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Local convergence comparison between two novel sixth order methods for solving equations

Abstract. The aim of this article is to provide the local convergence analysis of two novel competing sixth convergence order methods for solving equations involving Banach space valued operators. Earlier studies have used hypotheses reaching up to the sixth derivative but only the first derivative appears in these methods. These hypotheses limit the applicability of the methods. That is why we are motivated to present convergence analysis based only on the first derivative. Numerical examples where the convergence criteria are tested are provided. It turns out that in these examples the criteria in the earlier works are not satisfied, so these results cannot be used to solve equations but our results can be used.

1. Introduction

Let $\mathcal{B}_1$, $\mathcal{B}_2$ be Banach spaces and Ω be a convex subset of $\mathcal{B}_1$. Using Mathematical Modeling, numerous problems in Computational Sciences and also in Engineering, Mathematical Biology, Mathematical Economics and other disciplines can be written in the form of equation

$$F(x) = 0, \tag{1}$$

where $F\colon \Omega \subseteq \mathcal{B}_1 \to \mathcal{B}_2$ is a differentiable operator in the sense of Fréchet. The solutions of such equations cannot be found in closed form, in general. So, most of the solution methods for such equations are usually iterative.

Recently numerous researchers have presented fast convergence methods using only the first derivative or divided differences of order one [1, 2, 3, 4, 5, 6, 7, 8, 9, 10,

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12, 15, 13, 11, 14, 16, 17, 18, 19, 20, 21] in the method but not in the assumptions of the convergence, where much higher than order one derivatives are used. We consider a sixth order Jarratt-like method [12, 15, 20] for approximating a solution x^* of (1). Earlier studies of such methods make assumptions on the derivatives of F of order up to six although the method involves only the Fréchet derivative of order one. However, these methods are important for faster convergence, especially in cases of stiff systems of equations. So it is important to obtain the convergence of these methods using assumption only on the first order derivative of F.

In this article, we present the local convergence analysis of two competing sixth order methods by Wang [20] and Madhu [15], defined, respectively for each $n = 0, 1, 2, \dots$ by

$$\begin{aligned} y_n &= x_n - \alpha F'(x_n)^{-1}F(x_n),\\ z_n &= x_n - \frac{1}{2}(3F'(y_n) - F'(x_n))^{-1}(3F'(y_n) + F'(x_n))F'(x_n)^{-1}F(x_n),\\ x_{n+1} &= z_n - (\frac{3}{2}F'(y_n)^{-1} - \frac{1}{2}F'(x_n))^{-1})F(z_n) \end{aligned} \tag{2}$$

and

$$\begin{aligned} y_n &= x_n - \alpha F'(x_n)^{-1}F(x_n),\\ z_n &= y_n - \alpha_n F'(x_n)^{-1}F(y_n),\\ x_{n+1} &= z_n - \alpha_n F'(x_n))^{-1})F(z_n), \end{aligned} \tag{3}$$

where $x_0 \in \Omega$ is an initial point, $\alpha \in S$, $S = \mathbb{R}$ or $\mathbb{C}$, and $\alpha_n = 2I - F'(x_n)^{-1}F'(y_n)$. The sixth order of convergence was shown using hypothesis reaching up to sixth derivative of F and Taylor expansions in the special case when $\mathcal{B}_1 = \mathcal{B}_2 = \mathbb{R}^j$. These hypotheses limit the applicability of methods (2) and (3). As a motivational and academic example, define function F on $\Omega = [-\frac{1}{2}, \frac{5}{2}]$ by

$$F(x) = \begin{cases} x^3 \ln x^2 + x^5 - x^4, & x \neq 0,\\ 0, & x = 0. \end{cases}$$

We have that $x^* = 1$,

$$\begin{aligned} F'(x) &= 3x^2 \ln x^2 + 5x^4 - 4x^3 + 2x^2,\\ F''(x) &= 6x \ln x^2 + 20x^3 - 12x^2 + 10x \end{aligned}$$

and

$$F'''(x) = 6 \ln x^2 + 60x^2 - 24x + 22.$$

Function $F'''(x)$ is unbounded on Ω. Hence, the results in [12, 15, 20] cannot be applied to solve equation (1). We provide a local convergence analysis using only hypotheses on the first Fréchet-derivative. This way we expand the applicability of these methods (2) and (3). Moreover, we provide computable convergence radii, error bounds on the distances $\|x_n - x^*\|$ and uniqueness results based on Lipschitz-type functions. Such results were not given in [15] and [20]. Furthermore, we use the computational order of convergence (COC) and the approximate computational order of convergence (ACOC)(which do not depend on higher than one Fréchet-derivative) to determine the order of convergence of methods (2) and (3). Local results are important because they provide the degree of difficulty for choosing initial points. Our idea can be used on other iterative methods.

