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Nearly irreducibility of polynomials and the Newton diagrams

Abstract. Let f be a polynomial in two complex variables. We say that f is nearly irreducible if any two nonconstant polynomial factors of f have a common zero in $\mathbf{C}^2$. In the paper we give a criterion of nearly irreducibility for a given polynomial f in terms of its Newton diagram.

1. Introduction

Let $f(X,Y) = \sum c_{\alpha\beta}X^\alpha Y^\beta \in \mathbf{C}[X,Y]$ be a nonzero polynomial of positive degree. We say that the polynomial f is *quasi–convenient* if $c_{\alpha 0} \neq 0$ and $c_{0\beta} \neq 0$ for some integers $\alpha, \beta \geq 0$. Otherwise $f(X,Y) = X^s Y^t \widetilde{f}(X,Y)$ for some nonnegative integers s and t, where $\widetilde{f}$ is a quasi-convenient polynomial or it is a nonzero constant. Let $\text{supp} f := \{(\alpha,\beta) \in \mathbf{N}^2 : \ c_{\alpha\beta} \neq 0\}$. We define

$$\Delta_\infty(f) := \text{convex}(\{(0,0)\} \cup \text{supp} f).$$

The polygon $\Delta_\infty(f)$ is called *the Newton diagram at infinity* of the polynomial f.

For any nonzero vector $\vec{w} = [p,q]$ of the real plane $\mathbf{R}^2$ we put

$$\text{in}(f,\vec{w})(X,Y) := \sum_{p\alpha+q\beta=d_{\vec{w}}(f)} c_{\alpha\beta}X^\alpha Y^\beta,$$

where

$$d_{\vec{w}}(f) = \max\{p\alpha + q\beta : \ (\alpha,\beta) \in \text{supp} f\}.$$

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We call a quasi–convenient polynomial $f(X,Y) \in \mathbf{C}[X,Y]$ *nondegenerate at infinity* if for every real vector $\vec{w} = [p,q]$ such that $p > 0$ or $q > 0$ the system of equations

$$\mathrm{in}(f,\vec{w})(X,Y) = \frac{\partial}{\partial X}\mathrm{in}(f,\vec{w})(X,Y) = \frac{\partial}{\partial Y}\mathrm{in}(f,\vec{w})(X,Y) = 0$$

has no solutions in $\mathbf{C}^* \times \mathbf{C}^*$, where $\mathbf{C}^* = \mathbf{C} \setminus \{0\}$.

Definition 1.1
A polynomial $f(X,Y) \in \mathbf{C}[X,Y]$ of a positive degree is *nearly irreducible* if any two nonconstant polynomial factors $g(X,Y)$, $h(X,Y) \in \mathbf{C}[X,Y]$ of $f(X,Y)$ have a common zero in $\mathbf{C}^2$.

Note that every nearly irreducible polynomial has a connected zero-set. Note that nearly irreducible polynomial may be reducible (e.g. $f = XY$). It is easy to check that if f is nearly irreducible and $\mathrm{grad}\, f = (\frac{\partial f}{\partial X}, \frac{\partial f}{\partial Y}) \neq 0$ on the curve $f(X,Y) = 0$ then f is irreducible (see [17]).

The notion of nearly irreducibility of polynomials in two variables was introduced in [1] by S. Abhyankar and L. A. Rubel in connection with research of these authors on irreducibility of polynomials of the form $f(X) - g(Y)$. The main result of [1] was reproved by L. A. Rubel, A. Shinzel and H. Tverberg in [17]. Afterwards A. Płoski generalized the result of S. Abhyankar and L. A. Rubel by using the Newton diagram of a given polynomial (see [16], Theorem 2) which is Theorem 1.2 in this note.

Theorem 1.2 ([16], Theorem 2)
Let f be a quasi-convenient polynomial such that

1° *f is nondegenerate at infinity,*

2° *every face of the polygon $\Delta_\infty(f)$ not included in coordinate axes has a negative slope, (i.e. it is a segment included in the straight line of the form $p\alpha + q\beta = r$ for some $p, q > 0$).*

Then the polynomial f is nearly irreducible.

Our theorem (Theorem 1.3) generalizes the result of Płoski. We state

Theorem 1.3
Let f be a quasi-convenient polynomial such that

1° *f is nondegenerate at infinity,*

2° *if $\vec{w} = [p,q]$ is a nonzero vector such that $pq \leqslant 0$ then the system of equations $\mathrm{in}(f,\vec{w})(X,Y) = \mathrm{in}(f,-\vec{w})(X,Y) = 0$ has no solutions in $\mathbf{C}^* \times \mathbf{C}^*$.*

Then the polynomial f is nearly irreducible.

