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A subordination results for a class of analytic functions defined by q-differential operator

Abstract. In this paper, we derive several subordination results and integral means result for certain class of analytic functions defined by means of q-differential operator. Some interesting corollaries and consequences of our results are also considered.

1. Introduction and definitions

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic in the open unit disc $\Delta = \{z : |z| < 1\}$. Also denote by $\mathcal{T}$ a subclass of $\mathcal{A}$ consisting functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \qquad a_n \geq 0,\ z \in \Delta$$

introduced and studied by Silverman [22]. For two functions f and g given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = z + \sum_{n=2}^{\infty} c_n z^n$$

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their Hadamard product (or convolution) is defined by

$$(f * g)(z) := z + \sum_{n=2}^{\infty} a_n c_n z^n. \tag{2}$$

We briefly recall here the notion of *q-operators*, i.e. *q-difference operator* that plays vital role in the theory of hypergeometric series, quantum physics and in the operator theory. The application of *q-calculus* was initiated by Jackson [7] and Kanas and Răducanu [12] have used the fractional *q-calculus operators* in investigations of certain classes of functions which are analytic in Δ. For details on *q-calculus* one can refer [2, 3, 7, 12, 16, 11, 26] and also the reference cited therein. For the convenience, we provide some basic definitions and concept details of *q-calculus* which are used in this paper. We suppose throughout the paper that $0 < q < 1$.

For $0 < q < 1$ the Jackson's *q-derivative* of a function $f \in \mathcal{A}$ is, by definition, given as follows [7]

$$D_q f(z) = \begin{cases} \dfrac{f(z) - f(qz)}{(1-q)z} & \text{for } z \neq 0, \\ f'(0) & \text{for } z = 0, \end{cases} \tag{3}$$

and

$$D_q^2 f(z) = D_q(D_q f(z)).$$

From (3), we have

$$D_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1},$$

where

$$[n]_q = \frac{1-q^n}{1-q},$$

is sometimes called *the basic number n*. Observe that if $q \to 1^-$, then $[n]_q \to n$.

For a function $h(z) = z^n$, we obtain $D_q h(z) = D_q z^n = \frac{1-q^n}{1-q} z^{n-1} = [n]_q z^{n-1}$, and as $q \to 1^-$ we note

$$D_q h(z) = q \to 1^- \quad ([n]_q z^{n-1}) = n z^{n-1} = h'(z),$$

where h' is the ordinary derivative. Recently, for $f \in \mathcal{A}$, Govindaraj and Sivasubramanian [11] defined and discussed the Sălăgean *q-differential operator* as follows

$$\begin{aligned} \mathcal{D}_q^0 f(z) &= f(z), \\ \mathcal{D}_q^1 f(z) &= z\mathcal{D}_q f(z), \\ \mathcal{D}_q^m f(z) &= z\mathcal{D}_q^m(\mathcal{D}_q^{m-1} f(z)), \\ \mathcal{D}_q^m f(z) &= z + \sum_{n=2}^{\infty} [n]_q^m a_n z^n, \qquad m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\},\ z \in \Delta. \end{aligned}$$

We note that if $q \to 1^-$,

$$D^m f(z) = z + \sum_{n=2}^{\infty} n^m a_n z^n \qquad m \in \mathbb{N}_0,\ z \in \Delta$$

is the familiar Sălăgean derivative [21].

Now let

$$\begin{aligned}
\mathbb{D}^0 f(z) &= \mathcal{D}_q^m f(z),\\
\mathbb{D}_{\lambda,q}^{1,m} f(z) &= (1-\lambda)\mathcal{D}_q^m f(z) + \lambda z(\mathcal{D}_q^m f(z))'\\
&= z + \sum_{n=2}^{\infty} [n]_q^m [1+(n-1)\lambda] a_n z^n,\\
\mathbb{D}_{\lambda,q}^{2,m} f(z) &= (1-\lambda)\mathcal{D}_{\lambda,q}^{1,m} f(z) + \lambda z(\mathcal{D}_{\lambda,q}^{1,m} f(z) f(z))'\\
&= z + \sum_{n=2}^{\infty} [n]_q^m [1+(n-1)\lambda]^2 a_n z^n.
\end{aligned}$$

