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Jensen-type geometric shapes

Abstract. We present both necessary and sufficient conditions for a convex closed shape such that for every convex function the average integral over the shape does not exceed the average integral over its boundary.

It is proved that this inequality holds for n-dimensional parallelotopes, n-dimensional balls, and convex polytopes having the inscribed sphere (tangent to all its facets) with the centre in the centre of mass of its boundary.

1. Introduction

Dragomir and Pearce proved [4, Theorem 215] that if B_n is an n-dimensional ball then

$$\fint_{B_3} f(x)dx \le \fint_{\partial B_3} f(x)dx$$

for every convex function $f\colon B_3 \to \mathbb{R}$; here and below $\fint$ stands for the average integral (more precisely, $\fint_X f(x)dx := \frac{1}{|X|}\int_X f(x)dx$).

Later, Cal-Carcamo [1] and Cal-Carcamo-Escauriaza [2] proved that

$$\fint_{B_n} f(x)dx \le \fint_{\partial B_n} f(x)dx$$

for every $n \ge 2$ and a convex function $f\colon B_n \to \mathbb{R}$.

During *Conference on Inequalities and Applications 2016* Páles stated the problem whether for every convex and closed set X and every convex function $f\colon X \to \mathbb{R}$, the following inequality is valid

$$\fint_X f(x)dx \le \fint_{\partial X} f(x)dx. \tag{1}$$

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It is however easy to verify that for the triangle T with vertices $(0,-1)$, $(0,1)$, $(1,0)$ and the function $f\colon T \ni (x,y) \mapsto x$, inequality (1) is not satisfied (as inequality $\frac{1}{3} \le 1 - \frac{\sqrt{2}}{2}$ is not valid). Furthermore, this property is invariant under transition, scaling, changing orientation and reflection, whence it is a property of a shape. It turned out that the only triangles admitting this property are equilateral triangles (see Proposition 2 for details).

Author was notified that Fedor Nazarov proved the conjectured inequality in the case $n = 2$ and for a symmetric X.

This motivates us to introduce the following definition. A convex, closed and bounded set $X \subset \mathbb{R}^n$ ($n \in \mathbb{N}$) is called *Jensen-type* if for every convex function $f\colon X \to \mathbb{R}$, inequality (1) is satisfied. It is worth mentioning that if Y is similar to X then either both or none of them are of Jensen-type. Therefore this property can be considered as a property of convex shapes (i.e. classes of abstraction under a similarity relation).

Using this definition previous results can be expressed briefly as *n-dimensional ball is of Jensen-type* or *B_n is of Jensen-type*. The second example can be expressed by *a $45-45-90$ triangle is not of Jensen-type*.

Motivated by these preliminaries we are going to prove this property for regular polygons, parallelotopes (in all dimensions), and Platonic solids. Moreover, we will present an independent proof that all balls are of Jensen-type.

Finally, let us emphasize that this problem is related to Choquet theory (see Niculescu [5] and Niculescu–Persson [6, chap. 7]), where the following result was stated.

Theorem 1
Let μ be a probability measure on a metrizable compact convex subset K of a locally convex Hausdorff space. Then there exists a probability measure λ on K which has the same barycentre as μ; is null outside $\operatorname{Ext} K$ and verifies the inequality

$$\int_K f(x)d\mu(x) \le \int_{\operatorname{Ext} K} f(x)d\lambda(x);$$

for all continuous convex functions $f\colon K \to \mathbb{R}$. Here $\operatorname{Ext} K$ denotes the set of all extreme points of K.

2. Results

We begin with some necessary condition for X to be Jensen-type which generalizes the argumentation inspired by already presented result for a $45-45-90$ triangle.

Lemma 1
If X is of Jensen-type then centres of mass of X and ∂X coincide.

Proof. Let $\pi_i\colon \mathbb{R}^n \to \mathbb{R}$ be a projection on i-th coordinate ($i \in \{1,\dots,n\}$). Both π_i and $-\pi_i$ are convex so, as X is of Jensen-type, we get

$$⨍_X \pi_i(x)dx \le ⨍_{\partial X} \pi_i(x)dx \quad \text{and} \quad ⨍_X -\pi_i(x)dx \le ⨍_{\partial X} -\pi_i(x)dx.$$

Thus
$$\fint_X \pi_i(x)dx = \fint_{\partial X} \pi_i(x)dx \qquad \text{for } i \in \{1,\dots,n\}.$$
But centres of mass of X and ∂X equal to $\big(\fint_X \pi_i(x)dx\big)_{i=1}^n$ and $\big(\fint_{\partial X} \pi_i(x)dx\big)_{i=1}^n$, respectively. The above equality states that these points coincide.

Remark

We have presented some necessary condition for a shape to be of Jensen-type. Our conjecture is that every convex shape which satisfies this condition is of Jensen-type.

In the subsequent result we are going to prove that all parallelotopes and n-dimensional balls are of Jensen-type. Prior to this we characterize all Jensen-type triangles.

Proposition 2

Every triangle of Jensen-type is equilateral.

Proof. Let ABC be an arbitrary triangle which is of Jensen-type. Let us keep the standard notation $a = |BC|$, $b = |AC|$, $c = |AB|$.

