \quad R = R_0 \subsetneq R_1 \subsetneq \cdots \subsetneq R_{m-1} \subsetneq R_m = \overline{R}$$

of one dimensional semi-local noetherian rings such that for each $1 \leq i \leq m$, the ring R_i is obtained from R_{i-1} by blowing up the radical of R_{i-1} . For each maximal ideal $\mathfrak{n}$ of $\overline{R}$, every local ring $R'_i := (R_i)_{\mathfrak{n} \cap R_i}$ is called *infinitely near to R* . For each $i = 0, \dots, m$, the multiplicity and the residue field of the local ring R'_i are denoted by $e(R'_i)$ and k_i , respectively. The sequence $R'_0, R'_1, \dots, R'_m$ is called the *branch sequence of R along $\mathfrak{n}$* and the sequence of pairs $((e(R'_i), [k_i : k_0]), i = 0, \dots, m)$ is called the *multiplicity sequence of R* , where $[k_i : k_0]$ denotes the degree of the field extension $k_i | k_0$ (see [9, pp. 661, 669]). In particular, if R is analytically irreducible, residually rational and $R \neq \overline{R}$, then each R_i in (3.2.1) is also analytically irreducible, residually rational; if $\mathfrak{m}_i$ is the maximal ideal of R_i , then the ring R_i is obtained from R_{i-1} by blowing up $\mathfrak{m}_{i-1}$. Further, $R_i = R'_i$ for each $i = 0, \dots, m$, since $\overline{R}$ is local and $\mathfrak{n}$ is the only maximal ideal in $\overline{R}$.

A semi-local Cohen–Macaulay ring of dimension 1 is called an *Arf ring* if every integrally closed open ideal in R is stable, or, equivalently (see [9, Theorem 2.2]), if A is any local ring infinitely near to R , then A has maximal embedding dimension, i.e., $\text{embdim}(A) = e(A)$. In particular, if a local ring R is Arf, then R has maximal embedding dimension.

In the Proposition 3.3 below, we recall some conditions for a 1-dimensional Cohen–Macaulay local ring R which are equivalent to the equality $\text{embdim}(R) = e(R)$.

3.3. PROPOSITION

Let $(R, \mathfrak{m})$ be a one dimensional local Cohen–Macaulay ring and let $\mathfrak{a}$ be an $\mathfrak{m}$ -primary ideal. Then the following statements are equivalent:

- (i) $B(\mathfrak{a}) = (\mathfrak{a} : \mathfrak{a})$, i.e., $\mathfrak{a}$ is stable.
- (ii) There exists $z \in \mathfrak{a}$ such that $z\mathfrak{a} = \mathfrak{a}^2$.

In particular, the maximal ideal $\mathfrak{m}$ is stable $\iff \text{embdim}(R) = e(R) \iff \tau_R = e(R) - 1$.

Proof. For the equivalence of (i) and (ii) see [9, 1.8] and [12, 5.1]. If $\mathfrak{a} = \mathfrak{m}$, then the equivalence: $\mathfrak{m}$ is stable $\iff \text{embdim}(R) = e(R)$ is proved in [9, 1.8 and 1.10]. Therefore to complete the proof is it enough to show that: $\tau_R = e_0(R) - 1 \iff x\mathfrak{m} = \mathfrak{m}^2$ for some $x \in \mathfrak{m}$. Let $x \in \mathfrak{m}$ be a minimal reduction of $\mathfrak{m}$. Then, since R is Cohen–Macaulay, $\ell(R/xR) = e(R)$ and from $xR \subseteq \dots \subseteq (xR : \mathfrak{m}) \subseteq \dots \subseteq \mathfrak{m} \subsetneq R$ we have $\tau_R = \ell((R : \mathfrak{m})/R) = \ell((xR : \mathfrak{m})/xR) \leq \ell(R/xR) - 1 = e(R) - 1$. Moreover, the equality $\tau_R = e(R) - 1 \iff \ell((xR : \mathfrak{m})/xR) = \ell(R/xR) - 1 \iff \ell(R/(xR : \mathfrak{m})) = 1 = \ell(R/\mathfrak{m}) \iff (xR : \mathfrak{m}) = \mathfrak{m} \iff x\mathfrak{m} = \mathfrak{m}^2$.