In general, we have

$$\begin{aligned}
\mathbb{D}_{\lambda,q}^{\zeta,m} f(z) &= (1-\lambda)\mathcal{D}_{\lambda,q}^{\zeta-1,m_j} f(z) + \lambda z(\mathcal{D}_{\lambda,q}^{\zeta-1,m} f(z))'\\
&= z + \sum_{n=2}^{\infty} [n]_q^m [1+(n-1)\lambda]^{\zeta} a_n z^n, \qquad \lambda > 0,\ \zeta \in \mathbb{N}_0.
\end{aligned}$$

We note that when $q \to 1^-$, we get the differential operator

$$\mathbb{D}_{\lambda}^{\zeta,m} f(z) = z + \sum_{n=2}^{\infty} n^m [1+(n-1)\lambda]^{\zeta} a_n z^n \qquad \lambda > 0,\ m,\zeta \in \mathbb{N}_0.$$

We observe that for $m = 0$, we get the differential operator D^{ζ} defined by Al-Oboudi [5], and if $\zeta = 0$, we get Sălăgean differential operator D^m, see [21].

With the help of the differential operator $\mathbb{D}_{\lambda,q}^{\zeta,m}$, we say that a function $f \in \mathcal{A}$ is said to be in the class $\mathcal{S}_{\lambda,q}^{\zeta,m}(\alpha,\beta)$ if it satisfies

$$\Re\left(\frac{z(\mathbb{D}_{\lambda,q}^{\zeta,m} f(z))'}{\mathbb{D}_{\lambda,q}^{\zeta,m} f(z)} - \alpha\right) > \beta\left|\frac{z(\mathbb{D}_{\lambda,q}^{\zeta,m} f(z))'}{\mathbb{D}_{\lambda,q}^{\zeta,m} f(z)} - 1\right|, \qquad z \in \Delta,$$

where $-1 \le \alpha < 1$, $\beta \ge 0$, $\lambda > 0$, $m,\zeta \in \mathbb{N}_0$.

The family $\mathcal{S}_{\lambda,q}^{\zeta,m}(\alpha,\beta)$ contains many well-known as well as many new classes of analytic univalent functions. For $\beta = 0$, $\zeta = 0$ and $m = 0$ we obtain the family of starlike functions of order $\alpha (0 \le \alpha < 1)$ denoted by $\mathcal{S}^*(\alpha)$ and for $\beta = 0$, $\zeta = 0$ and $m = 1$ we have the family of convex functions of order $\alpha (0 \le \alpha < 1)$ denoted by $\mathcal{K}(\alpha)$. For $\zeta = 0$ and $m = 0$ we obtain the class $\beta - \mathcal{UST}(\alpha)$ and for $\zeta = 0$ and $m = 1$ we get the class $\beta - \mathcal{UKV}(\alpha)$. The classes $\beta - \mathcal{UST}(\alpha)$ and $\beta - \mathcal{UKV}(\alpha)$

were introduced by Rønning [19], [20]. We observe that $\beta-\mathcal{UST}(0)\equiv\beta-\mathcal{UST}$ the class of uniformly β-starlike functions and $\beta-\mathcal{UKV}(0)\equiv\beta-\mathcal{UKV}$ the class of uniformly β-convex functions introduced by Kanas and Wiśniowska [13], [14], see also the work of Kanas and Srivastava [15], Goodman [9], [10], Ma and Minda [18] and Gangadharan et al. [8].

Before we state and prove our main result we need the following definitions and lemmas.

Definition 1.1 (Subordination Principle)
Let g be analytic and univalent in Δ. If f is analytic in Δ, $f(0)=g(0)$ and $f(\Delta)\subset g(\Delta)$, then the function f is subordinate to g in Δ and we write $f\prec g$.

Definition 1.2 (Subordinating Factor Sequence)
A sequence $\{b_n\}_{n=1}^{\infty}$ of complex numbers is called a subordinating factor sequence if, whenever f is analytic , univalent and convex in Δ, we have the subordination given by

$$\sum_{n=2}^{\infty} b_n a_n z^n \prec f(z), \qquad z\in\Delta,\ a_1=1.$$

Lemma 1.3 ([28])
The sequence $\{b_n\}_{n=1}^{\infty}$ is a subordinating factor sequence if and only if

$$\Re\Big(1+2\sum_{n=1}^{\infty} b_n z^n\Big)>0, \qquad z\in\Delta.$$

Lemma 1.4
Assume that

$$\sum_{n=2}^{\infty}[n]_q^m[1+(n-1)\lambda]^{\zeta}[n(\beta+1)-(\alpha+\beta)]|a_n|\le 1-\alpha, \tag{4}$$

then $f\in\mathcal{S}_{\lambda,q}^{\zeta,m}(\alpha,\beta)$, where $-1\le\alpha<1$, $\beta\ge 0$, $\lambda>0$ and $m,\zeta\in\mathbb{N}_0$. The result is sharp for the function

$$f_n(z)=z-\frac{1-\alpha}{[n]_q^m[1+(n-1)\lambda]^{\zeta}[n(\beta+1)-(\alpha+\beta)]}z^n.$$