In comparison to Theorem 1.2, in Theorem 1.3 there are no restrictions on the shape of the polygon $\Delta_\infty(f)$. The proof of Theorem 1.3, based on the Kouchnirenko-Bernstein Theorem, is given in Section 3.

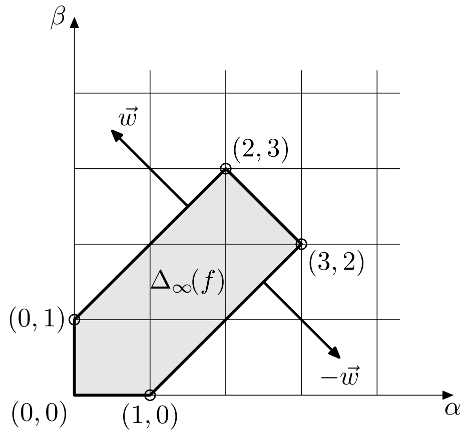

Fig. 1: Newton diagram at infinity of f.

Remark

If there is no pair of parallel faces of the polygon $\Delta_\infty(f)$ then for any $\vec{w} \neq \vec{0}$ at least one of the polynomials $\mathrm{in}(f, \vec{w})(X, Y)$ or $\mathrm{in}(f, -\vec{w})(X, Y)$ is a monomial and then condition 2° in our theorem trivially holds, so Theorem 1.3 implies Theorem 1.2 (see Example 3).

The examples presented below show that the assumption 2° in Theorem 1.3 is essential. In particular, Example 2 shows that nonnegative slope of boundary faces of $\Delta_\infty(f)$ not included in coordinate axes and nondegeneracy at infinity of f are not enough to nearly irreducibility property

Example 1

Let us consider the polynomial

$$f(X, Y) = X^3Y^2 + X^2Y^3 - X - Y = (XY - 1)(XY + 1)(X + Y),$$

whose Newton diagram at infinity is drawn in Figure 1.

It is easily seen that the polynomial f is nondegenerate at infinity and that it is not nearly irreducible. Note that condition 2° of Theorem 1.3 is not satisfied. Namely, if $\vec{w} = [-1, 1]$ then $\mathrm{in}(f, \vec{w})(X, Y) = Y(X^2Y^2 - 1)$ and $\mathrm{in}(f, -\vec{w})(X, Y) = X(X^2Y^2 - 1)$. The system

$$\mathrm{in}(f, \vec{w})(X, Y) = \mathrm{in}(f, -\vec{w})(X, Y) = 0$$

has a solution in $\mathbf{C}^* \times \mathbf{C}^*$.

Example 2

Let $f(X, Y) = (X - 1)(X + 1)(X + Y) = X^3 + X^2Y - X - Y$, whose Newton diagram at infinity is drawn in Figure 2. The polynomial f is nondegenerate at infinity and obviously f is not nearly irreducible. The assumption 2° of Theorem 1.3 does not hold because if $\vec{w} = [0, 1]$ then $\mathrm{in}(f, \vec{w})(X, Y) = Y(X^2 - 1)$ and $\mathrm{in}(f, -\vec{w})(X, Y) = X(X^2 - 1)$ have a common zero in $\mathbf{C}^* \times \mathbf{C}^*$. Note that for any $c \neq 0$ the polynomial $f(X, Y) + c$ satisfies 2°, so it is nearly irreducible.

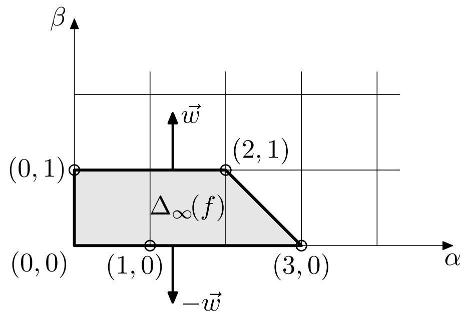

Fig. 2: Newton diagram at infinity of f.

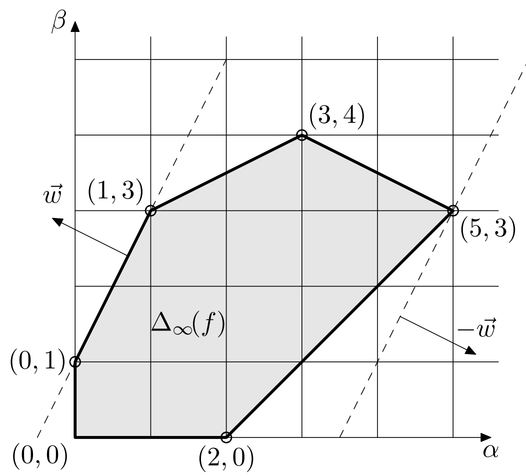

Fig. 3: Newton diagram at infinity of f.