The following result proved in [4] (see also [5]) shows how the property Arf is described by the type sequence of its value semigroup.

3.4. PROPOSITION ([4, Theorem 1.7-(5)])

Let $(R, \mathfrak{m})$ be a one dimensional noetherian local analytically irreducible, residually rational domain. Let v be the discrete valuation of $\overline{R}$ and let $v(R) = \{0 = v_0, v_1, \dots, v_{n-1}\} \cup \mathbb{N}_c$ be the value semigroup of R , where $0 = v_0 < v_1 < \dots < v_{n-1} < v_n = c$, $\mathfrak{C}$ is the conductor of $\overline{R}$ over R , $n := n(R) = \ell(R/\mathfrak{C})$ and $c = c(R) := \ell(\overline{R}/\mathfrak{C})$. If R is an Arf ring, then $t_i = v_i - v_{i-1} - 1$ is the i -th term in the type sequence of R .

Now we recall the following characterization of Arf rings given in [9].

3.5. PROPOSITION ([9, Theorem 2.2 and Corollary 3.8])

Let $(R, \mathfrak{m})$ be a one dimensional noetherian local analytically irreducible ring and let $R = R_0 \subsetneq R_1 \subsetneq \dots \subsetneq R_{m-1} \subsetneq R_m = \overline{R}$ be the branch sequence of R . Then R is an Arf ring if and only if $\text{embdim}(R_j) = e(R_j)$ for each $j = 0, \dots, m$. Moreover, if R is complete with algebraically residue field k , then R is an Arf ring if and only if the value semigroup $v(R)$ of R is $\{0, e(R_0), e(R_0) + e(R_1), \dots, e(R_0) + \dots + e(R_{m-2})\} \cup \mathbb{N}_c$, where $c = e(R_0) + \dots + e(R_{m-2}) + e(R_{m-1})$.

Under the assumptions of 3.5 we can characterize Arf rings using the type sequences of R and of each term in the branch sequence of R .

3.6. THEOREM

Let $(R, \mathfrak{m})$ be a complete local analytically irreducible domain with algebraically closed residue field k . Let $R = R_0 \subsetneq R_1 \subsetneq \dots \subsetneq R_{m-1} \subsetneq R_m = \overline{R}$ be the branch sequence of R . For each $j = 0, \dots, m-1$, let $\mathfrak{C}_j$ be the conductor of $\overline{R}$ over R_j , and let $n_j = n(R_j)$, $c_j = \ell(\overline{R}/\mathfrak{C}_j)$ and $t_i(R_j)$ be the i -th term in the type sequence of R_j . Then: R is an Arf ring if and only if for each $j = 0, \dots, m-1$ and $i = 1, \dots, n_j$, we have $n_j = m-j$ and $t_i(R_j) = e(R_{j+i-1}) - 1 = t_{i+1}(R_{j-1})$.

Proof. ($\Rightarrow$): By the assumptions on R and 3.5, for each $j = 0, \dots, m-1$ we have R_j is an Arf complete domain with integral closure $\overline{R}$, the same residue field k , $R_j \subsetneq R_{j+1} \subsetneq \dots \subsetneq R_{m-1} \subsetneq R_m = \overline{R}$ is the branch sequence of R_j and the value semigroup $v(R_j)$ is $\{0, v_{1,j}, v_{2,j}, \dots, v_{m-j-1,j}\} \cup \mathbb{N}_{c_j}$, where $v_{i,j} = e(R_j) + \dots + e(R_{j+i-1})$, $i = 1, \dots, m-j-1$ and $c_j = e(R_j) + \dots + e(R_{m-1})$. Therefore we have $n_j = n(R_j) = (m-j-1) + 1 = m-j$. Further, for each $j = 0, \dots, m-1$, if $\{t_i(R_j) \mid 1 \leq i \leq m-j\}$ is the type sequence of R_j , then by 3.4 we have $t_i(R_j) = v_{i,j} - v_{i-1,j} - 1 = e(R_{j+i-1}) - 1 = v_{i+1,j-1} - v_{i,j-1} - 1 = t_{i+1}(R_{j-1})$ for every $1 \leq i \leq m-j$.