Proof. It suffices to show that

$$\beta\left|\frac{z(\mathbb{D}_{\lambda,q}^{\zeta,m}f(z))'}{\mathbb{D}_{\lambda,q}^{\zeta,m}f(z)}-1\right|-\Re\left(\frac{z(\mathbb{D}_{\lambda,q}^{\zeta,m}f(z))'}{\mathbb{D}_{\lambda,q}^{\zeta,m}f(z)}-1\right)\le 1-\alpha.$$

We have

$$\beta\left|\frac{z(\mathbb{D}_{\lambda,q}^{\zeta,m}f(z))'}{\mathbb{D}_{\lambda,q}^{\zeta,m}f(z)}-1\right|-\Re\left(\frac{z(\mathbb{D}_{\lambda,q}^{\zeta,m}f(z))'}{\mathbb{D}_{\lambda,q}^{\zeta,m}f(z)}-1\right)$$

$$\leq(1+\beta)\left|\frac{z(\mathbb{D}_{\lambda}^{\zeta,m}f(z))'}{\mathbb{D}_{\lambda,q}^{\zeta,m}f(z)}-1\right|$$

$$\leq\frac{(1+\beta)\sum\limits_{n=2}^{\infty}[n]_q^m[1+(n-1)\lambda]^{\zeta}(n-1)|a_n||z|^{n-1}}{1-\sum\limits_{n=2}^{\infty}[n]_q^m[1+(n-1)\lambda]^{\zeta}|a_n||z|^{n-1}}$$

$$\leq\frac{(1+\beta)\sum\limits_{n=2}^{\infty}[n]_q^m[1+(n-1)\lambda]^{\zeta}(n-1)|a_n|}{1-\sum\limits_{n=2}^{\infty}[n]_q^m[1+(n-1)\lambda]^{\zeta}|a_n|}.$$

The last expression is bounded from above by $1-\alpha$ if

$$\sum_{n=2}^{\infty}[n]_q^m[1+(n-1)\lambda]^{\zeta}[n(\beta+1)-(\alpha+\beta)]|a_n|$$

holds. It is obvious that the function f_n satisfies the inequality (4), and thus $1-\alpha$ cannot be replaced by a larger number. Therefore we need only to prove that $f\in\mathcal{S}_{\lambda,q}^{\zeta,m}(\alpha,\beta)$. Since

$$\Re\left(\frac{1-\sum\limits_{n=2}^{\infty}[n]_q^m[1+(n-1)\lambda]^{\zeta}n\ a_nz^{n-1}}{1-\sum\limits_{n=2}^{\infty}[n]_q^m[1+(n-1)\lambda]^{\zeta}\ a_nz^{n-1}}-\alpha\right)$$

$$>\beta\left|\frac{\sum\limits_{n=2}^{\infty}[n]_q^m[1+(n-1)\lambda]^{\zeta}(n-1)\ a_nz^{n-1}}{1-\sum\limits_{n=2}^{\infty}[n]_q^m[1+(n-1)\lambda]^{\zeta}\ a_nz^{n-1}}\right|.$$

Letting $z\to 1$ along the real axis, we obtain the desired inequality given in (4). and the proof is complete.

Let $\mathcal{S}_{\lambda,q}^{*,\zeta,m}(\alpha,\beta)$ denote the class of functions $f\in\mathcal{A}$ whose coefficients satisfy the condition (4). We note that $\mathcal{S}_{\lambda,q}^{*,\zeta,m}(\alpha,\beta)\subseteq\mathcal{S}_{\lambda,q}^{\zeta,m}(\alpha,\beta)$.

2. Main Theorem

Employing the techniques used earlier by Srivastava and Attiya [27], Attiya [4] and Frasin [6], Singh [25] and others, we state and prove the following theorem.

