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On a paper of T.M.K. Davison

Abstract. In his paper [3] the author shows that Chebyshev polynomials of the first kind show up in relation with d'Alembert's functional equation. Here we point out a similar property of Chebyshev polynomials concerning the square norm equation and we exhibit that the reason is due to close relations with hypergroups.

1. Introduction

In [3] the author presents the following theorem.

THEOREM 1

Let R be a commutative ring and let $f : \mathbb{Z} \to R$ a function with $f(0) = 1$. Then

$$f(m+n) + f(m-n) = 2f(m)f(n)$$

holds for all m, n in $\mathbb{Z}$ if and only if

$$f(n) = T_{|n|}(f(1))$$

holds for all n in $\mathbb{Z}$, where $T_{|n|}$ denotes $|n|$-th Chebyshev polynomial of the first kind.

In this paper we show that this theorem is a consequence of the fact that d'Alembert's equation has a close connection to the exponentials of a special polynomial hypergroup, the Chebyshev hypergroup. In addition, we show a similar connection between the square-norm equation, the additive functions of the Chebyshev hypergroup and the derivatives of the Chebyshev polynomials of the first kind (simply: Chebyshev polynomials).

In this work concerning hypergroups and polynomial hypergroups we refer to [1], [2], [5]. Concerning exponential functions and additive functions on polynomial hypergroups see also [7], [8], [9], [10].

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2. Discrete polynomial hypergroups

An important special class of hypergroups is closely related to orthogonal polynomials.

Let $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ and $(c_n)_{n\in\mathbb{N}}$ be real sequences with the following properties: $c_n > 0$, $b_n \geqslant 0$, $a_{n+1} \geqslant 0$ for all n in $\mathbb{N}$, moreover $a_0 = 0$, and $a_n + b_n + c_n = 1$ for all n in $\mathbb{N}$. We define the sequence of polynomials $(P_n)_{n\in\mathbb{N}}$ by $P_0(x) = 1$, $P_1(x) = x$, and by the recursive formula

$$xP_n(x) = a_nP_{n-1}(x) + b_nP_n(x) + c_nP_{n+1}(x)$$

for all $n \geqslant 1$ and x in $\mathbb{R}$. The following theorem holds.

THEOREM 2

If the sequence of polynomials $(P_n)_{n\in\mathbb{N}}$ satisfies the above conditions, then there exist constants $c(n,m,k)$ for all n,m,k in $\mathbb{N}$ such that

$$P_nP_m = \sum_{k=|n-m|}^{n+m} c(n,m,k)P_k$$

holds for all n,m in $\mathbb{N}$.

Proof. By the theorem of Favard (see [4], [6]) the conditions on the sequence of polynomials $(P_n)_{n\in\mathbb{N}}$ imply that there exists a probability measure μ on $[-1,1]$ such that $(P_n)_{n\in\mathbb{N}}$ forms an orthogonal system on $[-1,1]$ with respect to μ. As P_n has degree n, we have

$$P_nP_m = \sum_{k=0}^{n+m} c(n,m,k)P_k$$

for all n,m in $\mathbb{N}$, where

$$c(n,m,k) = \frac{\int_{[-1,1]} P_kP_nP_m\,d\mu}{\int_{[-1,1]} P_k^2\,d\mu}$$

holds for all n,m,k in $\mathbb{N}$. The orthogonality of $(P_n)_{n\in\mathbb{N}}$ with respect to μ implies $c(n,m,k) = 0$ for $k > n+m$ or $n > m+k$ or $m > n+k$. Hence our statement is proved.

The formula in the theorem is called *linearization formula*, and the coefficients $c(n,m,k)$ are called *linearization coefficients*. The recursive formula for the sequence $(P_n)_{n\in\mathbb{N}}$ implies $P_n(1) = 1$ for all n in $\mathbb{N}$, hence we have

$$\sum_{k=|n-m|}^{n+m} c(n,m,k) = 1$$

for all n in $\mathbb{N}$. It may or may not happen that $c(n,m,k) \geqslant 0$ for all n,m,k

in $\mathbb{N}$. If it happens then we can define a hypergroup structure on $\mathbb{N}$ by the following rule:

$$\delta_n * \delta_m = \sum_{k=|n-m|}^{n+m} c(n,m,k)\delta_k$$

for all n, m in $\mathbb{N}$, with involution as the identity mapping and with e as 0. The resulting commutative hypergroup is called *the discrete polynomial hypergroup associated with the sequence* $(P_n)_{n\in\mathbb{N}}$.