The rest of the paper is structured as follows: Section 2 contains the local convergence analysis of method (2). The numerical examples are presented in the concluding Section 3.

2. Local Convergence Analysis I

The local convergence analysis of method (2) is based on some scalar functions and parameters. Let w_0 be a strictly continuous, increasing function defined on the interval $[0, +\infty)$ with values in $[0, +\infty)$ and satisfying $w_0(0) = 0$ and $\alpha \in S$. Suppose that equation

$$w_0(t) = 1 \tag{4}$$

has at least one positive solution. Denote by ρ the smallest such solution of (4).

Let also w, v be strictly continuous, increasing functions defined on the interval $[0, \rho)$ with values in $[0, +\infty)$ and satisfying $w(0) = 0$. Define functions g_1 and h_1 on the interval $[0, \rho)$ by

$$\begin{aligned} g_1(t) &= \frac{\int_0^1 w((1-\theta)t)d\theta + |1-\alpha| \int_0^1 v(\theta t)d\theta}{1 - w_0(t)}, \\ h_1(t) &= g_1(t) - 1. \end{aligned}$$

Suppose that

$$|1-\alpha| v(0) < 1. \tag{5}$$

We have by the definition of the scalar functions, ρ and (5) that $h_1(0) = |1 - \alpha| v(0) - 1 < 0$ and $h_1(t) \to +\infty$ as $t \to \rho^-$. By applying the intermediate value theorem on function h_1, we deduce that the equation $h_1(t) = 0$ has at least one solution in $(0, \rho)$. Denote by ρ_1 the smallest such solution.

Define functions p and q on $[0, \rho)$ by

$$\begin{aligned} p(t) &= \frac{1}{2}(3w_0(g_1(t)t) + w_0(t)), \\ q(t) &= p(t) - 1. \end{aligned}$$

We get $q(0) = -1$ and $q(t) \to +\infty$ as $t \to \rho^-$. Denote by ρ_q the smallest solution of equation $q(t) = 0$ in $(0, \rho)$. Set $\bar{\rho} = \min\{\rho, \rho_q\}$. Moreover, define functions g_2 and h_2 on the interval $[0, \bar{\rho})$ by

$$\begin{aligned} g_2(t) &= \frac{\int_0^1 w((1-\theta)t)d\theta}{1 - w_0(t)} + \frac{3}{4} \frac{(w_0(t) + w_0(g_1(t)t)) \int_0^1 v(\theta t)d\theta}{(1 - w_0(t))(1 - p(t))}, \\ h_2(t) &= g_2(t) - 1. \end{aligned}$$

We get again $h_2(0) = -1$ and $h_2(t) \to$ a positive number or $+\infty$ as $t \to \bar{\rho}^-$. Denote by ρ_1 the smallest solution of equation $h_2(t) = 0$ in $(0, \bar{\rho})$. Furthermore, define functions g_3 and h_3 on the interval $[0, \bar{\rho_1})$ by

$$
\begin{aligned}
g_3(t) = \Bigg[& \frac{\int_0^1 w((1-\theta)g_2(t)t)d\theta}{1 - w_0(g_2(t)t)} + \frac{(w_0(g_1(t)t) + w_0(g_2(t)t))}{(1 - w_0(g_1(t)t)(1 - w_0(g_2(t)t))} \\
& + \frac{w_0(t) + w_0(g_1(t)t)}{(1 - w_0(t))(1 - w_0(g_1(t)t)}\Bigg] \times \int_0^1 v(\theta g_2(t)t)d\theta g_2(t), \\
h_3(t) = {} & g_3(t) - 1.
\end{aligned}
$$