($\Leftarrow$): For each $j = 0, \dots, m-1$, by assumption, in particular, we have $\tau_{R_j} = t_1(R_j) = e(R_j) - 1$. Therefore $\text{emdim}(R_j) = e(R_j)$ by 3.3 and hence R is an Arf ring by 3.5.

In particular, for the ready reference we note the following formulas for the i -th term t_i in the type sequence of R , in terms of the types, the multiplicities and the lengths arising from the terms of the branch sequence of R .

3.7. COROLLARY

Let $(R, \mathfrak{m})$ be an Arf complete local domain with algebraically closed residue field k and let $R = R_0 \subsetneq R_1 \subsetneq \dots \subsetneq R_{m-1} \subsetneq R_m = \overline{R}$ be the branch sequence of R . Then: $m = n = n(R)$ and for each $i = 1, \dots, n$, the i -th term t_i in the type sequence of R is given by: $t_i = \tau(R_{i-1}) = e(R_{i-1}) - 1 = \ell(R_i/R_{i-1})$.

3.8. COROLLARY

Let $(R, \mathfrak{m})$ be an Arf complete local domain with algebraically closed residue field k and let $B = B(\mathfrak{m})$ be the blowing up of R along $\mathfrak{m}$. If $t_1, \dots, t_n$ is the type sequence of R , then $t_2, \dots, t_n$ is the type sequence of B .

Recall that several authors (see for example [6], [16] and references in them) have tried to characterize rings for which the inequality $\ell(\overline{R}/R) \leq \tau_R \cdot \ell(R/\mathfrak{C})$ is an equality or to give a classification of the rings according to the value of the integer $\ell^*(R) := \tau_R \cdot \ell(R/\mathfrak{C}) - \ell(\overline{R}/R)$. Now, using the special properties of Arf rings and 3.6 we give some relations between $\ell^*(R)$, the terms in the type sequence of R , $\ell^*(R_j)$ and $e(R_j)$, where $R = R_0 \subsetneq R_1 \subsetneq \dots \subsetneq R_{m-1} \subsetneq R_m = \overline{R}$ is the branch sequence of R . More precisely:

3.9. THEOREM

Let $(R, \mathfrak{m})$ be a complete local analytically irreducible domain with algebraically closed residue field k . Let $R = R_0 \subsetneq R_1 \subsetneq \dots \subsetneq R_{m-1} \subsetneq R_m = \overline{R}$ be the branch sequence of R and let $e_j = e(R_j)$ be the multiplicity of R_j , $j = 0, \dots, m$. Let $t_1, \dots, t_n$ be the type sequence of R . Then:

$$(1) \quad \ell^*(R_{m-1}) = 0 \text{ and } \ell^*(R_j) = \sum_{i=j+1}^{m-1} (m-i) \cdot (t_i - t_{i+1}) \text{ for } 1 \leq j \leq m-2.$$

- (2) For $j = 0, \dots, m-2$, we have $\ell^*(R) = \ell^*(R_j) + \sum_{i=1}^j (m-i) \cdot (t_i - t_{i+1}) = \ell^*(R_j) + \sum_{i=1}^j (m-i) \cdot (e_{i-1} - e_i)$.

Proof. We shall use the notation as in 3.6. Note that for every $0 \leq j \leq m$, $n_j = m - j$; in particular, $n = n(R) = n(R_0) = m$. Further, $t_{j+1}, \dots, t_m$ is the type sequence of R_j ; in particular, t_m is the type sequence of R_{m-1} and hence $n_{m-1} = n(R_{m-1}) = 1$ and $\ell^*(R_{m-1}) = 0$. Now, for $0 \leq j \leq m-2$, we have

$$\begin{aligned} \ell^*(R_j) &= \tau(R_j) \cdot \ell(R_j/\mathfrak{C}_j) - \ell(\overline{R}/R_j) = t_{j+1} \cdot n_j - \sum_{i=j+1}^m \ell(R_i/R_{i-1}) \\ &= t_{j+1}(m-j) - \sum_{i=j+1}^m t_i = \sum_{i=j+2}^m (t_{j+1} - t_i) = \sum_{i=j+1}^{m-1} (m-i) \cdot (t_i - t_{i+1}). \end{aligned}$$