It is well-known that the centre of mass of ABC equals (in barycentric coordinates) $\frac{1}{3}A + \frac{1}{3}B + \frac{1}{3}C$. Meanwhile the centre of mass of its boundary equals
$$\frac{1}{a+b+c}\Big(a\cdot\frac{B+C}{2} + b\cdot\frac{A+C}{2} + c\cdot\frac{A+B}{2}\Big)$$
$$= \frac{b+c}{2(a+b+c)}\cdot A + \frac{a+c}{2(a+b+c)}\cdot B + \frac{a+b}{2(a+b+c)}\cdot C.$$
Therefore, in view of Lemma 1 and the uniqueness of barycentric coordinates in a triangle, we obtain
$$\frac{b+c}{2(a+b+c)} = \frac{a+c}{2(a+b+c)} = \frac{a+b}{2(a+b+c)},$$
which after a simple transformation reduces to $a = b = c$.

The converse implication is a trivial corollary from more general Theorem 5 (which is proved at the end of this note).

Theorem 2

All parallelotopes are of Jensen-type.

Proof. Fix a parallelotope W of dimension n. Let $\{S_i\}_{i=1}^{2^n}$ be all its facets. Denote by S_i^* the facet opposite to S_i. In fact the facet S_i^* is simply S_i shifted by some vector $v_i \in \mathbb{R}^n$. Finally, for $y \in S_i$, let $y^* := y + v_i \in S_i^*$.

Fix a convex function $f\colon W \to \mathbb{R}$. By the Hermite-Hadamard inequality we have, for all admissible i and $y \in S_i$,
$$\int_y^{y^*} f(x)dx \le \operatorname{dist}(y,y^*)\cdot\frac{f(y)+f(y^*)}{2} = |v_i|\cdot\frac{f(y)+f(y^*)}{2}.$$

If we integrate both side over S_i, we obtain

$$\frac{\int_W f(x)dx}{\sin\angle(S_i,v_i)} = \int_{S_i}\int_y^{y^*} f(x)dxdy \le |v_i|\cdot\int_{S_i}\frac{f(y)+f(y^*)}{2}dy.$$

If we multiply both sides by $|S_i|\cdot\sin\angle(S_i,v_i)$ and use the well-known equality $|W| = |S_i|\cdot|v_i|\cdot\sin\angle(S_i,v_i)$, we obtain

$$|S_i|\int_W f(x)dx \le |W|\cdot\int_{S_i}\frac{f(y)+f(y^*)}{2}dy.$$

Finally, let us sum up the above inequality for $i\in\{1,2,\dots,2^n\}$. Then we get

$$|\partial W|\cdot\int_W f(x)dx \le |W|\int_{\partial W} f(y)dy,$$

which simplifies to

$$⨍_W f(x)dx \le ⨍_{\partial W} f(y)dy.$$

In the next theorem we will present an alternative proof (after [1, 2]) that all balls are Jensen-type.

Theorem 3
Let $n\ge 2$, then every n-dimensional ball is of Jensen-type.

Proof. Fix a convex function $f\colon B_n\to\mathbb{R}$. We have

$$\begin{aligned}⨍_{B_n} f(x)dx &= \frac{1}{|B_n|}\int_{B_n} f(x)dx = \frac{1}{|B_n|}\int_0^1 r^{n-1}\int_{S_{n-1}} f(rx)dxdr\\ &= \frac{1}{|B_n|}\int_0^1\frac{r^{n-1}}{2}\int_{S_{n-1}} f(rx)+f(-rx)dxdr.\end{aligned}$$

Applying Wright-convexity of f we get

$$\begin{aligned}⨍_{B_n} f(x)dx &\le \frac{1}{|B_n|}\int_0^1\frac{r^{n-1}}{2}\int_{S_{n-1}} f(x)+f(-x)dxdr\\ &\le \frac{1}{|B_n|}\int_0^1 r^{n-1}dr\int_{S_{n-1}} f(x)dx = \frac{1}{n|B_n|}\int_{S_{n-1}} f(x)dx.\end{aligned}$$

By the identity $n|B_n| = |S_{n-1}|$, we obtain desired inequality.

2.1. Convex polytopes having an inscribed sphere

We will now deal with convex polytopes. To avoid misunderstandings the *sphere inscribed* in a polytope is the sphere which is tangent to all its facets.