Suppose $\bar{\rho}_1$ and $\bar{\rho}_2$ are the smallest positive solutions of

$$g_1(t)t = 1 \tag{6}$$

and

$$g_2(t)t = 1, \tag{7}$$

respectively. Set $\bar{\bar{\rho}} = \min\{\bar{\rho}_1, \bar{\rho}_2\}$. We get that $h_3(0) = -1$ and $h_3(t) \to +\infty$ as $t \to \bar{\bar{\rho}}^-$. Denote by ρ_3 the smallest solution of equation $h_3(t) = 0$ in $(0, \bar{\bar{\rho}})$. Define the radius of convergence ρ^* by

$$\rho^* = \min\{\rho_i\}, \qquad i = 1, 2, 3. \tag{8}$$

Then, we have that for each $t \in [0, \rho^*)$,

$$0 \le g_i(t) < 1 \quad \text{and} \qquad 0 \le p(t) < 1. \tag{9}$$

Let $U(u, \varepsilon) = \{x \in \mathcal{B}_1 : \ \|x - u\| < \varepsilon\}$ for $u \in \mathcal{B}_1$ and $\varepsilon > 0$. Let also $\overline{U}(u, \varepsilon)$, stand, for its closure.

Next, we present the local convergence analysis of method (2) using the preceding notation. The proof follows in an analogous way as the corresponding ones in [5, 7] (see also [4, 8, 15, 14, 16, 20]).

Theorem 2.1
Let $F\colon \Omega \subset \mathcal{B}_1 \to \mathcal{B}_2$ be a continuously differentiable operator in the sense of Fréchet. Suppose:
there exists $x^* \in \Omega$, such that

$$F(x^*) = 0, \qquad F'(x^*)^{-1} \in \mathcal{L}(\mathcal{B}_2, \mathcal{B}_1);$$

there exists function $w_0\colon [0, +\infty) \to [0, +\infty)$ strictly continuous and increasing with $w_0(0) = 0$ such that for each $x \in \Omega$,

$$\|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \le w_0(\|x - x^*\|). \tag{10}$$

Set $\Omega_0 = \Omega \cap U(x^*, \rho^*)$, where ρ^* is defined previously. There exist $w, v\colon [0, +\infty) \to [0, +\infty)$ strictly continuous, increasing functions satisfying $w(0) = 0$ such that for each $x, y \in \Omega_0$, (4), (6), (7), (9),

$$\|F'(x^*)^{-1}(F'(x) - F'(y))\| \le w(\|x - y\|), \tag{11}$$

$$\|F'(x^*)^{-1}F'(x)\| \leq v(\|x - x^*\|) \tag{12}$$

and

$$\bar{U}(x^*, r) \subseteq \Omega \tag{13}$$

hold. Then, the sequence $\{x_n\}$ *generated for* $x_0 \in U(x^*, \rho) - \{x^*\}$ *by method* (2) *is well defined in* $U(x^*, \rho^*)$*, remains in* $U(x^*, \rho^*)$ *and converges to* x^* *so that for each* $n = 0, 1, 2, \ldots,$

$$\|y_n - x^*\| \leq g_1(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\| < \rho^*, \tag{14}$$

$$\|z_n - x^*\| \leq g_2(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\|, \tag{15}$$

and

$$\|x_{n+1} - x^*\| \leq g_3(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\|, \tag{16}$$

where functions g_i*,* $i = 1, 2, 3$ *are given previously. Moreover, if there is some* $R \geq \rho^*$ *such that*

$$\int_0^1 w_0(\theta R)d\theta < 1, \tag{17}$$

then the limit point x^* *is the only solution of equation* $F(x) = 0$ *in* $\Omega \cap \bar{U}(x^*, R)$*.*

Proof. We shall show using induction that sequence $\{x_k\}$ is well defined, remains in $U(x^*, \rho^*)$ and converges to x^* so that the estimates (14)–(16) hold. First, we show that y_0 is well defined and (17) holds for $n = 0$. To do this, by condition (10) and $x \in U(x^*, \rho^*)$, we have in turn that

$$\|F'(x^*)^{-1}(F'(x) - F'(x^*)\| \leq w_0(\|x - x^*\|) \leq w_0(\rho) < 1. \tag{18}$$