This proves (1). Now, since $t_i = e(R_{i-1}) - 1 = e_{i-1} - 1$ by 3.7, we have $t_i - t_{i+1} = e_{i-1} - e_i$ for every $1 \leq i \leq m-1$ and hence by (1), we have

$$\begin{aligned} \ell^*(R) &= \ell^*(R_0) = \sum_{i=1}^{m-1} (m-i) \cdot (t_i - t_{i+1}) = \sum_{i=1}^j (m-i) \cdot (t_i - t_{i+1}) + \ell^*(R_j) \\ &= \sum_{i=1}^j (m-i) \cdot (e_{i-1} - e_i) + \ell^*(R_j). \end{aligned}$$

This proves (2).

3.10. COROLLARY

With the same assumptions and notation as in 3.9, we have:

- (1) $e_j \leq e_{j-1}$ and $\ell^*(R_j) \leq \ell^*(R)$ for every $j = 1, \dots, m-1$.
 (2) For $1 \leq j \leq m-2$, $\ell^*(R_j) = \ell^*(R)$ if and only if $e_0 = \dots = e_{j-1} = e_j$.

Proof. Note that the inequality $e_j \leq e_{j-1}$ holds for every analytically irreducible domain. Therefore by 3.9-(2) $\ell^*(R_j) \leq \ell^*(R)$ for every $j = 1, \dots, m-2$ and by 3.9-(1) $\ell^*(R_{m-1}) = 0 \leq \ell^*(R)$.

(2) Since $m-i > 0$ for every $1 \leq i \leq j \leq m-2$, by 3.9-(2) $\ell^*(R_j) = \ell^*(R)$ if and only if $e_{j-1} = e_j$ for every $j = 1, \dots, m-2$.

Now for complete semigroup rings R such that $\ell^*(R) \leq \tau_R$ and $\tau_R = e(R) - 1$ using [6, Corollary 2.14], we give another characterization involving the type sequence of R and the type sequences of the rings R_j in the branch sequence of R , Arf rings, $\ell^*(R)$, $\ell^*(R_j)$, $1 \leq j \leq m-1$ (see 3.12 below). First we shall prove the following lemma concerning two special types of semigroup rings considered in [6, Corollary 2.14].

3.11. LEMMA

Let Γ be a numerical semigroup and let $R = K[[\Gamma]]$ be the semigroup ring of Γ over a field K . Let $R = R_0 \subsetneq R_1 \subsetneq \dots \subsetneq R_{m-1} \subsetneq R_m = \overline{R}$ be the branch sequence of R and let $e_j = e(R_j)$, $j = 0, \dots, m-1$.

- (1) Suppose that Γ is generated by $e, pe+1, pe+2, \dots, pe+(e-1)$, where e, p are positive integers with $e \geq 3$. Then $m = p$, R is an Arf ring and $e_j = e(R) = e$ for every $j = 0, \dots, p-1$.
- (2) Suppose that Γ is generated by $e, pe-a, pe-a+1, \dots, pe-a+(a-1)$, where e, p, a are positive integers with $e \geq 3, p \geq 2$ and $1 \leq a \leq e-1$. Then $m = p$, R is an Arf ring, $e_j = e(R) = e$ for every $j = 0, \dots, p-2$ and $e_{p-1} = e-a$.

Proof. (1) It is easy to check that $\text{emdim}(R) = e(R) = e$; in fact the e elements $e, pe+1, pe+2, \dots, pe+(e-1)$ form a minimal set of generators for the semigroup Γ and $e < pe+1$. For $j = 0, \dots, p-1$, let Γ_j be the semigroup generated by $e, (p-j)e+1, (p-j)e+2, \dots, (p-j)e+(e-1)$ and let $\Gamma_p = \mathbb{N}$. Then it is easy to verify that the sequence $R = K[[\Gamma_0]] \subsetneq K[[\Gamma_1]] \subsetneq \dots \subsetneq K[[\Gamma_{p-1}]] \subsetneq K[[\Gamma_p]] = \overline{R}$ is the branch sequence of R . Therefore $m = p$ and $\text{emdim}(R_j) = e = e_j$ for each $j = 0, \dots, p-1$ and hence R is Arf by 3.5.