Theorem 2.1
Let the function f be defined by (1) be in the class $\mathcal{S}^{*,\zeta,m}_{\lambda,q}(\alpha,\beta)$, where $-1 \leq \alpha < 1$, $\beta \geq 0$, $\lambda > 0$, $\zeta \in \mathbb{N}_0$. Also let $\mathcal{K}$ denote the familiar class of functions $f \in \mathcal{A}$ which are also univalent and convex in Δ. Then

$$\frac{(1+q)^m(1+\lambda)^\zeta(\beta+2-\alpha)}{2[1-\alpha+(1+q)^m(1+\lambda)^\zeta(\beta+2-\alpha)]}(f*g)(z) \prec g(z), \qquad z\in\Delta,\ g\in\mathcal{K}, \tag{5}$$

and

$$\Re(f(z)) > -\frac{1-\alpha+(1+q)^m(1+\lambda)^\zeta(\beta+2-\alpha)}{(1+q)^m(1+\lambda)^\zeta(\beta+2-\alpha)}, \qquad z\in\Delta. \tag{6}$$

The constant $\frac{(1+q)^m(1+\lambda)^\zeta(\beta+2-\alpha)}{2[1-\alpha+(1+q)^m(1+\lambda)^\zeta(\beta+2-\alpha)]}$ is the best estimate.

Proof. Let $f \in \mathcal{S}^{*,\zeta,m}_{\lambda,q}(\alpha,\beta)$ and let $g(z) = z + \sum_{n=2}^{\infty} c_n z^n \in \mathcal{K}$. Then

$$\begin{aligned}&\frac{(1+q)^m(1+\lambda)^\zeta(\beta+2-\alpha)}{2[1-\alpha+(1+q)^m(1+\lambda)^\zeta(\beta+2-\alpha)]}(f*g)(z)\\ &\qquad= \frac{(1+q)^m(1+\lambda)^\zeta(\beta+2-\alpha)}{2[1-\alpha+(1+q)^m(1+\lambda)^\zeta(\beta+2-\alpha)]}\Big(z+\sum_{n=2}^{\infty}a_nc_nz^n\Big).\end{aligned}$$

Thus, by Definition 1.2, the assertion of our theorem will hold if the sequence

$$\Big(\frac{(1+q)^m(1+\lambda)^\zeta(\beta+2-\alpha)}{2[1-\alpha+(1+q)^m(1+\lambda)^\zeta(\beta+2-\alpha)]}a_n\Big)_{n=1}^{\infty}$$

is a subordinating factor sequence, with $a_1 = 1$. In view of Lemma 1.3, this will be the case if and only if

$$\Re\Big(1+2\sum_{n=1}^{\infty}\frac{(1+q)^m(1+\lambda)^\zeta(\beta+2-\alpha)}{2[1-\alpha+(1+q)^m(1+\lambda)^\zeta(\beta+2-\alpha)]}a_nz^n\Big) > 0, \qquad z\in\Delta. \tag{7}$$

Now

$$\begin{aligned}&\Re\Big(1+\frac{(1+q)^m(1+\lambda)^\zeta(\beta+2-\alpha)}{1-\alpha+(1+q)^m(1+\lambda)^\zeta(\beta+2-\alpha)}\sum_{n=1}^{\infty}a_nz^n\Big)\\ &\quad= \Re\Big(1+\frac{(1+q)^m(1+\lambda)^\zeta(\beta+2-\alpha)}{1-\alpha+(1+q)^m(1+\lambda)^\zeta(\beta+2-\alpha)}z\\ &\qquad+\frac{1}{1-\alpha+(1+q)^m(1+\lambda)^\zeta(\beta+2-\alpha)}\\ &\qquad\quad\cdot\sum_{n=2}^{\infty}(1+q)^m(1+\lambda)^\zeta(\beta+2-\alpha)a_nz^n\Big)\\ &\quad\geq 1-\Big(\frac{[2]_q^m(1+\lambda)^\zeta(\beta+2-\alpha)}{1-\alpha+[2]_q^m(1+\lambda)^\zeta(\beta+2-\alpha)}r\end{aligned}$$

$$- \frac{1}{1-\alpha+[2]_q^m(1+\lambda)^\zeta(\beta+2-\alpha)} \cdot \sum_{n=2}^{\infty}[2]_q^m[1+(n-1)\lambda][n(\beta+1)-(\alpha+\beta)]a_n r^n\Big)$$

$$> 1 - \frac{(1+q)^m(1+\lambda)^\zeta(\beta+2-\alpha)}{1-\alpha+(1+q)^m(1+\lambda)^\zeta(\beta+2-\alpha)} r - \frac{1-\alpha}{1-\alpha+(1+q)^m(1+\lambda)^\zeta(\beta+2-\alpha)} r > 0, \qquad |z| = r.$$

Notice that the last but one inequality follows from the fact that $[2]_q^m \sum_{n=2}^{\infty}[1+(n-1)\lambda][n(\beta+1)-(\alpha+\beta)]$ is an increasing function of $n(n \geq 2)$). Thus (7) holds true in Δ. This proves the inequality (5). The inequality (6) follows by taking the convex function $g(z) = \frac{z}{1-z} = z + \sum\limits_{n=2}^{\infty} z^n$ in (5).