It follows from (18) and the Banach perturbation Lemma (see for example [2, 3, 13, 16]) that $F'(x)^{-1} \in \mathcal{L}(\mathcal{B}_2, \mathcal{B}_1)$ and

$$\|F'(x_0)^{-1}F'(x^*)\| \leq \frac{1}{1 - w_0(\|x - x^*\|)}. \tag{19}$$

In particular, y_0 exists by the first substep of method (2) and (19) for $x = x_0$ (since $x_0 \in U(x^*, \rho^*)$). Using the first substep of method (2), we obtain in turn that (12), (9) (for $i = 1$) and (19),

$$\begin{aligned} y_0 - x^* &= x_0 - x^* - F'(x_0)^{-1}F(x_0) + (1 - \alpha)F'(x_0)^{-1}F(x_0) \\ &= \int_0^1 F'(x_0)^{-1}F'(x^*)F'(x^*)^{-1}[F'(x^* + \theta(x_0 - x^*)) \\ &\qquad\qquad - F'(x_0)](x_0 - x^*)d\theta \\ &\quad + (1 - \alpha)F'(x_0)^{-1}F'(x^*)F'(x^*)^{-1}F(x_0), \end{aligned} \tag{20}$$

so

$$
\begin{aligned}
\|y_0 - x^*\| &\leq \|F'(x_0)^{-1}F'(x^*)\| \\
&\quad \times \left\|F'(x^*)^{-1}\int_0^1 [F'(x^* + \theta(x_0 - x^*)) - F'(x_0)](x_0 - x^*)d\theta\right\| \\
&\quad + |1-\alpha|\|F'(x_0)^{-1}F'(x^*)\|\|F'(x^*)^{-1}F(x_0)\| \\
&\leq \frac{\displaystyle\int_0^1 w((1-\theta)\|x_0 - x^*\|)d\theta\|x_0 - x^*\| + \int_0^1 v(\theta\|x_0 - x^*\|)d\theta}{1 - w_0(\|x_0 - x^*\|)} \qquad (21)\\
&\quad \times \|x_0 - x^*\| \\
&= g_1(\|x_0 - x^*\|)\|x_0 - x^*\| \leq \|x_0 - x^*\| < \rho^*,
\end{aligned}
$$

thus (14) holds for $n = 0$ and $y_0 \in U(x^*, \rho^*)$, where we also used the estimate

$$
\begin{aligned}
\|F'(x^*)^{-1}F(x_0)\| &= \|F'(x^*)^{-1}(F(x_0) - F(x^*))\| \\
&= \left\| \int_0^1 F'(x^*)^{-1}F'(x^* + \theta(x_0 - x^*))d\theta(x_0 - x^*)\right\| \qquad (22)\\
&\leq \int_0^1 v(\theta\|x_0 - x^*\|)d\theta\|x_0 - x^*\|,
\end{aligned}
$$

since $\|x^* + \theta(x_0 - x^*) - x^*\| = \theta\|x_0 - x^*\| \leq \rho^*$ (i.e. $x^* + \theta(x_0 - x^*) \in U(x^*, \rho^*)$ for each $\theta \in [0, 1]$).

Secondly, we show that z_0 is well defined and (18) holds for $n = 0$. To achieve this by the second substep of method (2) for $n = 0$, (11), (9) (for $i = 2$) and (22), we get in turn

$$
\begin{aligned}
z_0 - x^* &= x_0 - x^* - F'(x_0)^{-1}F(x_0) \\
&\quad + \Big(I - \frac{1}{2}(3F'(y_0) - F'(x_0))^{-1}(3F'(y_0) + F'(x_0))\Big) \\
&\quad \times F'(x_0)^{-1}F(x_0) \qquad (23)\\
&= x_0 - x^* - F'(x_0)^{-1}F(x_0) \\
&\quad + \frac{3}{2}(3F'(y_0) - F'(x_0))^{-1}(F'(y_0) - F'(x_0))F'(x_0)^{-1}F(x_0),
\end{aligned}
$$

so by (2) for $n = 0$, (9) (for $i = 2$), (20) and (23) get in turn that

$$
\begin{aligned}
\|z_0 - x^*\| &\leq g_1(\|x_0 - x^*\|)\|x_0 - x^*\| \\
&\quad + \frac{3}{4}\frac{(w_0(\|y_0 - x^*\|) + w_0(\|x_0 - x^*\|))}{(1 - p(\|x_0 - x^*\|))(1 - w_0(\|x_0 - x^*\|))} \\
&\quad \times \int_0^1 v(\theta\|x_0 - x^*\|)d\theta\|x_0 - x^*\| \\
&\leq g_2(\|x_0 - x^*\|)\|x_0 - x^*\| \leq \|x_0 - x^*\| < \rho^*,
\end{aligned}
$$