Example 3
Let $f(X,Y) = Y + X^2 + XY^3 + X^3Y^4 + X^5Y^3$. Its Newton diagram at infinity is given in Figure 3. The polynomial f is nondegenerate at infinity but we can not apply Theorem 1.2 because the polygon $\Delta_\infty(f)$ possesses faces with positive slope. Nevertheless, $\mathrm{in}(f,\vec{w})(X,Y)$ or $\mathrm{in}(f,-\vec{w})(X,Y)$ is a monomial for any $\vec{w} \neq \vec{0}$, hence after Theorem 1.3, the polynomial f is nearly irreducible.

2. Kouchnirenko-Bernstein Theorem

The famous Bézout theorem for affine curves states that two polynomial equations of given degrees m, $n > 0$ have at most mn common solutions provided that their number is finite. If additionally their Newton diagrams at infinity are known then we can give more precise estimation. Namely, we may replace the product mn by the Minkowski mixed area of these diagrams. Such results were proved in Kouchnirenko's and Bernstein's papers in 1970s [10, 11, 12, 4]. See also [3, 5, 8, 9]. Focusing only on two-dimensional case much more precise results are possible.

Let $f(X,Y)$, $g(X,Y) \in \mathbf{C}[X,Y]$ be polynomials of positive degrees. If $P = (a,b) \in \mathbf{C}^2$ is a solution of the system

$$f(X,Y) = 0, \quad g(X,Y) = 0,$$

then the symbol $(f,g)_P$ denotes the intersection multiplicity of f and g at P. We use the definition of the intersection multiplicity as in [7]. We have $(f,g)_P < +\infty$ if and only if P is an isolated solution of the given system.

A pair (f,g) of quasi–convenient polynomials is *nondegenerate at infinity* if for any real vector $\vec{w} = [p,q]$ such that $p > 0$ or $q > 0$ the system of equations $\mathrm{in}(f,\vec{w})(X,Y) = \mathrm{in}(g,\vec{w})(X,Y) = 0$ has no solutions in $\mathbf{C}^* \times \mathbf{C}^*$.

For a pair of quasi–convenient polynomials $f(X,Y)$, $g(X,Y) \in \mathbf{C}[X,Y]$ we denote by $\nu_\infty(f,g)$ the double Minkowski mixed area (see [18] for definition) of the diagrams $\Delta_\infty(f)$ and $\Delta_\infty(g)$, i.e.

$$\nu_\infty(f,g) := \mathrm{Area}(\Delta_\infty(f) + \Delta_\infty(g)) - \mathrm{Area}\Delta_\infty(f) - \mathrm{Area}\Delta_\infty(g),$$

where + denotes the Minkowski sum.

PROPOSITION 2.1 (Additivity of the Newton diagrams at infinity)
If polynomials f, $g \in \mathbf{C}[X,Y]$ are quasi–convenient then

$$\Delta_\infty(fg) = \Delta_\infty(f) + \Delta_\infty(g).$$

Proof of Proposition 2.1. The inclusion $\Delta_\infty(fg) \subset \Delta_\infty(f) + \Delta_\infty(g)$ is obvious and it holds for any pair of nonzero polynomials. To prove the opposite inclusion we consider for every nonzero polynomial h of a positive degree its *Newton diagram* $\Delta(h) := \mathrm{convex}(\mathrm{supp}\, h)$.

Let $u \in \Delta_\infty(f)$ and $v \in \Delta_\infty(g)$. It is easily seen that there exist points $u_1 \in \Delta(f)$, $v_1 \in \Delta(g)$ and real numbers $0 \le s,t \le 1$ such, that $u = su_1$ and $v = tv_1$. We need to show that $u + v = su_1 + tv_1 \in \Delta_\infty(fg)$. The following equality is well-known: $\Delta(f) + \Delta(g) = \Delta(fg)$ and it holds for every nonzero polynomials f and g (see [14]). Let us note that

$$u_1 + v_1 \in \Delta(f) + \Delta(g) = \Delta(fg) \subset \Delta_\infty(fg).$$

The quasi–convenience of the polynomials f and g means that their supports have common points with both coordinate axes. Therefore $\Delta(f) \subset \Delta_\infty(fg)$ and $\Delta(g) \subset \Delta_\infty(fg)$. Hence $(0,0)$, u_1, v_1, $u_1 + v_1 \in \Delta_\infty(fg)$. By convexity of $\Delta_\infty(fg)$ we see that $su_1 + tv_1 \in \Delta_\infty(fg)$.

