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On a Pexider type equation on Δ^+

Abstract. The Pexider equation in question is defined with the use of t -norm and is considered in the space Δ^+ of all non decreasing functions F from $\mathbb{R}^+$ into I that satisfy $F(0) = 0$, $F(\infty) = 1$ and are left-continuous on $(0, \infty)$. The general solution of the equation is found as well as, under certain regularity conditions solutions of more explicit form are exhibited.

1. Introduction

Cauchy's equation on the space of distance distribution functions, Δ^+ was investigated in two papers [7, 8], by one of the authors. In this paper we consider the Pexider equation on Δ^+ . Work on Pexider equations on general algebraic structures was done by M.A. Taylor [12] and A. Krapež and M.A. Taylor [4]. In their work, they assumed very little structure on the domain and no regularity on the functions and showed that if solutions exist, then some have to be in terms of homomorphisms of the underlying spaces. This paper makes certain regularity assumptions about the functions and, with the results of [7], obtains rather explicit solutions of the Pexider equation. For further references on the Pexider equation as well as functional equations in general, we refer to the classic works of Aczél [1], and Aczél and Dhombres [2].

This paper is divided into four sections, Section 1 being this introduction. In Section 2, we introduce the necessary notation and known results to keep this paper reasonably selfcontained. Section 3 introduces the functional equation, some general properties of the solutions and our main result. We conclude in Section 4 with by considering certain special cases, which yield explicit formulas for the solutions.

2. Preliminaries from Δ^+

We will denote by Δ^+ the space of all nondecreasing functions F from $\mathbb{R}^+$ into I that satisfy $F(0) = 0$, $F(\infty) = 1$, that are left-continuous on $(0, \infty)$.

The following elements of Δ^+ are of particular importance and therefore merit special symbols:

(i) For any a in $\mathbb{R}^+$, ε_a is the function in Δ^+ defined by

$$\varepsilon_a(x) = \begin{cases} 0, & \text{for } 0 \leq x \leq a, \\ 1, & \text{for } a < x \leq \infty, \end{cases} \quad \text{if } 0 \leq a < \infty,$$

and

$$\varepsilon_\infty(x) = \begin{cases} 0, & \text{for } 0 \leq x < \infty, \\ 1, & \text{for } x = \infty. \end{cases}$$

(ii) For any a in $\mathbb{R}^+$ and b in I , $\delta_{a,b}$ is the function in Δ^+ defined by

$$\delta_{a,b}(x) = \begin{cases} 0, & \text{for } 0 \leq x \leq a, \\ b, & \text{for } a < x < \infty, \\ 1, & \text{for } x = \infty, \end{cases} \quad \text{if } 0 \leq a < \infty, \text{ and } \delta_{\infty,b} = \varepsilon_\infty.$$

We will denote the set of all the $\delta_{a,b}$ by Δ_δ^+ and note that for all a in $\mathbb{R}^+$, $\delta_{a,1} = \varepsilon_a$ and $\delta_{a,0} = \varepsilon_\infty$. We also have:

LEMMA 2.1

For $0 \leq a, c < \infty$ and $0 < b, d \leq 1$, $\delta_{a,b} = \delta_{c,d}$ if and only if $a = c$ and $b = d$.

The set Δ^+ , partially ordered by

$$F \leq G \text{ if and only if } F(x) \leq G(x) \text{ for all } x \text{ in } \mathbb{R}^+,$$

forms a *complete lattice*, i.e., a partially ordered set in which every subset has a supremum and an infimum, see [3]. Here, for any subset S of Δ^+ , the supremum of S is the pointwise supremum of all functions in S and the infimum of S is the supremum of the set of all lower bounds of S . The latter refinement is necessary since the pointwise infimum of left-continuous functions need not be left-continuous.

We note that in particular we have,

$$\varepsilon_a \leq \varepsilon_b, \quad \text{whenever } a \geq b,$$

and

$$\delta_{a,b} \leq \delta_{c,d}, \quad \text{whenever } a \geq c \text{ and } b \leq d.$$

Moreover, ε_∞ and ε_0 are, respectively, the least and greatest elements in this partial order.

To generalize the triangle inequality to probabilistic metric spaces, one needs a binary operation on the space of distance distribution functions: A triangle function τ is a binary operation on Δ^+ that is commutative, associative, nondecreasing in each place, and has ε_0 as identity.