To prove the sharpness of the constant $\frac{(1+q)^m(1+\lambda)^\zeta(\beta+2-\alpha)}{2[1-\alpha+(1+q)^m(1+\lambda)^\zeta(\beta+2-\alpha)]}$, we consider the function $f_2 \in \mathcal{S}_{\lambda,q}^{*,\zeta,m}(\alpha,\beta)$ given by

$$f_2(z) = z - \frac{1-\alpha}{(1+q)^m(1+\lambda)^\zeta(\beta+2-\alpha)} z^2,$$

where $-1 \leq \alpha < 1$, $\beta \geq 0$, $\lambda > 0$, $m, \zeta \in \mathbb{N}_0$. Thus from (5) we have

$$\frac{(1+q)^m(1+\lambda)^\zeta(\beta+2-\alpha)}{2[1-\alpha+(1+q)^m(1+\lambda)^\zeta(\beta+2-\alpha)]} f_2(z) \prec \frac{z}{1-z}.$$

It can be easily verified that

$$\min\Big\{\Re\Big(\frac{(1+q)^m(1+\lambda)^\zeta(\beta+2-\alpha)}{2[1-\alpha+(1+q)^m(1+\lambda)^\zeta(\beta+2-\alpha)]} f_2(z)\Big)\Big\} = -\frac{1}{2}, \qquad z \in \Delta.$$

This shows that the constant $\frac{(1+q)^m(1+\lambda)^\zeta(\beta+2-\alpha)}{2[1-\alpha+(1+q)^m(1+\lambda)^\zeta(\beta+2-\alpha)]}$ is the best possible.

Putting $m = 0$ in Theorem 2.1 yields the following result obtained by Aouf et al. [1].

Corollary 2.2
Let f, defined by (1), be in the class $\mathcal{M}_\lambda^*(\zeta,\alpha,\beta)$, where $-1 \leq \alpha < 1$, $\beta \geq 0$, $\lambda > 0$, $\zeta \in \mathbb{N}_0$. Then

$$\frac{(1+\lambda)^\zeta(\beta+2-\alpha)}{2[1-\alpha+(1+\lambda)^\zeta(\beta+2-\alpha)]}(f*g)(z) \prec g(z) \qquad z \in \Delta,\ g \in \mathcal{K}$$

and

$$\Re(f(z)) > -\frac{1-\alpha+(1+\lambda)^\zeta(\beta+2-\alpha)}{(1+\lambda)^\zeta(\beta+2-\alpha)}, \qquad z \in \Delta.$$

The constant $\frac{(1+\lambda)^\zeta(\beta+2-\alpha)}{2[1-\alpha+(1+\lambda)^\zeta(\beta+2-\alpha)]}$ is the best estimate.

If we put $m = 0$ and $\zeta = 0$ in Theorem 2.1, we obtain the next two results obtained by Frasin [6].

Corollary 2.3
Let f, defined by (1)*, be in the class $\beta - \mathcal{UST}(\alpha)$. Then*

$$\frac{\beta+2-\alpha}{2(\beta+3-2\alpha)}(f*g)(z) \prec g(z), \qquad -1 \le \alpha < 1,\ \beta \ge 0,\ z \in \Delta,\ g \in \mathcal{K}$$

and

$$\Re(f(z)) > -\frac{\beta+3-2\alpha}{\beta+2-\alpha}, \qquad z \in \Delta.$$

The constant $\frac{\beta+2-\alpha}{2(\beta+3-2\alpha)}$ is the best estimate.

Corollary 2.4
Let f, defined by (1)*, be in the class $\beta - \mathcal{UKV}(\alpha)$. Then*

$$\frac{\beta+2-\alpha}{2\beta+5-3\alpha}(f*g)(z) \prec g(z), \qquad -1 \le \alpha < 1,\ \beta \ge 0,\ z \in \Delta,\ g \in \mathcal{K}$$

and

$$\Re(f(z)) > -\frac{2\beta+5-3\alpha}{2(\beta+2-\alpha)}, \qquad z \in \Delta.$$

The constant $\frac{\beta+2-\alpha}{2\beta+5-3\alpha}$ is the best estimate.