The assumption in Proposition 2.1 about quasi-convenience of the polynomials f and g is essential. For instance, if $f(X,Y) = X$ and $g(X,Y) = Y$, then $f(X,Y)\,g(X,Y) = XY$ and $\Delta_\infty(fg)$ is a segment, while $\Delta_\infty(f) + \Delta_\infty(g)$ is a square. By Proposition 2.1 we may write

$$\nu_\infty(f,g) = \mathrm{Area}\Delta_\infty(fg) - \mathrm{Area}\Delta_\infty(f) - \mathrm{Area}\Delta_\infty(g).$$

Let us present a useful version of the Kouchnirenko–Bernstein Theorem in two-dimensional case.

Theorem 2.2 (Kouchnirenko–Bernstein)
Let polynomials $f(X,Y)$, $g(X,Y) \in \mathbf{C}[X,Y]$ be quasi–convenient. It holds

1° *if f and g are coprime then* $\sum_{P\in\mathbf{C}^2}(f,g)_P \leq \nu_\infty(f,g)$,

2° $\sum_{P\in\mathbf{C}^2}(f,g)_P = \nu_\infty(f,g)$ *if and only if the pair (f,g) is nondegenerate at infinity.*

The first proof of this theorem (in multi-dimensional case) was given by Kouchnirenko in [10] under the additional assumption that the polynomials f and g have identical Newton diagrams at infinity.

The original Bernstein Theorem was formulated for Laurent polynomials in n variables without mentioning Kouchnirenko's assumption and it concerned counting isolated solutions of a system of such polynomials in the set $(\mathbf{C}^*)^n$ (see [4]). Theorem 2.2 follows from its local version due to Kouchnirenko (i.e. estimation of the intersection multiplicity of plane curves given in terms of their local Newton diagrams, see [10, 2, 15, 6, 13]) and from Bézout Theorem for projective curves. For the sake of completeness, we give the proof of Theorem 2.2 in Section 4.

3. Proof of Theorem 1.3

The proof of our theorem requires two lemmas. Second of them follows from well-known properties of the Minkowski mixed area (see [18], Theorem 5.1.7) but for the convenience of the reader we will give a proof.

Lemma 3.1
Let $f(X,Y)$ be a polynomial nondegenerate at infinity. If $g(X,Y), h(X,Y) \in \mathbf{C}[X,Y]$ are two coprime divisors of f then the pair (g,h) is nondegenerate at infinity.

Proof. Since $f(X,0)f(0,Y) \neq 0$, we have $g(X,0)g(0,Y)h(X,0)h(0,Y) \neq 0$ in $\mathbf{C}[X,Y]$. Therefore the polynomials g and h are quasi–convenient. Let us suppose, contrary to our claim, that the pair (g,h) is degenerate at infinity. By definition there exists a real vector $\vec{w} = [p,q]$, where $p > 0$ or $q > 0$, such that $\mathrm{in}(g,\vec{w})(x,y) = \mathrm{in}(h,\vec{w})(x,y) = 0$ for some $(x,y) \in \mathbf{C}^* \times \mathbf{C}^*$. Since $g(X,Y)$ and $h(X,Y)$ are coprime divisors of the polynomial $f(X,Y)$, there exists a polynomial $P(X,Y)$ such that $f(X,Y) = g(X,Y)h(X,Y)P(X,Y)$. Let us note that

$$\mathrm{in}(f,\vec{w})(X,Y) = \mathrm{in}(g,\vec{w})(X,Y)\mathrm{in}(h,\vec{w})(X,Y)\mathrm{in}(P,\vec{w})(X,Y),$$

hence

$$\mathrm{in}(f,\vec{w})(x,y) = \frac{\partial}{\partial X}\mathrm{in}(f,\vec{w})(x,y) = \frac{\partial}{\partial Y}\mathrm{in}(f,\vec{w})(x,y) = 0.$$

The above equalities contradict nondegeneracy at infinity of the polynomial f.

Lemma 3.2
If the polynomials $f, g \in \mathbf{C}[X,Y]$ of positive degrees are quasi–convenient then

1° $\nu_\infty(f,g) \geq 0$,

2° $\nu_\infty(f,g) = 0$ *if and only if the diagrams* $\Delta_\infty(f)$ *and* $\Delta_\infty(g)$ *form segments included in the same straight line passing through the origin.*

In the proof of Lemma 3.2 we need the following Brunn–Minkowski inequality (see [19], Theorem 6.5.3):

Theorem 3.3
If A and B are nonempty and measurable subsets of $\mathbf{R}^2$ then

$$(\mathrm{Area}(A+B))^{1/2} \geq (\mathrm{Area}A)^{1/2} + (\mathrm{Area}B)^{1/2},$$

where $A+B$ denotes the Minkowski sum of A and B.