As an immediate consequence we have that ε_∞ is a zero for τ , i.e., that

$$\tau(\varepsilon_\infty, F) = \varepsilon_\infty, \quad \text{for all } F \text{ in } \Delta^+,$$

whence (Δ^+, τ) is a semigroup with identity and zero.

We will be principally concerned with the class of triangle functions τ_T that are induced by left-continuous t -norms via:

$$\tau_T(F, G)(x) = \sup_{u+v=x} \{T(F(u), G(v))\}, \quad \text{for all } F, G \text{ in } \Delta^+ \text{ and } x \text{ in } \mathbb{R}^+.$$

A simple calculation yields that for all a, b in $\mathbb{R}^+$ and all c, d in I ,

$$\tau_T(\delta_{a,c}, \delta_{b,d}) = \delta_{a+b, T(c,d)}. \quad (1)$$

This implies that $(\Delta_\delta^+, \tau_T)$ is a subsemigroup of (Δ^+, τ_T) .

Moreover, it follows from (1), that for any $\delta_{a,b}$ in Δ_δ^+ , we have $\delta_{a,b} = \tau_T(\varepsilon_a, \delta_{0,b})$.

We also have the following basic lemma, which is due to R.C. Powers [6]:

LEMMA 2.2

Let F be in Δ^+ , then

$$F = \sup_{a \in \mathbb{R}^+} \delta_{a, F(a)}. \quad (2)$$

DEFINITION 2.3

A function φ from Δ^+ into Δ^+ is said to be *sup-continuous* if, for any index set J and any collection $\{F_j\}$ such that F_j is in Δ^+ for all j in J , we have

$$\varphi(\sup_{j \in J} F_j) = \sup_{j \in J} \varphi(F_j).$$

The next lemma is due to R.M. Tardiff [11] (see also [10, Sec. 12.9]).

LEMMA 2.3

If T is a continuous t -norm, then τ_T is sup-continuous in each variable.

Thus we have that a sup-continuous function on Δ^+ as well as any τ_T , with continuous T , is completely determined by its values on Δ_δ^+ . This is the key observation for solving functional equations for sup-continuous functions on Δ^+ .

3. Pexider equation

DEFINITION 3.1

We say that the triple $(\varphi_1, \varphi_2, \varphi_3)$ is a solution of the Pexider equation if

$$\varphi_1(\tau_T(F, G)) = \tau_T(\varphi_2(F), \varphi_3(G)), \quad \text{for all } F, G \in \Delta^+. \quad (3)$$

We begin with some obvious observations.

If φ is a solution of Cauchy's equation, see [7, 8], then $(\varphi, \varphi, \varphi)$ is a solution triple of the Pexider equation. Furthermore, if φ is a solution of Cauchy's equation and we let $\varphi_2(F) = \tau_T(\varphi(F), H_0)$, $\varphi_3(F) = \tau_T(\varphi(F), H_1)$, and $\varphi_1(F) = \tau_T(\varphi(F), \tau_T(H_0, H_1))$, then $(\varphi_1, \varphi_2, \varphi_3)$ is a solution triple of the Pexider equation. The latter includes all constant solutions by choosing $\varphi(F) = \varepsilon_0$. Furthermore, if $H_0 = \varepsilon_\infty$, then $\varphi_1 = \varphi_2 = \varepsilon_\infty$ and φ_3 is arbitrary; similarly for $H_1 = \varepsilon_\infty$.

This yields a large class of solutions of the Pexider equation and the rest of this section will be devoted to showing that, among sup-continuous solutions satisfying a few additional conditions, these are the only solutions of the Pexider equation on Δ^+ . Krapež and Taylor, [4], showed that such solutions have to occur if the Pexider equation has a solution; they did so on spaces with very little structure and without regularity assumptions. To show that all solutions of a certain class are of this form, we need, however, some regularity assumptions.

From now on we will make more stringent assumptions on the class of solutions of (3). As was pointed out, the assumption that the solutions are sup-continuous allows us to reduce the problem to solving (3) on Δ_δ^+ . We will now make two further assumption: First we assume that T is a *strict t -norm* and second, that the solutions will map Δ_δ^+ into a cancellative subsemigroup of (Δ^+, τ_T) . To this end we need the following results and definitions, see [10]:

THEOREM 3.2

If T is a strict t -norm then

$$T(x, y) = g^{-1}(g(x) + g(y)), \quad \text{for all } x, y \text{ in } I,$$

where g is a continuous, strictly decreasing function from I onto $\mathbb{R}^+ = [0, \infty]$, with $g(1) = 0$.