which shows (15) for $n = 0$ and $z_0 \in U(x^*, \rho^*)$, where we also used the estimate

$$\begin{aligned}
\|(2F'(x^*))^{-1}(3(F'(y_0)-F'(x^*))-(F'(x_0)-F'(x^*))\| \\
&\le \frac{1}{2}[3w_0(\|y_0-x^*\|)+w_0(\|x_0-x^*\|)] \\
&\le p(\|x_0-x^*\|)<p(\rho^*)<1,
\end{aligned}$$

so

$$\|(3F'(y_0)-F'(x_0))^{-1}F'(x^*)\| \le \frac{1}{2(1-p(\|x_0-x^*\|))}. \tag{24}$$

Thirdly, we show that x_1 is well defined and (19) hold for $n=0$. By the third substep of method (2), (24), (9) (for $i=3$), (19), (24) and (17) we get in turn that

$$\begin{aligned}
x_1-x^* &= z_0-x^*-F'(z_0)^{-1}F(z_0)+F'(z_0)^{-1}F(z_0) \\
&\quad -\Big[\frac{3}{2}F'(y_0)^{-1}-\frac{1}{2}F'(x_0)^{-1}\Big]F(z_0) \\
&= z_0-x^*-F'(z_0)^{-1}F(z_0) \\
&\quad +F'(z_0)^{-1}(F'(y_0)-F'(z_0))F'(y_0)^{-1}F(z_0) \\
&\quad +\frac{1}{2}F'(x_0)^{-1}(F'(y_0)-F'(x_0))F'(y_0)^{-1}F(z_0),
\end{aligned}$$

so

$$\begin{aligned}
\|x_1-x^*\| &\le \frac{\int_0^1 w((1-\theta)\|z_0-x^*\|)d\theta\|z_0-x^*\|}{1-w_0(\|z_0-x^*\|)} \\
&\quad +\frac{(w_0(\|z_0-x^*\|)+w_0(\|y_0-x^*\|)}{(1-w_0(\|z_0-x^*\|))(1-w_0(\|y_0-x^*\|))} \\
&\quad \times \int_0^1 v(\theta\|z_0-x^*\|)d\theta\|z_0-x^*\| \\
&\quad +\frac{1}{2}\frac{(w_0(\|x_0-x^*\|)+w_0(\|y_0-x^*\|))}{(1-w_0(\|x_0-x^*\|))(1-w_0(\|y_0-x^*\|))} \\
&\quad \times \int_0^1 v(\theta\|z_0-x^*\|)d\theta\|z_0-x^*\| \\
&\le g_3(\|x_0-x^*\|)\|x_0-x^*\| \le \|x_0-x^*\| < \rho^*,
\end{aligned}$$

which shows (16) for $n=0$ and $x_1 \in U(x^*,\rho^*)$. Then, substitute x_0, y_0, z_0 and x_1 by x_k, y_k, z_k and x_{k+1}, resp., in the preceding estimates, to complete the induction for (14)–(16). Then, in view of the estimate

$$\|x_{k+1}-x^*\| \le c\|x_k-x^*\| < \rho^*, \tag{25}$$

where $c=g_3(\|x_0-x^*\|)\in[0,1)$, we deduce that $\lim_{k\to\infty} x_k = x^*$ and $x_{k+1}\in U(x^*,\rho^*)$.

Finally, we show the uniqueness of the solution result. Let $y^* \in \Omega \cap \bar{U}(x^*, R)$ be such that $F(y^*) = 0$. Set

$$T = \int_0^1 F'(x^* + \theta(y^* - x^*))d\theta.$$

Then, using (10) and (17), we get that

$$\begin{aligned}\|F'(x^*)^{-1}(T - F'(x^*))\| &\leq \int_0^1 w_0(\theta\|x^* - y^*\|)d\theta \\ &\leq \int_0^1 w_0(\theta R)d\theta < 1,\end{aligned}$$

so, $T^{-1} \in \mathcal{L}(\mathcal{B}_2, \mathcal{B}_1)$. Then, from the identity $0 = F(y^*) - F(x^*) = T(y^* - x^*)$, we conclude that $x^* = y^*$ completing the uniqueness of the solution part and the proof of the theorem.

