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A proof of S. Rolewicz's conjecture

Abstract. S. Rolewicz's conjecture concerning solutions of the functional-differential inequality (P) is proved.

1. Introduction

We aim at showing the following conjecture of S. Rolewicz:

The only even, nonnegative and differentiable functions $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfying the inequality

$$f(t) - f(s) - f'(s)(t - s) \geq f(t - s), \quad t, s \in \mathbb{R}, \quad (\text{P})$$

are given by the formula $f(t) = Ct^2$, $t \in \mathbb{R}$, $C \geq 0$.

The problem was posed by S. Rolewicz (oral communication) as arising in his study of Φ -subdifferentials of a real-valued function defined on a metric space and strongly monotone multifunctions, cf. [7] and also [3] for more details.

If, moreover, f is twice differentiable in a neighborhood of the origin and it satisfies an initial condition, S. Rolewicz's conjecture was confirmed by the first author, cf. [1].

2. Consequences of (P)

Most of the assertions that follow were essentially proved in [1]. For the sake of completeness of the argument we repeat them here with suitable comments, changes and additions.

If an even and differentiable function $f : \mathbb{R} \rightarrow \mathbb{R}$ is a solution to (P), then it also satisfies the inequality

$$f(s) - f(t) + f'(t)(t - s) \geq f(t - s), \quad s, t \in \mathbb{R}. \quad (1)$$

(exchange t with s in (P)). If, moreover, f is nonnegative then

$$f(0) = f'(0) = 0. \quad (2)$$

(put $t = s = 0$ in (P) to get $f(0) = 0$ and then $s = 0$ in (P) to obtain $f'(0)t \leq 0$, $t \in \mathbb{R}$, yielding $f'(0) = 0$) and the following inequalities hold true:

$$[f'(t) - f'(s)](t - s) \geq 2f(t - s), \quad t, s \in \mathbb{R}, \quad (3)$$

(add the corresponding sides of (P) and (1)),

$$f(t) - f(s) + f'(s)(t + s) \geq f(t + s), \quad t, s \in \mathbb{R}, \quad (4)$$

(put $-s$ in place of s in (P) and use the oddness of f'),

$$2t f'(t) \geq f(2t), \quad t \in \mathbb{R}, \quad (5)$$

(put $s = t$ in (4)), and

$$f'(s) \geq \frac{2}{s}f(s), \quad s > 0, \quad (6)$$

(thanks to (2), with $t = 0$ in (3)).

Now we are in position to prove

LEMMA 1

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be an even, nonnegative and differentiable function satisfying inequality (P). Then

- a) f is of class C^1 in $\mathbb{R}$,
- b) f is either the zero function or it does not vanish in $\mathbb{R} \setminus \{0\}$,
- c) if, moreover,

$$f(1) = 1. \quad (T)$$

then the following inequalities are satisfied:

$$f(x) \leq x^2, \quad x \in [-1, 1]; \quad f(x) \geq x^2, \quad x \in \mathbb{R} \setminus [-1, 1]. \quad (7)$$

Proof. a) Inequality (P) and nonnegativity of f imply (see (3)) that f' is increasing in $\mathbb{R}$. Thus f is convex, whence f' is continuous (cf. M. Kuczma [5], Th. 3, p. 157).

b) Assume that $f(s_0) = 0$ for some $s_0 > 0$. Since f' is increasing and $f'(0) = 0$ (cf. (2)) we infer that $f(s) = 0$ for every $s \in [0, s_0]$. Consequently $f(s) = 0$ for all $s \in [-s_0, s_0]$, because f is even. Inequality (5) and oddness of f' then yield $f(2t) \leq 0$ for $|t| \leq s_0$, i.e., $f(s) = 0$ for $|s| \leq 2s_0$. On repeating this argument we get $f(s) = 0$ for $|s| \leq 2^n s_0$, $n \in \mathbb{N}$, and f vanishes in $\mathbb{R}$. If $s_0 < 0$, we have $f(-s_0) = 0$ as well, and the above argument works.

c) Because of (T) and b) we have $f(s) > 0$ for $s > 0$ and (6) yields

$$\frac{f'(s)}{f(s)} \geq \frac{2}{s} \quad s > 0.$$

Integrating this inequality first in the interval $[1, x]$, $x > 1$, then in $[x, 1]$, $x \in (0, 1)$, by virtue of (1) we obtain

$$f(x) \geq x^2, \quad x > 1; \quad f(x) \leq x^2, \quad x \in (0, 1).$$

Since f is even, we get inequalities (7), and the proof of the lemma is finished.