(2) For $j = 0, \dots, p-2$, let Γ_j be the semigroup generated by $e, (p-j)e-a, (p-j)e-a+1, \dots, (p-j)e-a+(e-1)$ (note that this is a minimal set of generators for Γ_j), Γ_{p-1} generated by $e-a, e-a+1, \dots, e, e+1, \dots, 2e-a-1$ (note that $e-a < e$ and that $e-a, e-a+1, 2e-2a-1$ is a minimal set of generators for Γ_{p-1}) and let $\Gamma_p = \mathbb{N}$. Then it is easy to verify that the sequence $R = K[[\Gamma_0]] \subsetneq K[[\Gamma_1]] \subsetneq \dots \subsetneq K[[\Gamma_{p-2}]] \subsetneq K[[\Gamma_{p-1}]] \subsetneq K[[\Gamma_p]] = \overline{R}$ is the branch sequence of R and $\text{emdim}(R_j) = e = e_j$ for each $j = 0, \dots, p-2$, $\text{emdim}(R_{p-1}) = e-a = e_{p-1}$ and hence R is Arf by 3.6.

3.12. THEOREM

Let Γ be a numerical semigroup of multiplicity e and type τ_Γ . Let $R = K[[\Gamma]]$ be the semigroup ring of Γ over a field K and let $R = R_0 \subsetneq R_1 \subsetneq \dots \subsetneq R_{m-1} \subsetneq R_m = \overline{R}$ be the branch sequence of R . Let $t_1 = \tau_\Gamma, t_2, \dots, t_n$ be the type sequence of R . For a natural number $a \leq t_1$, the following statements are equivalent:

- (i) $\ell^*(R) = a$ and $\text{emdim}(R) = e(R)$.
- (ii) R is an Arf ring and

$$t_i = \begin{cases} e-1, & \text{if } 1 \leq i \leq m \text{ and } a = 0, \\ e-1, & \text{if } 1 \leq i \leq m-1 \text{ and } a > 0, \\ e-a-1, & \text{if } i = m \text{ and } a > 0. \end{cases}$$

(iii) R is an Arf ring and

$$\ell^*(R) = \ell^*(R_j) = \begin{cases} 0, & \text{if } 1 \leq j \leq m-1 \text{ and } a = 0, \\ a, & \text{if } 1 \leq j \leq m-2 \text{ and } a > 0, \end{cases}$$

and if $a > 0$, then $\ell^*(R_{m-1}) = 0$.

Proof. (i) $\Rightarrow$ (ii): Note that by 3.3 $\text{emdim}(R) = e(R) \iff \tau_R = e(R) - 1$. Therefore by [6, Corollary 2.14] the value semigroup of R is:

$$v(R) = \Gamma = \begin{cases} \mathbb{N}e + \sum_{i=1}^{e-1} \mathbb{N}(pe + i), & \text{if } a = 0 \text{ (see 3.11-(1))}, \\ \mathbb{N}e + \sum_{i=0}^{a-1} \mathbb{N}(pe - a + i), & \text{if } a > 0 \text{ (see 3.11-(2))}. \end{cases}$$

In particular, $n = n(R) = m = p$ and R is an Arf ring (see 3.11). Further, by 3.7 and 3.11, i -th term t_i in the type sequence of R is given by

$$t_i = \begin{cases} e - 1, & \text{if } 1 \leq i \leq m \text{ and } a = 0, \\ e - 1, & \text{if } 1 \leq i \leq m - 1 \text{ and } a > 0, \\ e - a - 1, & \text{if } i = m \text{ and } a > 0. \end{cases}$$

(ii) $\Rightarrow$ (iii): If $a = 0$, then $\ell^*(R) = 0$ and by 3.9-(2) $\ell^*(R_j) = 0$ for every $j = 1, \dots, m-1$. If $a > 0$, then by 3.9, we have $\ell^*(R_{m-1}) = 0$ and $\ell^*(R) = t_{m-1} - t_m = a = \ell^*(R_j)$ for every $j = 1, \dots, m-2$.