Proof of Lemma 3.2. Note that $\Delta_\infty(fg) = \Delta_\infty(f) + \Delta_\infty(g)$ (see Proposition 2.1). Using Brunn–Minkowski inequality for the sets $A = \Delta_\infty(f)$ and $B = \Delta_\infty(g)$ we have

$$(\mathrm{Area}\Delta_\infty(fg))^{1/2} \geq (\mathrm{Area}\Delta_\infty(f))^{1/2} + (\mathrm{Area}\Delta_\infty(g))^{1/2},$$

hence

$$\mathrm{Area}\Delta_\infty(fg) \geq \mathrm{Area}\Delta_\infty(f) + \mathrm{Area}\Delta_\infty(g) + 2[(\mathrm{Area}\Delta_\infty(f))(\mathrm{Area}\Delta_\infty(g))]^{1/2}.$$

This proves 1°.

Suppose now that in 1° the equality holds. Last inequality implies that $\mathrm{Area}\Delta_\infty(f) = 0$ or $\mathrm{Area}\Delta_\infty(g) = 0$. Suppose, without loss of generality, that $\mathrm{Area}\Delta_\infty(f) = 0$. Since the set $\Delta_\infty(f)$ is convex, $(0,0) \in \Delta_\infty(f)$, and $\deg f > 0$, we get that $\Delta_\infty(f)$ is a segment included in a straight line passing through the origin. Moreover,

$$\mathrm{Area}\Delta_\infty(fg) = \mathrm{Area}\Delta_\infty(g).$$

It is easy to check that the diagram $\Delta_\infty(g)$ does not contain a point not belonging to the straight line including $\Delta_\infty(f)$. Indeed, otherwise we would have $\mathrm{Area}\Delta_\infty(fg) > \mathrm{Area}\Delta_\infty(g)$. This last observation proves 2°.

Proof of Theorem 1.3. Let us suppose, contrary to our claim, that there exist polynomials $g(X,Y)$, $h(X,Y) \in \mathbf{C}[X,Y]$ of positive degrees being divisors of the polynomial $f(X,Y)$ such that

$$\sum_{P \in \mathbf{C}^2} (g,h)_P = 0.$$

Obviously, the polynomials $g(X,Y)$ and $h(X,Y)$ are coprime and they are quasi–convenient. From Lemma 3.1 it follows that the pair (g,h) is nondegenerate at infinity. Using now Kouchnirenko–Bernstein Theorem (Theorem 2.2) we state that

$$\nu_\infty(g,h) = 0.$$

Therefore, Lemma 3.2 implies that the diagrams $\Delta_\infty(g)$ and $\Delta_\infty(h)$ are segments included in the same straight line $p\alpha + q\beta = 0$, where $\vec{w} = [p, q] \neq \vec{0}$ and $pq \leqslant 0$. So, we have

$$\mathrm{in}(g, \vec{w})(X, Y) = \mathrm{in}(g, -\vec{w})(X, Y) = g(X, Y),$$
$$\mathrm{in}(h, \vec{w})(X, Y) = \mathrm{in}(h, -\vec{w})(X, Y) = h(X, Y).$$

There exists a polynomial $P(X, Y)$ such that

$$f(X, Y) = g(X, Y)h(X, Y)P(X, Y),$$

hence

$$\mathrm{in}(f, \vec{w})(X, Y) = \mathrm{in}(g, \vec{w})(X, Y)\mathrm{in}(h, \vec{w})(X, Y)\mathrm{in}(P, \vec{w})(X, Y),$$
$$\mathrm{in}(f, -\vec{w})(X, Y) = \mathrm{in}(g, -\vec{w})(X, Y)\mathrm{in}(h, -\vec{w})(X, Y)\mathrm{in}(P, -\vec{w})(X, Y),$$

so

$$\mathrm{in}(f, \vec{w})(X, Y) = g(X, Y)h(X, Y)\mathrm{in}(P, \vec{w})(X, Y),$$
$$\mathrm{in}(f, -\vec{w})(X, Y) = g(X, Y)h(X, Y)\mathrm{in}(P, -\vec{w})(X, Y).$$

By condition 2° of our assumptions we see that $\{g(X, Y) = 0\} \subset \{XY = 0\}$ and $\{h(X, Y) = 0\} \subset \{XY = 0\}$. Let us recall that the polynomials g and h are coprime. Using Hilbert Nullstellensatz we conclude that the polynomials g and h are powers (up to a constant) of different variables. Therefore the point $(0, 0)$ is the solution of the system $g(X, Y) = h(X, Y) = 0$, which is a contradiction.

