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Projections of measures with small supports

Abstract. In this paper, we use a characterization of the mutual multifractal Hausdorff dimension in terms of auxiliary measures to investigate the projections of measures with small supports.

1. Introduction

Dimensional properties of projections of sets and measures have been investigated for decades. The first significant work in this area was the result of Marstrand [16], to which the Hausdorff dimension of a planar set is preserved under typical orthogonal projections. This result was later generalized to higher dimensions by Kaufman [14] and Mattila [17] and they obtain similar results for the Hausdorff dimension of a measure. Falconer and Mattila [12] and Falconer and Howroyd [10, 11] have proved that the packing dimension of the projected set or measure are the same for almost all projections.

O'Neil [20] has compared the generalized Hausdorff and packing dimensions of a subset $E \subseteq \mathbb{R}^n$ with respect to a measure μ with those of their projections onto m-dimensional subspaces. In [4, 28], the authors studied the multifractal analysis of the orthogonal projections on m-dimensional linear subspaces of singular measures on $\mathbb{R}^n$ satisfying the multifractal formalism. These results were later generalized by Selmi et al. in [7, 8, 25, 27, 35, 37].

Recently, mutual (mixed) multifractal spectra have generated an enormous interest in the mathematical literature. Many authors were interested in mutual multifractal spectra and their applications [3, 6, 18, 38, 39]. Previously, only the

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scaling behaviour

$$\lim_{r\to 0}\frac{\log\mu(B(x,r))}{\log r}$$

of a single measure μ has been investigated (see for example [4, 20, 22]). However, mutual multifractal analysis of two Borel probability measures μ and ν on $\mathbb{R}^n$ investigates the simultaneous scaling behaviour

$$\lim_{r\to 0}\frac{\log\mu(B(x,r))}{\log r},\qquad \lim_{r\to 0}\frac{\log\nu(B(x,r))}{\log r}.$$

It combines local characteristics which depend simultaneously on various different aspects of the underlying dynamical system and provides the basis for a significantly better understanding of the underlying dynamics. Olsen [23] conjectured a mutual multifractal formalism which links the mutual spectrum to the Legendre transform of mixed Rényi dimensions. General upper bound has been obtained and proved to be an equality if both measures are self-similar with same contracting similarities. Later, in [18], a mixed multifractal formalism associated with the mixed multifractal generalizations of Hausdorff and packing measures and dimensions is proved in some cases based on a generalization of the well known large deviation formalism.

In [8, 9, 30, 36], the authors studied the mutual multifractal analysis of the orthogonal projections on m-dimensional linear subspaces. More specifically, they investigated the relationship between $f_{\mu,\nu}(\alpha,\beta)$ and $f_{\mu_V,\nu_V}(\alpha,\beta)$, where

$$f_{\mu,\nu}(\alpha,\beta)=\dim_\zeta(\mathscr{B}_{\mu,\nu}(\alpha,\beta)),$$

$$\mathscr{B}_{\mu,\nu}(\alpha,\beta)=\Big\{x:\ \lim_{r\to 0}\frac{\log\mu(B(x,r))}{\log r}=\alpha,\ \lim_{r\to 0}\frac{\log\nu(B(x,r))}{\log r}=\beta\Big\}$$

and $\zeta\in\{H,P\}$. Here $\dim_H$ and $\dim_P$ denote, respectively, the Hausdorff dimension and the packing dimension. In addition, if we write for $\gamma\geq 0$,

$$E_{\mu,\nu}(\gamma)=\Big\{x\in\operatorname{supp}\mu\cap\operatorname{supp}\nu:\ \lim_{r\to 0}\frac{\log(\mu(B(x,r)))}{\log(\nu(B(x,r))}=\gamma\Big\}.$$

Then,

$$\bigcup_{\substack{(\alpha,\beta)\in\mathbb{R}_+\times\mathbb{R}_+^*,\\ \frac{\alpha}{\beta}=\gamma}}\mathscr{B}_{\mu,\nu}(\alpha,\beta)\subseteq E_{\mu,\nu}(\gamma).$$