As an example we consider the hypergroup associated with the Chebyshev polynomials. The corresponding recurrence relation is

$$xT_n(x) = \frac{1}{2}\big(T_{n+1}(x) + T_{n-1}(x)\big)$$

for all $n \geqslant 1$ and x in $\mathbb{R}$. By induction it follows that

$$T_m(x)T_n(x) = \frac{1}{2}\big(T_{n+m}(x) + T_{|n-m|}(x)\big)$$

holds for all m, n in $\mathbb{N}$ and x in $\mathbb{R}$. Hence in this case the linearization coefficients are nonnegative, and the resulting hypergroup associated with the Chebyshev polynomials is the *Chebyshev hypergroup.*

On any discrete polynomial hypergroup associated with the sequence of polynomials $(P_n)_{n\in\mathbb{N}}$ and for any function $f : \mathbb{N} \to \mathbb{C}$ one defines

$$f(m * n) = \sum_{k=|m-n|}^{m+n} c(n,m,k)f(k)$$

for all n, m in $\mathbb{N}$. Using this concept we call the function $f : \mathbb{N} \to \mathbb{C}$ an *exponential*, if $f(0) = 1$ and

$$f(m * n) = f(m)f(n)$$

for all m, n in $\mathbb{N}$, and *additive*, if

$$f(m * n) = f(m) + f(n)$$

holds for all m, n in $\mathbb{N}$. Observe that we can give a meaning to $f(m * n)$ also in the case if f has its range in a commutative ring with unit, assuming that the coefficients $c(n, m, k)$ are nonnegative integers. This happens in the case of the Chebyshev hypergroup. Another special property of the Chebyshev polynomials of the first kind is that their coefficients are integers which makes it possible to define them in the obvious way on an arbitrary commutative ring R with unit. In this case $T_0(x) = e$, the unit of the ring, T_1 is the identity mapping on R and for T_{n-1}, T_n and T_{n+1} the above recursive relation holds.

It follows that for any complex valued function f defined on the Chebyshev hypergroup and having values in the commutative ring R with unit e we have

$$f(m*n)=\frac{1}{2}\big(f(m+n)+f(|m-n|)\big)$$

for all m,n in $\mathbb{N}$. Hence R-valued exponentials on the Chebyshev hypergroup are characterized by $f(0)=e$ together with the functional equation

$$\varphi(m+n)+\varphi(|m-n|)=2\varphi(m)\varphi(n),$$

and R-valued additive functions on the Chebyshev hypergroup are characterized by the functional equation

$$a(m+n)+a(|m-n|)=2a(m)+2a(n).$$

Here we recall the following two theorems which motivate our observations.

THEOREM 3

Let $K=(\mathbb{N},*)$ be the polynomial hypergroup associated with the sequence of polynomials $(P_n)_{n\in\mathbb{N}}$. The function $\varphi:\mathbb{N}\to\mathbb{C}$ is an exponential on K if and only if there exists a complex number z such that

$$\varphi(n)=P_n(z)$$

holds for all n in $\mathbb{N}$.

THEOREM 4

Let $K=(\mathbb{N},*)$ be the polynomial hypergroup associated with the sequence of polynomials $(P_n)_{n\in\mathbb{N}}$. The function $a:\mathbb{N}\to\mathbb{C}$ is an additive function on K if and only if there exists a complex number c such that

$$a(n)=cP_n'(1)$$

holds for all n in $\mathbb{N}$.

For the proofs of these theorems see e.g. [2], [7], [8], [9], [10]. We can easily extend these results to R-valued functions on the Chebyshev hypergroup.

THEOREM 5

Let R be a commutative ring with unit. The function $\varphi:\mathbb{N}\to R$ is an exponential with respect to the Chebyshev hypergroup if and only if there exists an element r in R such that

$$\varphi(n)=T_n(r)$$

holds for all n in $\mathbb{N}$.

Proof. Denote by e the unit of R. Let r be a fixed element of R and let $\varphi(n)=T_n(r)$ for all n in $\mathbb{N}$. By the linearization formula for the Chebyshev polynomials of the first kind we have

$$\varphi(m*n)=\frac{1}{2}\big(\varphi(m+n)+\varphi(|m-n|)\big)$$

$$= \frac{1}{2}\big(T_{m+n}(r) + T_{|m-n|}(r)\big) = T_m(r)T_n(r)$$
$$= \varphi(m)\varphi(n)$$

for all m, n in $\mathbb{N}$, that is, φ is an exponential. Conversely, let $\varphi : \mathbb{N} \to R$ be an exponential and we define $\psi(n) = T_n(\varphi(1))$ for all n in $\mathbb{N}$. Then $\psi(0) = e = \varphi(0)$ and $\psi(1) = \varphi(1)$. On the other hand, for any positive integer n we have

$$\psi(n * 1) = \frac{1}{2}\big(\psi(n+1) + \psi(n-1)\big)$$
$$= \frac{1}{2}\big(T_{n+1}(\varphi(1)) + T_{n-1}(\varphi(1))\big) = T_n(\varphi(1))T_1(\varphi(1))$$
$$= \psi(n)\varphi(1),$$

which means

$$\psi(n+1) = 2\psi(n)\varphi(1) - \psi(n-1).$$

Similarly, it follows for all positive integer n that

$$\varphi(n+1) = 2\varphi(n)\varphi(1) - \varphi(n-1).$$

From this we infer that $\varphi = \psi$ and the theorem is proved.