4. Proof of Theorem 2.2

Let $f(X, Y) = \sum c_{\alpha\beta} X^\alpha Y^\beta \in \mathbf{C}[X, Y]$ be a nonzero polynomial of positive degree. Recall that *the Newton diagram* of the polynomial f is, by definition, $\Delta(f) := \mathrm{convex}(\mathrm{supp} f)$.

For every quasi–convenient polynomial we consider additionally its *Newton diagram at zero*, which is the closure of the set $\Delta_\infty(f) \setminus \Delta(f)$ in the natural topology of the real plane. We denote it by $\Delta_0(f)$.

Obviously $\Delta_\infty(f) = \Delta_0(f) \cup \Delta(f)$. If $(0, 0) \in \mathrm{supp} f$, then $\Delta_\infty(f) = \Delta(f)$ and $\Delta_0(f) = \emptyset$ (see Figure 4).

For any quasi-convenient polynomial f we consider polygons $\Delta_I(f)$ and $\Delta_{II}(f)$ such that the triangle with vertices $(0, 0)$, $(\deg f, 0)$, $(0, \deg f)$ is the union of the sets $\Delta_\infty(f)$, $\Delta_I(f)$ and $\Delta_{II}(f)$, whose interiors are pairwise disjoint and such that $(\deg f, 0) \in \Delta_I(f) \setminus \Delta_\infty(f)$ or $\Delta_I(f) = \emptyset$ and $(0, \deg f) \in \Delta_{II}(f) \setminus \Delta_\infty(f)$ or $\Delta_{II}(f) = \emptyset$ (see Figure 5).

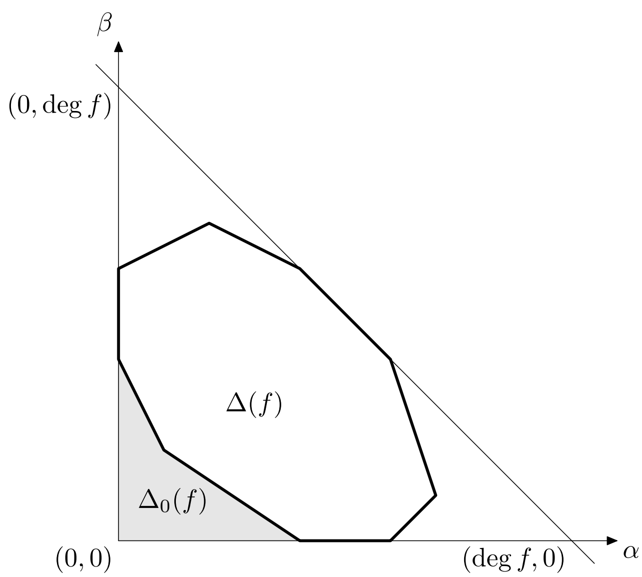

Fig. 4: Newton diagram at zero of f.

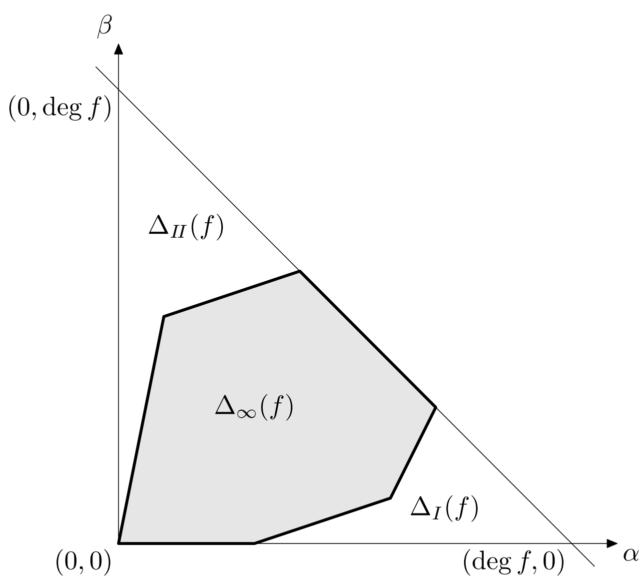

Fig. 5: Polygons $\Delta_I(f)$ and $\Delta_{II}(f)$.

Let $F(X,Y,Z)$ be the *homogenization of the polynomial* $f(X,Y)$ given by the formula $F(X,Y,Z) = Z^m f(X/Z, Y/Z)$, where $m = \deg f$. It is well-known that the projective curve $F(X,Y,Z) = 0$ is *a projective closure* of the affine curve $f(X,Y) = 0$ and it is natural to consider the affine curves $F(1,Y,Z) = 0$ and $F(X,1,Z) = 0$. If the polynomial $f(X,Y)$ is quasi–convenient, then the polynomials $f_1(Y,Z) := F(1,Y,Z)$ and $f_2(X,Z) := F(X,1,Z)$ are also quasi–convenient and the polynomial $F(X,Y,Z)$ is the homogenization of each of them.