The latter union is composed by an uncountable number of pairwise disjoint nonempty sets. Then, the Hausdorff and packing dimensions of $E_{\mu,\nu}(\gamma)$ are fully carried by some subset $\mathscr{B}_{\mu,\nu}\big(\alpha,\beta\big)$, for which the Hausdorff dimension of is evaluated by the Legendre transform of the multifractal Hausdorff function (see for example [1, 2, 5, 15, 29, 30, 31, 33, 34]). Also, Selmi et al. investigated the projection properties of the ν-Hausdorff, and the ν-packing dimensions of $E_{\mu,\nu}(\gamma)$ in [7]. They derived global bounds on the relative multifractal dimensions of a projection of a measure in terms of its original relative multifractal dimensions. It is more difficult to obtain a lower and upper bound for the dimension of the set $E_{\mu_V,\nu_V}(\gamma)$, where V is a linear subspace of $\mathbb{R}^n$.

As a continuity of these researches, we will start by introducing the mutual multifractal Hausdorff measure which differs slightly from those introduced in [18, 19], especially in [38, 39]. Also, we use a characterization of the mutual multifractal Hausdorff dimension in terms of auxiliary measures. We treat the mutual multifractal Hausdorff dimension of a Borel set using a characterization in terms of appropriately formed energy integrals. In particular, we obtain an inequality relating the mutual multifractal Hausdorff dimension of the original measure to those of its projection.

2. Mutual multifractal Hausdorff measure and function

Our main reason for modifying Svetova's definition is to allow us to prove results for non necessary doubling measures. One main cause and motivation is the fact that such characteristics are not in fact preserved under projections. Let μ, ν be two compactly supported probability measures on $\mathbb{R}^n$ with common support equal to K, $E \subseteq K$ and $\delta > 0$. For $\mathbf{q} = (q,t) \in \mathbb{R}^2$, $s \in \mathbb{R}$ and $\boldsymbol{\mu} = (\mu,\nu)$, we define the mutual Hausdorff measure,

$$\mathscr{H}^{\mathbf{q},s}_{\boldsymbol{\mu},\delta}(E) = \inf\left\{ \sum_i \mu(B(x_i,3r_i))^q \nu(B(x_i,3r_i))^t \; r_i^s \right\},$$

where the infimum is taken over all δ-coverings of E, and

$$\mathscr{H}^{\mathbf{q},s}_{\boldsymbol{\mu}}(E) = \sup_{\delta>0} \mathscr{H}^{\mathbf{q},s}_{\boldsymbol{\mu},\delta}(E).$$

Notice that the centers of the balls in the admissible covers need not be in the set E. This differs from the definition of Svetova [38, 39] and allows us to apply the Method II of Rogers [24] more easily. For $q \leq 0$ and $t \leq 0$ it is straightforward to verify that this measure is equivalent to Svetova's mutual multifractal Hausdorff measures and when μ and ν satisfy a global doubling condition, the mutual multifractal Hausdorff measures are equivalent for other cases. We observe also that $\mathscr{H}^{\mathbf{q},s}_{\boldsymbol{\mu}}$-measure is a Method II measure [24, Theorems 15 and 23] and that we would obtain the same measures if we worked with covers by open balls instead.

The function $\mathscr{H}^{\mathbf{q},s}_{\boldsymbol{\mu}}$ is σ-subadditive and increasing, which induces a measure on Borel subsets of $\mathbb{R}^n$. It assigns a dimension to each subset E of $\mathbb{R}^n$ denoted by

$$b^{\mathbf{q}}_{\boldsymbol{\mu}}(E) = \sup\{s \in \mathbb{R} : \; \mathscr{H}^{\mathbf{q},s}_{\boldsymbol{\mu}}(E) = \infty\} = \inf\{s \in \mathbb{R} : \; \mathscr{H}^{\mathbf{q},s}_{\boldsymbol{\mu}}(E) = 0\}.$$

Then, we define the mutual multifractal function $b_{\boldsymbol{\mu}} \colon \mathbb{R}^2 \to [-\infty,+\infty]$ by

$$b_{\boldsymbol{\mu}}(\mathbf{q}) = b^{\mathbf{q}}_{\boldsymbol{\mu}}(K).$$

Remark 2.1
In the special case where $q = 0$ or $t = 0$, the mutual multifractal function $b_{\boldsymbol{\mu}}(\mathbf{q})$ is strictly related to O'Neil's multifractal function [20]. The function $b_{\boldsymbol{\mu}}(\mathbf{q})$ is an obvious multifractal analogue of the Hausdorff dimension $\dim_H(K)$ of K, i.e. in the special case when $\mathbf{q} = (0,0)$, we have $b_{\boldsymbol{\mu}}(\mathbf{q}) = \dim_H(K)$.