3. Key lemma

The following lemma is crucial for the proof of our theorem. It says that if f is a solution of (P) which is not a quadratic function then f' exceeds by a constant the derivative of the latter, at least over an interval contained in $(\frac{1}{2}, +\infty)$.

LEMMA 2

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be an even, nonnegative and differentiable function satisfying inequality (P) and condition (T). Then we have the following assertions.

a) Either

$$f(x) = x^2, \quad x \in \mathbb{R}, \quad (8)$$

or there are: an $\varepsilon > 0$ and $a, b \in \mathbb{R}$, $\frac{1}{2} < a < b$, such that

$$f'(x) > 2x + \varepsilon, \quad x \in [a, b]. \quad (9)$$

b) If there exists a sequence $(x_n)_{n \in \mathbb{N}}$ converging to zero; $x_n > 0$, $n \in \mathbb{N}$; such that

$$f'(x_n) \geq 2x_n, \quad n \in \mathbb{N}, \quad (10)$$

then (8) holds true.

Proof. a) Assume that the "or" statement is not true. Then in every interval $[a, b] \subset (\frac{1}{2}, \infty)$ there exists a sequence $(x_n)_{n \in \mathbb{N}}$ such that

$$f'(x_n) \leq 2x_n + \frac{1}{n}, \quad n \in \mathbb{N}.$$

Since the sequence is bounded, we may assume that $(x_n)_{n \in \mathbb{N}}$ converges to an $x_0 \in [a, b]$.

By Lemma 1a) f' is continuous. Therefore $f'(x_0) \leq 2x_0$. On the other hand, it follows from (5) and (7) that $f'(x_0) \geq \frac{1}{2x_0} f(2x_0) \geq 2x_0$. This proves that the set $A := \{x : f'(x) = 2x\}$ is dense in $(\frac{1}{2}, \infty)$. From the continuity of f' we infer that $f'(x) = 2x$ for all $x \in (\frac{1}{2}, \infty)$, whence

$$f'(1) = 2. \quad (T')$$

According to (4) and (7) we have

$$f(s) - f(t) + f'(t)(t + s) \geq (t + s)^2, \quad t + s \geq 1.$$

After exchanging s with t and summing up the corresponding sides of the two inequalities we get

$$f'(s) - 2s \geq 2t - f'(t), \quad t + s \geq 1,$$

On putting $t = 1$ here, on account of (T'), we obtain

$$f'(s) \geq 2s, \quad s \geq 0.$$

Integrating this inequality in the interval $[0, x]$, $x > 0$, we have (cf. (2))

$$f(x) \geq x^2, \quad x \geq 0.$$

Define the function $g : \mathbb{R} \rightarrow \mathbb{R}$ by the formula

$$g(x) = f(x) - x^2, \quad x \in \mathbb{R}. \quad (11)$$

It is seen that g is an even, nonnegative and differentiable solution of inequality (P) fulfilling the condition $g(1) = 0$. By Lemma 1b) we have $g(x) = 0$ for every $x \in \mathbb{R}$, which means that (8) is satisfied.

b) Define the function $g : \mathbb{R} \rightarrow \mathbb{R}$ by formula (11). This is an even, differentiable solution of inequality (P). The thing to prove is that g is nonnegative. For, from Lemma 1c) we know that f satisfies inequalities (7), whence

$$g(x) \leq 0 \quad \text{for } x \in [-1, 1] \quad \text{and} \quad g(x) \geq 0 \quad \text{for } x \in \mathbb{R} \setminus [-1, 1].$$

Putting $t = u + v$ and $f = g$ in (P) we have

$$g(u + v) - g(u) - g(v) \geq g'(u)v, \quad u, v \in \mathbb{R}, \quad (12)$$

whence

$$\frac{g(v + x_n) - g(v)}{x_n} - \frac{g(x_n)}{x_n} \geq \frac{g'(x_n)}{x_n}v, \quad v \in \mathbb{R}.$$

Consequently, for any $v > 0$ we get $g'(v) \geq 0$, as $g(0) = g'(0) = f'(0) = 0$, cf. (2), and $g'(x_n) = f'(x_n) - 2x_n \geq 0$, cf. (10). This means that $f'(x) \geq 2x$, $x \geq 0$, whence $f(x) \geq x^2$, $x > 0$, which implies that

$$g(x) \geq 0, \quad x \in \mathbb{R},$$

as g is even. Moreover, by (T) and (11) we have $g(1) = 0$. According to Lemma 1b) the function g vanishes everywhere, and (8) follows.