(iii) $\Rightarrow$ (i): Clearly $\ell^*(R) = a$ by (iii) and since R is an Arf ring, we have $\text{emdim}(R) = e(R)$.

4. Examples

In this section we give some examples of Arf rings and some of not Arf rings. In the following examples R denote the semigroup ring $K[[\Gamma]]$ of the semigroup Γ over a field K . Note that in this case each term R_j in the branch sequence of R is also semigroup ring; in fact, if Γ is generated by $n_1, n_2, \dots, n_p$ with $n_1 < n_2 < \dots < n_p$, then $R_1 = K[[\Gamma_1]]$, where $\Gamma_1 = v(R_1)$ is generated by $n_1, n_2 - n_1, \dots, n_p - n_1$; by repeating this argument we get the result for R_j , $j \geq 2$.

4.1. EXAMPLE

Let $e, r, r' \in \mathbb{N}$ with $e \geq 3$, $1 \leq r$, $1 \leq r'$, $r + r' \leq e - 1$ and let Γ be the semigroup generated by the sequence $e, e + r, e + r + r', e + r + r' + 1, \dots, 2e + r + r' - 1$. We consider the four cases (i) $r' = r = 1$; (ii) $r' = 1$, $r \geq 2$; (iii) $1 < r' \leq r$; (iv) $r < r'$ separately.

(a) We first compute the type sequence of R .

CASE (i): $(r', r) = (1, 1)$: This case is considered in 3.11-(1) ($p = 1$). In this case $t_1 = e - 1$ is the type sequence of R .

CASE (ii): $r' = 1$ and $r \geq 2$: In this case $c = e + r$ and $\Gamma \setminus \mathbb{N}_c = \{0, e\}$. Therefore $n = 2$ and $v_1 = e$. Further, $\Gamma(1) \setminus \Gamma(0) = T(\Gamma) = [r, e - 1] \cup [e + 1, e + r - 1]$ and $\Gamma(2) \setminus \Gamma(1) = [1, r - 1]$. Therefore $t_1 = \tau_R = e - 1$, $t_2 = r - 1$ and the type sequence of Γ is $e - 1, r - 1$. Therefore, R is almost Gorenstein if and only if $r = 2$.

CASE (iii): $1 < r' \leq r$: In this case $c = e + r + r'$ and $\Gamma \setminus \mathbb{N}_c = \{0, e, e + r\}$. Therefore $n = 3$ and $v_1 = e$, $v_2 = e + r$. Further, we have

$$\begin{aligned} \Gamma(1) \setminus \Gamma(0) &= T(\Gamma) = \{r\} \cup [r + r', e + r + r' - 1] \setminus \{e, e + r\}, \\ \Gamma(2) \setminus \Gamma(1) &= \begin{cases} [r + 1, r + r' - 1], & \text{if } r = r', \\ [r', r + r' - 1] \setminus \{r\}, & \text{if } r' < r, \end{cases} \end{aligned}$$

and

$$\Gamma(3) \setminus \Gamma(2) = \begin{cases} [1, r - 1], & \text{if } r' = r, \\ [1, r' - 1], & \text{if } r' < r. \end{cases}$$

Therefore

$$t_1 = \tau_R = e - 1, \quad t_2 = \begin{cases} r' - 1, & \text{if } r' = r, \\ r - 1, & \text{if } r' < r, \end{cases} \quad t_3 = \begin{cases} r - 1, & \text{if } r' = r, \\ r' - 1, & \text{if } r' < r, \end{cases}$$

and the type sequence of Γ is

$$\begin{cases} e - 1, r' - 1, r - 1, & \text{if } r' = r, \\ e - 1, r - 1, r' - 1, & \text{if } r' < r. \end{cases}$$

Therefore, R is almost Gorenstein if and only if $(r', r) = (2, 2)$.