3. Main result

Let m be an integer with $0 < m < n$ and $G_{n,m}$ the Grassmannian manifold of all m-dimensional linear subspaces of $\mathbb{R}^n$. Denote by $\gamma_{n,m}$ the invariant Haar measure on $G_{n,m}$ such that $\gamma_{n,m}(G_{n,m}) = 1$. For $V \in G_{n,m}$, we define the projection map $\pi_V : \mathbb{R}^n \to V$ as the usual orthogonal projection onto V. Then, the set $\{\pi_V,\ V \in G_{n,m}\}$ is compact in the space of all linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$ and the identification of V with π_V induces a compact topology for $G_{n,m}$. Also, for a Borel probability measure ν with compact support $\operatorname{supp}\nu \subset \mathbb{R}^n$ and for $V \in G_{n,m}$, we denote by ν_V, the projection of ν onto V, i.e.

$$\nu_V(A) = \nu \circ \pi_V^{-1}(A) \qquad \text{for all } A \subseteq V.$$

Since ν is compactly supported and $\operatorname{supp}\nu_V = \pi_V(\operatorname{supp}\nu)$ for all $V \in G_{n,m}$, then, for any continuous function $f : V \to \mathbb{R}$, we have

$$\int_V f d\nu_V = \int f(\pi_V(x)) d\nu(x),$$

whenever these integrals exist.

In order to proceed in our investigation of the behaviour of $\boldsymbol{\mu}$ under projection we need to introduce an assumption on the structure of its support: we need to be able to assume the existence of a uniform measure with a support containing the support of $\boldsymbol{\mu}$.

Definition 3.1
Let $E \subseteq \mathbb{R}^n$ and $0 < s < +\infty$. We say that E is s-Ahlfors regular if it is closed and if there exists a Borel measure ν on $\mathbb{R}^n$ and a constant $1 \leq C_E < +\infty$, such that $\nu(E) > 0$ and

$$C_E^{-1} r^s \leq \nu(B(x,r)) \leq C_E r^s \qquad \text{for all } x \in E \text{ and } 0 < r \leq 1.$$

Remark 3.1
We observe that $\mathbb{R}^n$ is n-Ahlfors regular and any m-dimensional subspace V is m-Ahlfors regular. It is easy to see that an s-Ahlfors regular set has packing dimension less than or equal to s.

The reason for introducing this notion is that it allows us to derive growth estimates on measures supported on Ahlfors regular sets. Following the method in [11, 13, 20] it is now straightforward to show that no measure can have too many points where the measure of a ball grows too quickly. The main use of this method is that it allows us to estimate, for ν supported on an s-Ahlfors regular set where $s \leq m$, the value of $\int_V \nu_V(B(x_V, r)) d\gamma_{n,m}$ from above by $\nu(B(x,r))$. In the following theorem, we concentrate on investigating the behaviour of the mutual multifractal dimension of a projection of measures in terms of the original mutual multifractal dimension of $E \subseteq K$ such that K is a s-Ahlfors regular set. The approach we use here was first used by Falconer and O'Neil in [13] and further developed by O'Neil in [20]. Throughout this paper, we denote $\boldsymbol{\mu}_V = (\mu_V, \nu_V)$.

THEOREM 3.1
Fix $0 < m < n$, $\boldsymbol{q} \in \mathbb{R}_+ \times \mathbb{R}_+$ and $0 < s \leq m$ such that K is s-Ahlfors regular. We have for all compact set $E \subseteq K$ with $\mu(E) > 0$ and $\nu(E) > 0$ and $\gamma_{n,m}$-almost every m-dimensional subspace V,

$$b^{\boldsymbol{q}}_{\boldsymbol{\mu}_V}(\pi_V(E)) \geq b^{\boldsymbol{q}}_{\boldsymbol{\mu}}(E).$$

REMARK 3.2
It is straightforward that any self-similar compact set which satisfies a strong separation is automatically s-Ahlfors regular for s equal to its packing dimension. For example, if μ is a self-similar (quasi self-similar) measure on $\mathbb{R}^n$ with support K of packing dimension $s \leq m$, then K is s-Ahlfors regular (for more details, see [20, 21]).