Remark 2.2

(a) Let $w_0(t) = L_0 t$, $w(t) = Lt$. The radius $\tilde{\rho}_1 = \frac{2}{2L_0+L}$ was obtained by Argyros as the convergence radius for Newton's method under condition (10)–(12). Notice that the convergence radius for Newton's method given independently by Rheinboldt [18] and Traub [19] is given by

$$\tilde{\rho} = \frac{2}{3L} < \tilde{\rho}_1.$$

Let $f(x) = e^x - 1$. Then $x^* = 0$. Set $\Omega = U(0, 1)$. Then, we have that $L_0 = e - 1 < L = e^{\frac{1}{L_0}}$, so $\tilde{\rho} = 0.24252961 < \tilde{\rho}_1 = 0.3827$.

Moreover, the new error bounds [2, 3, 4, 5, 6] are

$$\|x_{n+1} - x^*\| \leq \frac{L}{1 - L_0\|x_n - x^*\|}\|x_n - x^*\|^2,$$

whereas the old ones [18, 19]

$$\|x_{n+1} - x^*\| \leq \frac{L}{1 - L\|x_n - x^*\|}\|x_n - x^*\|^2.$$

Clearly, the new error bounds are more precise, if $L_0 < L$. Clearly, the radius of convergence of method (2) given by ρ^* is smaller than $\tilde{\rho}_1$.

(b) Method (2) stays the same if we use the new instead of the old conditions [12, 15, 20]. We can use the computational order of convergence (COC)

$$\xi = \frac{\ln \frac{\|x_{n+2} - x_{n+1}\|}{\|x_{n+1} - x_n\|}}{\ln \frac{\|x_{n+1} - x_n\|}{\|x_n - x_{n-1}\|}} \qquad \text{for each } n = 1, 2, \ldots$$

or the approximate computational order of convergence (ACOC)

$$\xi^* = \frac{\ln \frac{\|x_{n+2} - x^*\|}{\|x_{n+1} - x^*\|}}{\ln \frac{\|x_{n+1} - x^*\|}{\|x_n - x^*\|}} \qquad \text{for each } n = 0, 1, 2, \ldots.$$

(c) Using (10) and

$$\begin{aligned}\|F'(x^*)^{-1}F'(x)\| &= \|F'(x^*)^{-1}(F'(x)-F'(x^*))+I\|\\ &\le 1+\|F'(x^*)^{-1}(F'(x)-F'(x^*))\|\\ &\le 1+q_0(\|x-x^*\|)\end{aligned}$$

condition (13) can be replaced by

$$w(t)=1+w_0(t)$$

or

$$w=1+w_0(\rho^*).$$

(d) If we restrict method (2) to the first two substeps and replace z_n by x_{n+1}, then we obtain the results for the Jarratt method [12].

3. Local Convergence Analysis-II

The local convergence analysis of method (3) is given in an analogous way to method (2). Let w_0, w, v, ρ and R be as in Section 2. Define functions G_1, H_1, G_2 and H_2 on the interval $[0,\rho)$ by

$$\begin{aligned}G_1(t) &= \frac{\displaystyle\int_0^1 w((1-\theta)t)d\theta}{1-w_0(t)},\\ H_1(t) &= G_1(t)-1,\\ G_2(t) &= G_1(t)+\frac{\displaystyle\int_0^1 v(\theta t)d\theta+\int_0^1 v(\theta G_1(t)t)d\theta+\varphi(t)}{1-w_0(t)},\\ H_2(t) &= G_2(t)-1,\end{aligned}$$

where

$$\varphi(t)=\frac{(w_0(G_1(t)t)+w_0(t))\displaystyle\int_0^1 v(\theta G_1(t)t)d\theta G_1(t)}{1-w_0(t)}.$$

We have $H_1(0)=H_2(0)=-1$ and $H_1(t)\to+\infty$ as $t\to\rho^-$, $H_2(t)\to$ a positive constant or $+\infty$ as $t\to\rho^-$. Denote by r_1,r_2 the smallest solutions of equations $H_1(t)=0$ and $H_2(t)=0$, respectively. Suppose that equation

$$G_1(t)t=1$$

has at least one positive positive solution. Denote by $\tilde{\tilde{\rho}}$ the smallest such solution. Moreover, define functions G_3 and H_3 on the interval $[0,\tilde{\tilde{\rho}})$ by