CASE (iv): $r < r'$: In this case $c = e + r + r'$ and $\Gamma \setminus \mathbb{N}_c = \{0, e, e + r\}$. Therefore $n = 3$ and $v_1 = e$, $v_2 = e + r$. Further, we have $\Gamma(1) \setminus \Gamma(0) = T(\Gamma) = [r + r', e + r + r' - 1] \setminus \{e, e + r\}$, $\Gamma(2) \setminus \Gamma(1) = [r', r + r' - 1]$ and $\Gamma(3) \setminus \Gamma(2) = [1, r' - 1]$. Therefore $t_1 = \tau_R = e - 2$, $t_2 = r$, $t_3 = r' - 1$ and the type sequence of Γ is $e - 2, r, r' - 1$. Therefore, R is almost Gorenstein if and only if $(r, r') = (1, 2)$.

(b) Now we shall show that R is an Arf ring in cases (i), (ii), (iii) and R is not Arf in case (iv).

CASE (i): $(r', r) = (1, 1)$: in this case R is an Arf ring (see 3.11-(1) ($p = 1$)).

CASE (ii): $r' = 1$ and $r \geq 2$: In this case, let $\Gamma_0 := \Gamma$, Γ_1 be the numerical semigroup generated by $[r, 2r - 1]$, $\Gamma_2 := \mathbb{N}$ and let $R_j := K[[\Gamma_j]]$ for $j = 0, 1, 2$. Then it is easy to see that $e(R_0) = e = \text{embdim}(R_0)$, $e(R_1) = r = \text{embdim}(R_1)$, $e(R_2) = 1 = \text{embdim}(R_2)$, $\Gamma = \Gamma_0 \subsetneq \Gamma_1 \subsetneq \Gamma_2 = \mathbb{N}$ and $R = R_0 \subsetneq R_1 \subsetneq R_2 = \bar{R}$ is the branch sequence of R . Therefore R is an Arf ring by 3.5.

CASE (iii): $1 < r' \leq r$: In this case, let $\Gamma_0 := \Gamma$, Γ_1 be the numerical semigroup generated by $\{r\} \cup [r + r', 2r + r' - 1]$ (note that Γ_1 is minimally generated by $\{r\} \cup ([r + r', 2r + r' - 1] \setminus \{2r\})$), Γ_2 be the numerical semigroup generated by $[r', 2r' - 1]$, $\Gamma_3 := \mathbb{N}$ and let $R_j := K[[\Gamma_j]]$ for $j = 0, 1, 2, 3$. Then it is easy to see that $e(R_0) = e = \text{embdim}(R_0)$, $e(R_1) = r = \text{embdim}(R_1)$, $e(R_2) = r' = \text{embdim}(R_2)$, $e(R_3) = 1 = \text{embdim}(R_3)$, $\Gamma = \Gamma_0 \subsetneq \Gamma_1 \subsetneq \Gamma_2 \subsetneq \Gamma_3 = \mathbb{N}$ and $R = R_0 \subsetneq R_1 \subsetneq R_2 \subsetneq R_3 = \overline{R}$ is the branch sequence of R . Therefore R is an Arf ring by 3.5.

CASE (iv): $1 < r' \leq r$: $r < r'$: In this case, since $e(R) = ne > e - 1 = \text{embdim}(R)$, R is not an Arf ring by 3.5.

4.2. EXAMPLE

Let $m, d, p \in \mathbb{N}$, $m \geq 2$, $p \geq 1$, $d \geq 1$, $\gcd(m, d) = 1$, Γ be the semigroup generated by an arithmetic sequence $m, m + d, \dots, m + pd$ and let $R = K[[\Gamma]]$. Let B be the blowing-up of R along the maximal ideal of R . Then (see 3.1) $B = K[[\Gamma']]$, where Γ' is the semigroup generated by m, d , and so $\text{embdim}(B) = 2$. Further, by 3.5:

- (i) If $d = 1$, then R is Arf if and only if $\text{embdim}(R) = m$ (in fact, in this case, $B = K[[T]]$). The case $d = 1$ is also contained in Proposition 4.4 of the article [3].
- (ii) If $d = 2$ or $m = 2$, then for every $j \geq 2$ the j -th term in the branch sequence of R is $R_j = K[[\Gamma_j]]$, where Γ_j is the semigroup generated by $2, 2n + 1$ for some integer $n \geq 1$ and so $\text{embdim}(R_j) = e(R_j)$ for every $j \geq 1$. Therefore, R is an Arf ring if and only if $\text{embdim}(R) = m$; in particular, if $m = 2$, then R is an Arf ring.
- (iii) If $d \geq 3$ and $m \geq 3$, then $e(B) \geq 3$, $\text{embdim}(B) < e(B)$ and hence R is not an Arf ring.