Lemma 3
Let $n \in \mathbb{N}$, $\Delta \subset \mathbb{R}^n$ be a convex $(n-1)$-dimensional set, and $s \in \mathbb{R}^n \setminus \Delta$ such that $G = \operatorname{conv}\{\Delta, s\}$ is n-dimentional. Then for every convex function $f\colon G \to \mathbb{R}$,

$$\fint_G f(x)dx \le \frac{n}{n+1} \fint_\Delta f(x)dx + \frac{1}{n+1} f(s).$$

Proof. For every $\theta \in (0,1]$ let T_θ be a homothetic transformation of Δ with centre s and scale θ. Denote its image by Δ_θ (obviously $\Delta_1 = \Delta$). Moreover, set $H := \operatorname{dist}(s, \Delta)$ and let $\pi\colon G \to \Delta$ be a projection such that $\pi|_{\Delta_\theta} = T_\theta^{-1}$. We know that

$$x = \theta \cdot \pi(x) + (1-\theta) \cdot s \qquad \text{for all } \theta \in (0,1] \text{ and } x \in \Delta_\theta.$$

Whence,

$$\begin{aligned}\int_G f(x)dx &= H \cdot \int_0^1 \int_{\Delta_\theta} f(x)dxd\theta \\ &= H \cdot \int_0^1 \int_{\Delta_\theta} f(\theta \cdot \pi(x) + (1-\theta)s)dxd\theta. \\ &= H \cdot \int_0^1 \int_\Delta \theta^{n-1} f(\theta \cdot x + (1-\theta)s)dxd\theta\end{aligned}$$

Thus, by Jensen's and Fubini's inequalities,

$$\int_G f(x)dx \le H \cdot \int_\Delta \Big(\int_0^1 \theta^n d\theta \cdot f(x) + \int_0^1 \theta^{n-1}(1-\theta)d\theta \cdot f(s) \Big) dx.$$

Since $\int_0^1 \theta^n d\theta = \frac{1}{n+1}$ and $\int_0^1 \theta^{n-1}(1-\theta)d\theta = \frac{1}{n(n+1)}$ we obtain

$$\begin{aligned}\int_G f(x)dx &\le H \cdot \Big(\frac{1}{n+1} \int_\Delta f(x)dx + \frac{1}{n(n+1)} |\Delta| \cdot f(s) \Big) \\ &= \frac{H \cdot |\Delta|}{n} \cdot \Big(\frac{n}{n+1} \fint_\Delta f(x)dx + \frac{1}{n+1} \cdot f(s) \Big).\end{aligned}$$

To finish the proof we can use the classical equality $|G| = \frac{1}{n} \cdot H \cdot |\Delta|$.

Theorem 4
Let W be an n-dimensional convex polytope having an inscribed sphere with centre s. Then

$$\fint_W f(x)dx \le \frac{n}{n+1} \fint_{\partial W} f(x)dx + \frac{1}{n+1} f(s) \tag{2}$$

for every convex function $f\colon W \to \mathbb{R}$.

Proof. Let r be the radius of the inscribed sphere. Denote all facets of W by $\{A_1, \ldots, A_k\}$. Moreover, let $G_i = \operatorname{conv}\{A_i, s\}$. We have $|G_i| = \frac{r}{n} \cdot |A_i|$, in particular $|W| = \frac{r}{n} |\partial W|$. By Lemma 3 we have (for all $i \in \{1, \ldots, k\}$),

$$\begin{aligned} \int_{G_i} f(x)dx &\le \frac{n}{n+1} \cdot |G_i| \cdot \fint_{A_i} f(x)dx + \frac{1}{n+1} \cdot |G_i| \cdot f(s) \\ &= \frac{n}{n+1} \cdot \frac{r}{n} \cdot |A_i| \cdot \fint_{A_i} f(x)dx + \frac{1}{n+1} \cdot |G_i| \cdot f(s) \\ &= \frac{r}{n+1} \cdot \int_{A_i} f(x)dx + \frac{1}{n+1} \cdot |G_i| \cdot f(s). \end{aligned}$$

Summing this inequality (side-by-side for $i \in \{1, \cdots, k\}$) we obtain

$$\begin{aligned} \int_W f(x)dx &\le \frac{r}{n+1} \cdot \int_{\partial W} f(x)dx + \frac{1}{n+1} \cdot |W| \cdot f(s) \\ &= \frac{r \cdot |\partial W|}{n+1} \cdot \fint_{\partial W} f(x)dx + \frac{1}{n+1} \cdot |W| \cdot f(s) \end{aligned}$$

To finish the proof note that

$$\frac{r \cdot |\partial W|}{n+1} = \frac{n}{n+1} \cdot \frac{r \cdot |\partial W|}{n} = \frac{n}{n+1} \cdot |W|.$$

We can now present some simple corollary.

Corollary 1
Let W be a convex n-dimensional polytope having the inscribed sphere with centre s and let m be the centre of mass of ∂W. Then

$$\fint_W f(x)dx \le \fint_{\partial W} f(x)dx + \frac{1}{n+1}(f(s) - f(m))$$

for every convex function $f \colon W \to \mathbb{R}$.

Indeed, by Jensen's inequality [3, Theorem 2.6.2] we have

$$f(m) = f\Big(\fint_{\partial W} x dx\Big) \le \fint_{\partial W} f(x)dx,$$

thus

$$0 \le \frac{1}{n+1}\Big(\fint_{\partial W} f(x)dx - f(m)\Big).$$

We can now sum this inequality with (2) side-by-side to obtain desired inequality.

As a trivial particular case we obtain some sufficient condition for W to be of Jensen-type.

Theorem 5
Let W be a convex polytope having the inscribed sphere. If the centre of this sphere coincide with the centre of mass ∂W, then W is of Jensen-type.

Obviously this result implies that all regular polygons and Platonic solids are of Jensen-type.

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