Putting $m = 0$, $\zeta = 0$ and $\beta = 0$ in Theorem 2.1, we obtain the next two results obtained by Frasin [6].

Corollary 2.5
Let f, defined by (1)*, be in the class $\mathcal{S}^*(\alpha)$. Then*

$$\frac{2-\alpha}{6-4\alpha}(f*g)(z) \prec g(z), \qquad z \in \Delta,\ g \in \mathcal{K}$$

and

$$\Re(f(z)) > -\frac{3-2\alpha}{2-\alpha}, \qquad z \in \Delta.$$

The constant $\frac{2-\alpha}{6-4\alpha}$ is the best estimate.

Corollary 2.6
Let f, defined by (1)*, be in the class $\mathcal{K}(\alpha)$. Then*

$$\frac{2-\alpha}{5-3\alpha}(f*g)(z) \prec g(z,) \qquad z \in \Delta,\ g \in \mathcal{K}$$

and

$$\Re(f(z)) > -\frac{5-3\alpha}{2(2-\alpha)}, \qquad z \in \Delta.$$

The constant $\frac{2-\alpha}{5-3\alpha}$ is the best estimate.

3. Integral Means Inequalities

Lemma 3.1 ([17])
If the functions f and g are analytic in Δ with $g \prec f$, then for $\eta > 0$, and $0 < r < 1$,

$$\int_0^{2\pi} |g(re^{i\theta})|^\eta d\theta \le \int_0^{2\pi} |f(re^{i\theta})|^\eta d\theta.$$

In [22], Silverman found that the function $f_2(z) = z - \frac{z^2}{2}$ is often extremal over the family $\mathcal{T}$ and applied this function to resolve his integral means inequality, conjectured in [23] and settled in [24], that

$$\int_0^{2\pi} |f(re^{i\theta})|^\eta d\theta \le \int_0^{2\pi} |f_2(re^{i\theta})|^\eta d\theta,$$

for all $f \in \mathcal{T}$, $\eta > 0$ and $0 < r < 1$. In [24], Silverman also proved his conjecture for the subclasses $\mathcal{T}^*(\alpha)$ and $\mathcal{K}(\alpha)$ of $\mathcal{T}$.

Applying Lemma 3.1 and Lemma 1.4, we prove the following result.

Theorem 3.2
Suppose $f \in \mathcal{S}_{\lambda,q}^{\zeta,m}(\alpha,\beta)$, $\eta > 0$, and f_2 is defined by

$$f_2(z) = z - \frac{1-\alpha}{(1+q)^m[1+\lambda]^\zeta[\beta+2-\alpha]} z^2.$$

Then for $z = re^{i\theta}$, $0 < r < 1$ we have

$$\int_0^{2\pi} |f(z)|^\eta \, d\theta \le \int_0^{2\pi} |f_2(z)|^\eta \, d\theta. \tag{8}$$

Proof. Observe that for $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n$ inequality (8) is equivalent to

$$\int_0^{2\pi} \left|1 - \sum_{n=2}^{\infty} |a_n| z^{n-1}\right|^\eta d\theta \le \int_0^{2\pi} \left|1 - \frac{1-\alpha}{[2]_q^m[1+\lambda]^\zeta[\beta+2-\alpha]} z\right|^\eta d\theta.$$

By Lemma 3.1, it suffices to show that

$$1 - \sum_{n=2}^{\infty} |a_n| z^{n-1} \prec 1 - \frac{1-\alpha}{[2]_q^m[1+\lambda]^\zeta[\beta+2-\alpha]} z.$$

Setting

$$1 - \sum_{n=2}^{\infty} |a_n| z^{n-1} = 1 - \frac{1-\alpha}{[2]_q^m[1+\lambda]^\zeta[\beta+2-\alpha]} w(z),$$

and using (4), we obtain that $w(z)$ is analytic in Δ, $w(0)=0$ and

$$|w(z)| = \left| \sum_{n=2}^{\infty} \frac{[2]_q^m [1+\lambda]^{\zeta} [\beta+2-\alpha]}{1-\alpha} |a_n| z^{n-1} \right|$$

$$\leq |z| \sum_{n=2}^{\infty} \frac{[n]_q^m [1+(n-1)\lambda]^{\zeta} [n(\beta+1)-(\alpha+\beta)]}{1-\alpha} |a_n| \leq |z|.$$

This completes the proof of Theorem 3.2.

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