$$G_3(t) = \frac{\int_0^1 w((1-\theta)G_2(t)t)d\theta G_2(t)}{1-w_0(G_2(t)t)}$$
$$+ \frac{(w_0(t)+w_0(G_2(t)t))\int_0^1 v(\theta G_2(t)t)d\theta G_2(t)}{(1-w_0(G_2(t)t))(1-w_0(t))}$$
$$+ \frac{(w_0(G_1(t)t)+w_0(t))\int_0^1 v(\theta G_2(t)t)d\theta G_2(t)}{(1-w_0(t))^2}$$

and

$$H_3(t) = G_3(t) - 1.$$

We get $H_3(0) = -1$ and $H_3(t) \to +\infty$ as $t \to \tilde{\tilde{\rho}}^-$. Denote by r_3 the smallest solution of equation $H_3(t) = 0$ in $(0, \tilde{\tilde{\rho}})$. Further, define the radius of convergence r by

$$r^* = \min\{r_i\}, \qquad i = 1, 2, 3. \tag{26}$$

Then, for each $t \in [0, r^*)$,

$$0 \leq G_i(t) < 1 \tag{27}$$

and

$$0 \leq w_0(G_2(t)t) < 1.$$

Theorem 3.1
Suppose that hypotheses of Theorem 2.1 but with ρ^*, replaced by r^* given by (26) *and R by R^*. Then, the conclusions of Theorem 2.1 hold but with method* (2) *replaced by method* (3).

Proof. As in Theorem 2.1 but for method (3), we obtain in turn from the three substeps of method (3) that

$$\|y_n - x^*\| \leq G_1(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\| < r^*$$

and

$$\begin{aligned}
z_n - x^* &= x_n - x^* - (2I - F'(x_n)^{-1}F'(y_n))F'(x_n)^{-1}F(y_n)\\
&= x_n - x^* - F'(x_n)^{-1}F(x_n) + F'(x_n)^{-1}(F(x_n) - F(y_n))\\
&\quad + F'(x_n)^{-1}[F'(y_n)F'(x_n)^{-1} - F'(x_n)^{-1}]F(y_n)\\
&= y_n - x^* + F'(x_n)^{-1}[(F(x_n) - F(y_n))\\
&\quad + (F'(y_n) - F'(x_n))F'(x_n)^{-1}F(y_n)],
\end{aligned}$$

so

$$\|z_n - x^*\| \leq \|y_n - x^*\| + \frac{\int_0^1 v(\theta\|x_n - x^*\|)d\theta\|x_n - x^*\| + \int_0^1 v(\theta\|y_n - x^*\|)d\theta\|y_n - x^*\|}{1 - w_0(\|x_n - x^*\|)}$$

$$+\frac{\psi(\|x_n-x^*\|,\|y_n-x^*\|)}{1-w_0(\|x_n-x^*\|)}$$
$$\le G_2(\|x_n-x^*\|)\|x_n-x^*\|$$
$$\le \|x_n-x^*\|,$$

where

$$\psi(s,t)=\frac{(w_0(t)+w_0(s))\int_0^1 v(\theta t)d\theta t}{1-w_0(t)}$$

and from

$$\begin{aligned}x_{n+1}-x^* &= z_n-x^*-F'(z_n)^{-1}F(z_n)\\ &\quad+F'(z_n)^{-1}F(z_n)-\alpha_nF'(x_n)^{-1}F(z_n)\\ &= z_n-x^*-F'(z_n)^{-1}F(z_n)\\ &\quad+F'(z_n)^{-1}(F'(x_n)-F'(z_n))F'(x_n)^{-1}F(z_n)\\ &\quad+(F'(x_n)^{-1})^2(F'(y_n)-F'(x_n))F(z_n),\end{aligned}$$

leading to

$$\begin{aligned}\|x_{n+1}-x^*\| &\le \frac{\int_0^1 w((1-\theta)\|z_n-x^*\|)d\theta\|z_n-x^*\|}{1-w_0(\|z_n-x^*\|)}\\ &\quad+\frac{(w_0(\|x_n-x^*\|)+w_0(\|z_n-x^*\|))\int_0^1 v(\theta\|z_n-x^*\|)d\theta\|z_n-x^*\|}{(1-w_0(\|x_n-x^*\|)(1-w_0(\|z_n-x^*\|))}\\ &\quad+\frac{(w_0(\|y_n-x^*\|)+w_0(\|x_n-x^*\|))\int_0^1 v(\theta\|z_n-x^*\|)d\theta\|z_n-x^*\|}{1-w_0(\|x_n-x^*\|)}\\ &\le G_3(\|x_n-x^*\|)\|x_n-x^*\|)\le\|x_n-x^*\|.\end{aligned}$$

4. Numerical Examples

The numerical examples are presented in this section.