The function g is called an *inner additive generator* (briefly, a generator); and it is well-known that g and h generate the same t -norm if and only if there is a $k > 0$ such that

$$g(x) = k \cdot h(x), \quad \text{for all } x \text{ in } I. \quad (4)$$

DEFINITION 3.3

Let T be a strict t -norm and g any inner additive generator of T . Then we let

$$\Delta_T^+ = \{F \text{ in } \Delta^+ \mid g \circ F \text{ is convex on } (b_F, \infty)\},$$

where $b_F = \sup_{x \in \mathbb{R}^+} \{F(x) = 0\}$.

In view of (4), the set Δ_T^+ does not depend on the choice of generator g . Furthermore, $(\Delta_T^+ \setminus \{\varepsilon_\infty\}, \tau_T)$ is a cancellative subsemigroup of (Δ^+, τ_T) , [10, Theorem 7.8.11].

We note here that the set Δ_T^+ is often referred to as the set of T -log-concave elements of Δ^+ . This terminology is due to R.A. Moynihan [5] (see also [10, Sec. 7.8]) and stems from the fact that he used multiplicative generators to define this set.

Clearly $b_{\delta_{a,b}} = a$ and, since $\delta_{a,b}$ is constant on (a, ∞) , it follows that $g \circ \delta_{a,b}$ is convex, whence

$$\Delta_\delta^+ \subseteq \Delta_T^+.$$

With this we have the following lemma:

LEMMA 3.4

Assume that $\varphi_i(\Delta_\delta^+ \setminus \{\varepsilon_\infty\}) \subset \Delta_T^+ \setminus \{\varepsilon_\infty\}$, for $i = 1, 2, 3$ and that for $i = 2$ or $i = 3$, we have $\varphi_i(F) \leq \varphi_i(\varepsilon_0)$ and $\varphi_i(\varepsilon_0) \in \Delta_\delta^+ \setminus \{\varepsilon_\infty\}$. Then $(\varphi_1, \varphi_2, \varphi_3)$ is a solution triple of (3) on Δ_δ^+ , if and only if there is a function φ with

$$\varphi(\tau_T(\delta_{a,c}, \delta_{b,d})) = \tau_T(\varphi(\delta_{a,c}), \varphi(\delta_{b,d}))$$

(i.e. a solution of Cauchy's equation on Δ_δ^+), such that

$$\varphi_1(F) = \tau_T(\varphi(F), \tau_T(\varphi_2(\varepsilon_0), \varphi_3(\varepsilon_0)))$$

$$\varphi_2(F) = \tau_T(\varphi(F), \varphi_2(\varepsilon_0))$$

$$\varphi_3(F) = \tau_T(\varphi(F), \varphi_3(\varepsilon_0))$$

Proof. The only if part follows from the observations in the previous sections. We will follow an approach similar to that of the proof in [4, Theorem 10]. We let $G = \varepsilon_0$ in (3) to get

$$\varphi_1(F) = \tau_T(\varphi_2(F), \varphi_3(\varepsilon_0)). \quad (5)$$

For $F \in \Delta_T^+$ we have that φ_1 maps into the coset $S_3 = \tau_T(\Delta_T^+, \varphi_3(\varepsilon_0))$. Similarly, letting $F = \varepsilon_0$ we have that φ_1 maps into the coset $S_2 = \tau_T(\Delta_T^+, \varphi_2(\varepsilon_0))$. Since $\varphi_2(F) \leq \varphi_2(\varepsilon_0)$ and $\varphi_2(\varepsilon_0) = \delta_{a,b}$, (or the same holds for φ_3), we can write $\varphi_2(F) = \tau_T(H, \delta_{a,b})$, which is $\varphi_2(F) = T(H(x - a), b)$, so that