4. Solution of (P)

Now we are in position to prove the theorem announced.

THEOREM 1

The only solutions $f : \mathbb{R} \rightarrow \mathbb{R}$ of problem (P) are given by the formula

$$f(x) = Cx^2, \quad x \in \mathbb{R}, \quad (S)$$

where C is a nonnegative constant.

Proof. Assume (T) and that (8) does not hold. By Lemma 2b) it is enough to show that there exists a sequence $(x_n)_{n \in \mathbb{N}}$ converging to zero, $x_n > 0$, $n \in \mathbb{N}$, such that condition (10) is fulfilled. Suppose it is not the case. Then there is a $\delta > 0$ such that

$$f'(x) < 2x, \quad x \in (0, \delta). \quad (13)$$

Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be function defined by (11).

According to (12) and (13) we have

$$g(u+v) - g(u) - g(v) \geq 0, \quad u \in (0, \delta), \quad v < 0.$$

Therefore

$$\frac{g(v) - g(u+v)}{u} \leq -\frac{g(u)}{u}, \quad u \in (0, \delta), \quad v < 0.$$

Since (8) does not hold, by Lemma 2a) there are an $\varepsilon > 0$ and a proper interval $[a, b] \subset (\frac{1}{2}, \infty)$ such that inequality (9) is fulfilled. From (2) and (11) we infer that $g'(0) = 0 = g(0)$, so there exists a $\gamma \in (0, \delta)$ such that

$$-\frac{g(u)}{u} < \varepsilon, \quad u \in (0, \gamma).$$

Then

$$\frac{g(v) - g(u+v)}{u} < \varepsilon, \quad u \in (0, \gamma), \quad v < 0. \quad (14)$$

Fix a $u \in (0, \min\{\gamma, \frac{1}{2}(b-a)\})$ and put here $v := -(n+1)u$, with $n \in \mathbb{N}$ so chosen that $nu, (n+1)u \in [a, b]$, to get

$$\frac{g((n+1)u) - g(nu)}{u} < \varepsilon.$$

because g is even. By virtue of the Mean-Value Theorem the left hand side of the above inequality equals $g'(\xi)$ for some $\xi \in [nu, (n+1)u] \subset [a, b]$. Hence on account of (9) and (14)

$$\varepsilon < f'(\xi) - 2\xi = g'(\xi) < \varepsilon,$$

a contradiction. Thus (8) holds true which is (S) with $C = 1$, as claimed.

If (T) does not hold, then we put $C := f(1)$. When $C = 0$, formula (S) follows by Lemma 2b). Assume that $C > 0$. Then the function $f^* : \mathbb{R} \rightarrow \mathbb{R}$ given by the formula

$$f^*(x) = \frac{1}{C}f(x), \quad x \in \mathbb{R},$$

is an even, nonnegative and differentiable solution of the inequality (1) satisfying the condition $f^*(1) = 1$ and our assertion follows from the first part of the proof.

5. Remarks

- a) A shorter proof of Rolewicz's conjecture was found by R. Girgensohn when he got acquainted with our result. A joint work which contains also

solutions of a Pexider type inequality and a functional equation related to (P) is to appear online in SIAM Problems & Solutions.

- b) (added in proof). The paper mentioned above (and presented on the 39th International Symposium on Functional Equations, Denmark, August 2001; cf. [2]) is that by R. Girgensohn and the present authors [4]. It was to be published online together with [3] containing the formulation of the problem only. However, it was not, supposedly because of file transmission problems, so that [4] appeared for some time with the label "unsolved". This caused that prior to [4] there is published in SIAM P&S another solution of Rolewicz's problem, due to M. Renardy [6]. His proof is a shortened version of that of Theorem 1 in [4].

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