After [13, Corollary 2.2, page 183] we have

$$\mathrm{Area}\Delta_I(f) = \mathrm{Area}\Delta_0(f_1) \quad \text{and} \quad \mathrm{Area}\Delta_{II}(f) = \mathrm{Area}\Delta_0(f_2).$$

For a pair of quasi–convenient polynomials $f(X,Y)$, $g(X,Y) \in \mathbf{C}[X,Y]$ we denote $\nu_0(f,g) := \mathrm{Area}\Delta_0(fg) - \mathrm{Area}\Delta_0(f) - \mathrm{Area}\Delta_0(g)$. The pair (f,g) is *nondegenerate at zero* if for any real vector $\vec{w} = [p,q]$ such that $p < 0$ and $q < 0$ the system of equations $\mathrm{in}(f,\vec{w})(X,Y) = \mathrm{in}(g,\vec{w})(X,Y) = 0$ has no solutions in $\mathbf{C}^* \times \mathbf{C}^*$.

Let us recall the estimation of the intersection multiplicity due to Kouchnirenko (see [10, Theorem 1] and [15, Theorem 1.2]). By $(f,g)_0$ we denote the intersection multiplicity of f and g at $O = (0,0) \in \mathbf{C}^2$.

Theorem 4.1 (Kouchnirenko)
If $f(X,Y)$, $g(X,Y)$ are quasi–convenient polynomials then

1° $(f,g)_0 \geq \nu_0(f,g)$,

2° $(f,g)_0 = \nu_0(f,g)$ *if and only if the pair (f,g) is nondegenerate at zero.*

A short and elegant proof of Theorem 4.1 is given in [15], which is based on the Newton–Puisseux theorem. Let us note that the above estimation is interesting only for a pair of quasi-convenient polynomials without constant terms. Indeed, if $f(0,0) \neq 0$ or $g(0,0) \neq 0$, then $(f,g)_0 = \nu_0(f,g) = 0$ and at the same time the condition of nondegeneracy is trivially satisfied.

Apart from Theorem 4.1, in order to prove Theorem 2.2, we need the next lemma, which follows immediately from Bézout theorem for projective curves.

Lemma 4.2 (Bézout)
Let $f(X,Y)$, $g(X,Y) \in \mathbf{C}[X,Y]$ be coprime polynomials of degree m and n, respectively, and let $F(X,Y,Z)$ and $G(X,Y,Z)$ be their homogenizations. If $P_1 = (1{:}0{:}0)$, $P_2 = (0{:}1{:}0) \in \mathbf{P^2}(\mathbf{C})$, where $\mathbf{P^2}(\mathbf{C})$ is the projective plane, then

1° $\sum_{P \in \mathbf{C}^2} (f,g)_P \leq mn - (F,G)_{P_1} - (F,G)_{P_2}$, *where $(F,G)_{P_1}$ and $(F,G)_{P_2}$ denote the intersection multiplicity of the projective curves $F(X,Y,Z) = 0$ and $G(X,Y,Z) = 0$ at the points P_1 and P_2, respectively,*

2° $\sum_{P \in \mathbf{C}^2} (f,g)_P = mn - (F,G)_{P_1} - (F,G)_{P_2}$ *if and only if the projective curves $F(X,Y,Z) = 0$ and $G(X,Y,Z) = 0$ intersect simultaneously the line at infinity $L_\infty = \{(x{:}y{:}z) \in \mathbf{P^2}(\mathbf{C}) : \ z = 0\}$ at most at the points P_1 and P_2.*

Proof of Theorem 2.2. Let $f(X,Y)$, $g(X,Y) \in \mathbf{C}[X,Y]$ be quasi-convenient polynomials of positive degrees m and n, respectively.

We may write

$$\begin{aligned}\frac{m^2}{2} &= \mathrm{Area}\Delta_\infty(f) + \mathrm{Area}\Delta_I(f) + \mathrm{Area}\Delta_{II}(f)\\ &= \mathrm{Area}\Delta_\infty(f) + \mathrm{Area}\Delta_0(f_1) + \mathrm{Area}\Delta_0(f_2),\end{aligned}$$

$$\begin{aligned}\frac{n^2}{2} &= \mathrm{Area}\Delta_\infty(g) + \mathrm{Area}\Delta_I(g) + \mathrm{Area}\Delta_{II}(g)\\ &= \mathrm{Area}\Delta_\infty(g) + \mathrm{Area}\Delta_0(g_1) + \mathrm{Area}\Delta_0(g_2)\end{aligned}$$

and

$$\begin{aligned}\frac{(m+n)^2}{2} &= \mathrm{Area}\Delta_\infty(fg) + \mathrm{Area}\Delta_I(fg) + \mathrm{Area}\Delta_{II}(fg)\\ &= \mathrm{Area}\Delta_\infty(fg) + \mathrm{Area}\Delta_0(f_1g_1) + \mathrm{Area}\Delta_0(f_2g_2).\end{aligned}$$