THEOREM 6

Let R be a commutative ring with unit. The function $: \mathbb{N} \to R$ is additive with respect to the Chebyshev hypergroup if and only if there exists an element c in R such that

$$a(n) = cT_n'(1)$$

holds for all n in $\mathbb{N}$.

Proof. First of all we remark that $T_n'(1)$ is an integer for all n in $\mathbb{N}$. This follows from the equation

$$T_m'(1) + T_n'(1) = \frac{1}{2}\big(T_{m+n}'(1) + T_{|m-n|}'(1)\big)$$

which can be obtained from the linearization formula by differentiation and then substituting $x = 1$. Denote by e the unit of R. Let c be a fixed element of R and let $a(n) = cT_n'(1)$ for all n in $\mathbb{N}$. The above formula shows that a is additive. Conversely, let $a : \mathbb{N} \to R$ be additive and we define $\psi(n) = a(1)T_n'(1)$ for all n in $\mathbb{N}$. Then $\psi(0) = 0 = a(0)$ and $\psi(1) = a(1)$. On the other hand, for any positive integer n we have

$$\psi(n * 1) = \frac{1}{2}\big(\psi(n+1) + \psi(n-1)\big)$$
$$= \frac{1}{2}a(1)\big(T_{n+1}'(1) + T_{n-1}'(1)\big) = a(1)\big(T_n'(1) + T_1'(1)\big)$$
$$= \psi(n) + a(1),$$

which means that

$$\psi(n+1) = 2\psi(n) - \psi(n-1) + 2a(1).$$

Similarly, it follows for all positive integer n that

$$a(n+1) = 2a(n) - a(n-1) + 2a(1).$$

From this we infer that $a = \psi$ and the theorem is proved.

3. Exponential and additive functions on the Chebyshev hypergroup

Let R be a commutative rung with unit e. As we have seen, a function $\varphi : \mathbb{N} \to R$ is an exponential on the Chebyshev hypergroup if and only if it satisfies $\varphi(0) = e$ and

$$\varphi(m+n) + \varphi(|m-n|) = 2\varphi(m)\varphi(n)$$

for all m, n in $\mathbb{N}$, which is a slight modification of d'Alembert's functional equation. Similarly, a function $a : \mathbb{N} \to R$ is an additive function on the Chebyshev hypergroup if and only if it satisfies

$$a(m+n) + a(|m-n|) = 2a(m) + 2a(n)$$

for all m, n in $\mathbb{N}$, which is a slight modification of the square-norm functional equation. Consequently, according to the above results we have the following result, which is Theorem 1 above from [3].

THEOREM 7

Let R be a commutative ring with unit e. Let $f : \mathbb{Z} \to R$ be a function with $f(0) = e$. Then

$$f(m+n) + f(m-n) = 2f(m)f(n)$$

holds for all m, n in $\mathbb{Z}$ if and only if

$$f(n) = T_{|n|}(f(1))$$

holds for all n in $\mathbb{Z}$, where $T_{|n|}$ denotes $|n|$-th Chebyshev polynomial of the first kind.

Proof. It is obvious that f is even, hence it satisfies

$$f(m+n) + f(|m-n|) = 2f(m)f(n)$$

for all m, n in $\mathbb{N}$. This means, that f is an R-valued exponential of the Chebyshev hypergroup, and by Theorem 5 we have

$$f(n) = T_n(r)$$

for all n in $\mathbb{N}$ with some complex r in R. For $n = 1$ we have $r = f(1)$ and the theorem is proved.

The case of additive functions can be treated similarly.

THEOREM 8
Let R be a commutative ring with unit. Let $f : \mathbb{Z} \to R$ be a function. Then

$$f(m+n) + f(m-n) = 2f(m) + 2f(n)$$

holds for all m, n in $\mathbb{Z}$ if and only if

$$f(n) = f(1)T'_{|n|}(1)$$

holds for all n in $\mathbb{Z}$, where $T_{|n|}$ denotes $|n|$-th Chebyshev polynomial of the first kind.

Proof. It is obvious that $f(0) = 0$, which implies that f is even, hence it satisfies

$$f(m+n) + f(|m-n|) = 2f(m) + 2f(n)$$

for all m, n in $\mathbb{N}$. This means, that f is an additive function on the Chebyshev hypergroup, and by Theorem 6 we have

$$f(n) = cT'_n(1)$$

for all n in $\mathbb{N}$ with some c in R. For $n = 1$ we have $c = f(1)$ and the theorem is proved.

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