$$H(x) = g^{-1}(g(\varphi_2(F)(x + a)) - g(b)) \in \Delta_T^+$$

and the range of φ_1 is contained in the coset $S_{2,3} = \tau_T(\Delta_T^+, \tau_T(\varphi_2(\varepsilon_0), \varphi_3(\varepsilon_0)))$. Now we let $M_i(F) = \tau_T(F, \varphi_i(\varepsilon_0))$ for $i = 2, 3$. Using the fact that τ_T is cancellative on Δ_T^+ , we have that M_i is invertible on the coset $\tau_T(\Delta_T^+, \varphi_i(\varepsilon_0))$ for $i = 2$ and $i = 3$, respectively. The associativity and commutativity of τ_T yield that $M_2(M_3(F)) = M_3(M_2(F))$ and $M_i(\tau_T(F, G)) = \tau_T(M_i(F), G) = \tau_T(F, M_i(G))$. Similarly, we have $M_2^{-1}(M_3^{-1}(F)) = M_3^{-1}(M_2^{-1}(F))$, for $F \in S_{2,3}$ and $M_i^{-1}(\tau_T(G, H)) = \tau_T(M_i^{-1}(G), H)$, for $G \in S_i$, $i = 2, 3$. Using these properties of the M_i , we can now proceed and let $F = \varepsilon_0$ in (3), to get

$$\varphi_1(F) = M_3(\varphi_2(F)) \quad \text{and} \quad \varphi_1(F) = M_2(\varphi_3(F)),$$

which by the invertibility of the M_i , yields:

$$\varphi_2(F) = M_3^{-1}(\varphi_1(F)) \quad \text{and} \quad \varphi_3(F) = M_2^{-1}(\varphi_1(F)). \quad (6)$$

Substituting (6) into (3) yields

$$\varphi_1(\tau_T(F, G)) = \tau_T(M_3^{-1}(\varphi_1(F)), M_2^{-1}(\varphi_1(G))).$$

Applying $M_2^{-1}M_3^{-1}$ to both sides and simplifying yields that $\varphi = M_2^{-1}M_3^{-1}\varphi_1$ satisfies Cauchy's equation on Δ_δ^+ . Using this and equations (5) and (6), gives the desired result.

Lemma 3.4 together with Lemmas 2.2 and 2.3 immediately yield our main result:

THEOREM 3.5

For $i = 1, 2, 3$, let $\varphi_i : \Delta^+ \rightarrow \Delta^+$, be sup-continuous. Further, assume that $\varphi_i(\Delta_\delta^+ \setminus \{\varepsilon_\infty\}) \subset \Delta_T^+ \setminus \{\varepsilon_\infty\}$, for $i = 1, 2, 3$ and that for $i = 2$ or $i = 3$, we have $\varphi_i(F) \leq \varphi_i(\varepsilon_0)$ and $\varphi_i(\varepsilon_0) \in \Delta_\delta^+ \setminus \{\varepsilon_\infty\}$. Then $(\varphi_1, \varphi_2, \varphi_3)$ is a solution triple of (3) on Δ^+ , if and only if there is a function φ with

$$\varphi(\tau_T(F, G)) = \tau_T(\varphi(F), \varphi(G)), \quad \text{for all } F, G \in \Delta^+$$

(i.e. a solution of Cauchy's equation on Δ^+), such that

$$\varphi_1(F) = \tau_T(\varphi(F), \tau_T(\varphi_2(\varepsilon_0), \varphi_3(\varepsilon_0)))$$

$$\varphi_2(F) = \tau_T(\varphi(F), \varphi_2(\varepsilon_0))$$

$$\varphi_3(F) = \tau_T(\varphi(F), \varphi_3(\varepsilon_0))$$

4. Explicit solutions

Using hypotheses similar to those of Theorem 3.5, we can use the following theorem (see [7, Theorems 6.6 and 6.7]), to obtain specific explicit solutions of the Pexider equation.

THEOREM 4.1

Let T be a strict t -norm with generator g and let φ be a sup-continuous solution of Cauchy's equation for τ_T such that $\varphi(\Delta_\delta^+) \subset \Delta_T^+$. Then, given any $c \in (0, 1)$, we have for all F in Δ^+ ,

$$\varphi(F) = \sup_{t \in \mathbb{R}^+} \tau_T \left([\varphi(\varepsilon_1)]^t, [\varphi(\delta_{0,c})]^{kg(F(t))} \right), \quad (7)$$

where $k = \frac{1}{g(c)}$.