Example 4.1
Let $\mathcal{B}_1=\mathcal{B}_2=\mathbb{R}^3$, $\Omega=\bar{U}(0,1)$, $x^*=(0,0,0)^T$. Define function F on Ω for $u=(x,y,z)^T$ by

$$F(u)=(e^x-1,\frac{e-1}{2}y^2+y,z)^T.$$

Then, the Fréchet-derivative is given by

$$F'(v)=\begin{bmatrix}e^x & 0 & 0\\ 0 & (e-1)y+1 & 0\\ 0 & 0 & 1\end{bmatrix}.$$

Notice that using the (12)-(16), conditions, we get $w_0(t) = (e-1)t$, $w(t) = e^{\frac{1}{e-1}}t$ and $v(t) = e^{\frac{1}{e-1}}$.

Then using the definition of ρ^* and r^*, we have that (see also (8) and (27)),

$$\rho^* = 0.14444885915244823348935199192056,$$
$$r^* = 0.041513536254307446815570159515119.$$

Example 4.2
Let $\mathcal{B}_1 = \mathcal{B}_2 = C[0,1]$, the space of continuous functions defined on $[0,1]$ and be equipped with the max norm. Let $\Omega = \overline{U}(0,1)$. Define function F on Ω by

$$F(\varphi)(x) = \varphi(x) - 5\int_0^1 x\theta\varphi(\theta)^3 d\theta.$$

We have that

$$F'(\varphi(\xi))(x) = \xi(x) - 15\int_0^1 x\theta\varphi(\theta)^2\xi(\theta)d\theta \qquad \text{for each } \xi \in \Omega.$$

Then, we get that $x^* = 0$, $w_0(t) = 7.5t$, $w(t) = 15t$ and $v(t) = 15$. This way, we have that

$$\rho^* = 0.0029787165027481215216764720565834,$$
$$r^* = 0.00025772162389070053020029282819792.$$

Example 4.3
Let us return back to the motivational example. Then, we get that $w_0(t) = w(t) = 147t$ and $v(t) = 147$. So, we obtain

$$\rho^* = 0.0000022805995203038406502924970165\,04,$$
$$r^* = 0.00000015632156094549084653745141\,60.$$

Example 4.4
Let $\mathcal{B}_1 = \mathcal{B}_2 = C[0,1], \Omega = \bar{U}(x^*,1)$ and consider the nonlinear integral equation of the mixed Hammerstein-type [1, 2, 7, 9, 10, 12, 16] defined by

$$x(s) = \int_0^1 G(s,t)\Big(x(t)^{3/2} + \frac{x(t)^2}{2}\Big)dt,$$

where the kernel G is the Green's function defined on the interval $[0,1]\times[0,1]$ by

$$G(s,t) = \begin{cases} (1-s)t, & t \le s, \\ s(1-t), & s \le t. \end{cases}$$

The solution $x^*(s) = 0$ is the same as the solution of equation (1), where the mapping $F\colon C[0,1] \to C[0,1])$ is defined by

$$F(x)(s) = x(s) - \int_0^1 G(s,t)\Big(x(t)^{3/2} + \frac{x(t)^2}{2}\Big)dt.$$

Notice that

$$\Big\| \int_0^1 G(s,t)dt \Big\| \leq \frac{1}{8}.$$

Then, we have that

$$F'(x)y(s) = y(s) - \int_0^1 G(s,t)\Big(\frac{3}{2}x(t)^{1/2} + x(t)\Big)dt,$$

so since $F'(x^*(s)) = I$,

$$\|F'(x^*)^{-1}(F'(x) - F'(y))\| \leq \frac{1}{8}\Big(\frac{3}{2}\|x-y\|^{1/2} + \|x-y\|\Big).$$

Then, we get that $w_0(t) = w(t) = \frac{1}{8}(\frac{3}{2}t^{1/2} + t)$, $v(t) = 1 + w_0(t)$. So, we obtain

$$\rho^* = 0.740685070945967027888912070920 92,$$
$$r^* = 0.578955318897242277031978119339 33.$$

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Local convergence comparison between two novel sixth order methods **[19]**

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