References

- [1] V. Barucci, D.E. Dobbs, M. Fontana, *Maximality properties in numerical semigroups and applications to one-dimensional analytically irreducible local domains*, Mem. Amer. Math. Soc. **125** (1997), no. 598, x+78 pp.
- [2] V. Barucci, R. Fröberg, *One-dimensional almost Gorenstein rings*, J. Algebra **188** (1997), no. 2, 418-442.
- [3] M. Bras-Amorós, *Acute semigroups, the order bound on the minimum distance, and the Feng-Rao improvements*, IEEE Trans. Inform. Theory **50** (2004), no. 6, 1282-1289.
- [4] M. D'Anna, *Canonical module and one-dimensional analytically irreducible Arf domains*, Commutative ring theory (Fès, 1995), 215-225, Lecture Notes in Pure and Appl. Math., **185**, Dekker, New York, 1997.

- [5] M. D'Anna, D. Delfino, *Integrally closed ideals and type sequences in one-dimensional local rings*, Rocky Mountain J. Math. **27** (1997), no. 4, 1065-1073.
- [6] D. Delfino, *On the inequality $\lambda(\overline{R}/R) \leq t(R)\lambda(R/\mathfrak{G})$ for one-dimensional local rings*, J. Algebra **169** (1994), no. 1, 332-342.
- [7] R. Fröberg, C. Gottlieb, R. Häggkvist, *On numerical semigroups*, Semigroup Forum **35** (1987), no. 1, 63-83.
- [8] D. Katz, *On the number of minimal prime ideals in the completion of a local domain*, Rocky Mountain J. Math. **16** (1986), no. 3, 575-578.
- [9] J. Lipman, *Stable ideals and Arf rings*, Amer. J. Math. **93** (1971), 649-685.
- [10] E. Matlis, *The multiplicity and reduction number of a one-dimensional local ring*, Proc. London Math. Soc. (3) **26** (1973), 273-288.
- [11] T. Matsuoka, *On the degree of singularity of one-dimensional analytically irreducible noetherian local rings*, J. Math. Kyoto Univ. **11** (1971), 485-494.
- [12] A. Ooishi, *Genera and arithmetic genera of commutative rings*, Hiroshima Math. J. **17** (1987), no. 1, 47-66.
- [13] D.P. Patil, I. Sengupta, *Minimal set of generators for the derivation module of certain monomial curves*, Comm. Algebra **27** (1999), no. 11, 5619-5631.
- [14] D.P. Patil, B. Singh, *Generators for the derivation modules and the relation ideals of certain curves*, Manuscripta Math. **68** (1990), no. 3, 327-335.
- [15] D.P. Patil, G. Tamone, *On the type sequences of one dimensional rings*, Preprint, Department of Mathematics, Indian Institute of Science, Bangalore, 2005.
- [16] D.P. Patil, G. Tamone, *On the length equalities for one-dimensional rings*, J. Pure Appl. Algebra **205** (2006), 266-278.
- [17] J.C. Rosales, M. B. Branco, *Numerical semigroups that can be expressed as an intersection of symmetric numerical semigroups*, J. Pure Appl. Algebra **171** (2002), no. 2-3, 303-314.
- [18] G. Scheja, U. Storch, *Regular Sequences and Resultants*, Research Notes in Mathematics **8**, A K Peters, Ltd., Natick, Massachusetts 2001.

*Department of Mathematics
Indian Institute of Science
Bangalore 560 012
India
E-mail: patil@math.iisc.ernet.in*

*Dipartimento di Matematica
Università di Genova
via Dodecaneso 35
I-16146 Genova
Italy
E-mail: tamone@dima.unige.it*

*Received: 16 November 2006; final version: 30 January 2007;
available online: 28 February 2007.*