Here F^μ is the “ μ -th τ_T power” of F (for F in $\Delta_T^+ \setminus \{\varepsilon_\infty\}$), given by

$$F^\mu(x) = g^{-1} \left(\mu \cdot g \left(F \left(\frac{x}{\mu} \right) \right) \right), \quad \text{for } 0 < \mu < \infty,$$

$$F^0 = \lim_{\mu \rightarrow 0} F^\mu = \varepsilon_0, \quad F^\infty = \lim_{\mu \rightarrow \infty} F^\mu = \begin{cases} \varepsilon_\infty, & \text{for } F \neq \varepsilon_0, \\ \varepsilon_0, & \text{for } F = \varepsilon_0. \end{cases}$$

Assuming now that φ satisfies the hypotheses of Theorem 4.1 and in particular, that $\varphi(\varepsilon_1) = \varepsilon_a$ and $\varphi(\delta_{0,c}) = \delta_{0,b}$, we see that

$$\varphi(\delta_{u,v}) = \delta_{au, \theta(v)}; \quad \theta(v) = g^{-1} \left(\frac{g(b)}{g(c)} g(v) \right).$$

This, in turn, implies by the sup-continuity of φ that φ is an order automorphism (see [6]), that is

$$\varphi(F)(x) = \theta(F(\gamma(x))) \quad \text{for all } F \in \Delta^+ \text{ and all } x \in \mathbb{R}^+.$$

Here $\gamma(u) = \frac{u}{a}$.

Thus the question arises whether each function of the triple $(\varphi_1, \varphi_2, \varphi_3)$ (a solution of (3)) is also induced by left and right composition. This is not quite the case as the following theorem shows:

THEOREM 4.2

For $i = 1, 2, 3$, let $\varphi_i : \Delta_\delta^+ \rightarrow \Delta_\delta^+ \setminus \{\varepsilon_\infty\}$, be sup-continuous with

$$\varphi_i(\varepsilon_0) = \delta_{u_i, v_i}, \quad \varphi_i(\varepsilon_1) = \delta_{a+u_i, v_i}, \quad \varphi_i(\delta_{0,c}) = \delta_{u_i, T(v_i, b)} \quad \text{for } i = 2, 3.$$

Then $(\varphi_1, \varphi_2, \varphi_3)$ is a solution triple of (3) if and only if there is a φ , given by (7), with $\varphi(\varepsilon_1) = \varepsilon_a$ and $\varphi(\delta_{0,c}) = \delta_{0,b}$ such that

$$\varphi_i(F)(x) = \begin{cases} 0, & \text{if } x = 0, \\ g^{-1} \left(\frac{g(b)}{g(c)} \cdot g \left(F \left(\frac{x - v_i}{a} \right) \right) + g(u_i) \right), & \text{if } 0 < x < \infty, \\ 1, & \text{if } x = \infty; \end{cases}$$

for $i = 2, 3$ and $\varphi_1(F) = \tau_T(\varphi_2(F), \varphi_3(\varepsilon_0))$. Here we use for ease of notation that $F(x) = 0$, for $x < 0$.

Thus we see that these solutions are not induced on Δ^+ .

In a similar manner, we can obtain order automorphism solutions of the second type (see [6]), where $\varphi(\varepsilon_1) = \delta_{0,b}$ and $\varphi(\delta_{0,c}) = \varepsilon_a$. In this case we have that the left and right compositions of the order automorphism can only differ by a multiplicative constant since they need to generate the same T -norm.

Thus solutions of this type are given, for $i = 2, 3$, by

$$(\varphi_i(F))(x) = \begin{cases} 0, & \text{if } x = 0, \\ (g^{-1}F^\vee g^{-1})(k \cdot x - b_i), & \text{if } 0 < x < \infty, \\ 1, & \text{if } x = \infty; \end{cases}$$

and $\varphi_1(F) = \tau_T(\varphi_2(F), \varphi_3(\varepsilon_0))$; with the same notation as above and $F^\vee$ is the right-continuous *quasi-inverse* of F which is given by

$$F^\vee(y) = \begin{cases} 0, & \text{for } y = 0, \\ \inf\{x \mid F(x) > y\}, & \text{for } 0 < y < 1, \\ \infty, & \text{for } y = 1. \end{cases} \quad (8)$$

In conclusion we note that the case of non-strict T -norms requires another approach; solutions of the type of those just presented are still possible, but new methods will be needed to describe all sup-continuous solutions in that case. We further note that the restriction to triangle functions of the form τ_T is not as restrictive as it may seem, since, as was noted in [9], triangle functions that are sup-continuous and map $\Delta_\delta^+ \times \Delta_\delta^+$ into Δ_δ^+ are essentially of this form.

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