Hence

$$\nu_\infty(f,g) = mn - \nu_0(f_1,g_1) - \nu_0(f_2,g_2). \tag{1}$$

Since $(F,G)_{P_1} = (f_1,g_1)_0$ and $(F,G)_{P_2} = (f_2,g_2)_0$, using the estimation of intersection multiplicity (Theorem 4.1, 1°) we state that $(F,G)_{P_1} = (f_1,g_1)_0 \geq \nu_0(f_1,g_1)$ and $(F,G)_{P_2} = (f_2,g_2)_0 \geq \nu_0(f_2,g_2)$. By Bézout Lemma (Lemma 4.2, 1°) and the equality (1) we conclude that

$$\sum_{P\in\mathbf{C}^2} (f,g)_P \leq mn - \nu_0(f_1,g_1) - \nu_0(f_2,g_2) = \nu_\infty(f,g),$$

provided that the polynomials f and g are coprime. So, we proved estimation 1° of Theorem 2.2. To prove condition 2° let us note that for any real vector $\vec{w} = [p,q] \neq \vec{0}$ we have

$$\mathrm{in}(f_1,\vec{u})(Y,Z) = Z^m \mathrm{in}(f,\vec{w})\Big(\frac{1}{Z},\frac{Y}{Z}\Big)$$

and

$$\mathrm{in}(g_1,\vec{u})(Y,Z) = Z^n \mathrm{in}(g,\vec{w})\Big(\frac{1}{Z},\frac{Y}{Z}\Big),$$

where $\vec{u} = [q-p,-p]$. Hence, the pair (f_1,g_1) is nondegenerate at zero if and only if for any real vector $\vec{w} = [p,q]$ such that $p > 0$ and $p > q$, the system of equations $\mathrm{in}(f,\vec{w})(X,Y) = \mathrm{in}(g,\vec{w})(X,Y) = 0$ has no solutions in $\mathbf{C}^* \times \mathbf{C}^*$. Similarly, we have

$$\mathrm{in}(f_2,\vec{v})(X,Z) = Z^m \mathrm{in}(f,\vec{w})\Big(\frac{X}{Z},\frac{1}{Z}\Big)$$

and

$$\mathrm{in}(g_2,\vec{v})(X,Z) = Z^n \mathrm{in}(g,\vec{w})\Big(\frac{X}{Z},\frac{1}{Z}\Big),$$

where $\vec{v} = [p-q, -q]$. Hence, the pair (f_2, g_2) is nondegenerate at zero if and only if for any real vector $\vec{w} = [p, q]$ such that $q > 0$ and $q > p$, the system of equations $\mathrm{in}(f, \vec{w})(X, Y) = \mathrm{in}(g, \vec{w})(X, Y) = 0$ has no solutions in $\mathbf{C}^* \times \mathbf{C}^*$.

Moreover, let us note that if $\vec{w} = [p, q]$ and $p = q > 0$ then the sytem of equations $\mathrm{in}(f, \vec{w})(X, Y) = \mathrm{in}(g, \vec{w})(X, Y) = 0$ has no solutions in $\mathbf{C}^* \times \mathbf{C}^*$ if and only if the projective curves $\{F(X, Y, Z) = 0\}$ and $\{G(X, Y, Z) = 0\}$ intersect simultaneously the line at infinity L_∞ at most at the points P_1 and P_2.

Therefore, one can see that the pair (f, g) is nondegenerate at infinity if and only if both pairs (f_1, g_1), (f_2, g_2) are nondegenerate at zero and it holds the inclusion $\{F(X, Y, Z) = 0\} \cap \{G(X, Y, Z) = 0\} \cap L_\infty \subset \{P_1, P_2\}$.

Note that if the pair (f, g) is nondegenerate at infinity then f and g are coprime. Since $(F, G)_{P_1} = (f_1, g_1)_0$ and $(F, G)_{P_2} = (f_2, g_2)_0$, to finish the proof of 2° of Theorem 2.2, it is enough to apply 2° of Theorem 4.1 to the pairs (f_1, g_1) and (f_2, g_2) and to use Bézout Lemma (Lemma 4.2, 2°) and the equality (1).

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