4. Proof of the main result

We present the tools, as well as the intermediate results, which will be used in the proof of our main result. We first use a characterization of the mutual multifractal Hausdorff dimension in terms of auxiliary measures. We treat the mutual multifractal Hausdorff dimension of a set E using a characterization of $b^{\mathbf{q}}_{\boldsymbol{\mu}}(E)$ in terms of appropriate energy integrals. Moreover, we obtain an inequality relating the mutual multifractal Hausdorff dimension of the original measure to the one of its projection.

4.1. Some characterizations of the mutual multifractal function

Denote by $\mathscr{P}(E)$ the family of finite Borel measures with compact support contained in $E \subseteq \mathbb{R}^n$. For compactly supported Borel probability measures μ, ν on $\mathbb{R}^n$ with common support K and a set $E \subseteq K$ with $\mu(E) > 0$ and $\nu(E) > 0$, we define

$$\mathfrak{P}^{\mathbf{q},s}_{\boldsymbol{\mu}}(E) = \big\{\theta \in \mathscr{P}(E) :\ \text{for } 0 < r \leq 1,\ \theta(B(x,r)) \leq f^{\mathbf{q},s}_{\boldsymbol{\mu}}(x,r) \text{ for } \theta\text{-a.e. } x\big\},$$

where $f^{\mathbf{q},s}_{\boldsymbol{\mu}}(x,r) = \mu(B(x,3r))^q \nu(B(x,3r))^t r^s$.

The next theorem is essentially a restatement of [20, Theorem 5.1] in a general case.

THEOREM 4.1
For a compact set $E \subseteq K$ with $\mu(E) > 0$ and $\nu(E) > 0$, and $\boldsymbol{q} \in \mathbb{R}^2$, we have

$$b^{\boldsymbol{q}}_{\boldsymbol{\mu}}(E) = \sup\big\{s \in \mathbb{R} :\ \exists\theta \in \mathfrak{P}^{\boldsymbol{q},s}_{\boldsymbol{\mu}}(E),\ \theta(E) > 0\big\}.$$

Proof. See [20, Theorem 5.1] for the key ideas needed to prove this theorem.

REMARK 4.1
For all $\mathbf{q} \in \mathbb{R}^2$, we define the $(\mathbf{q}, \boldsymbol{\mu})$-upper density of order s of θ at x by

$$d^{\mathbf{q},s}_{\boldsymbol{\mu}}(\theta, x) = \limsup_{r\to 0} \frac{\theta(B(x,r))}{\mu(B(x,3r))^q \nu(B(x,3r))^t r^s}$$

and the $(\mathbf{q}, \boldsymbol{\mu})$-local multifractal Hausdorff dimension, $b_{\boldsymbol{\mu}}^{\mathbf{q}}(\theta, x)$, of a measure θ at a point x by

$$\begin{aligned}
b_{\boldsymbol{\mu}}^{\mathbf{q}}(\theta, x) &= \liminf_{r \to 0} \frac{\log \theta(B(x,r)) - q \log \mu(B(x,3r)) - t \log \nu(B(x,3r))}{\log r} \\
&= \sup\big\{ s \in \mathbb{R} : \ d_{\boldsymbol{\mu}}^{\mathbf{q},s}(\theta, x) = 0 \big\}.
\end{aligned}$$

Similar techniques to those used in [26, 27, 32] allow us to reformulate Theorem 4.1, as

$$\begin{aligned}
b_{\boldsymbol{\mu}}^{\mathbf{q}}(E) &= \sup\big\{ \operatorname*{ess\,inf}_{x \in E} b_{\boldsymbol{\mu}}^{\mathbf{q}}(\theta, x) : \ 0 \neq \theta \in \mathscr{P}(E) \big\} \\
&= \sup\big\{ b_{\boldsymbol{\mu}}^{\mathbf{q}}(\theta) : \ 0 \neq \theta \in \mathscr{P}(E) \big\},
\end{aligned} \tag{1}$$

where the essential bounds being related to the measure θ, and

$$b_{\boldsymbol{\mu}}^{\mathbf{q}}(\theta) = \sup\big\{ s \in \mathbb{R} : \ b_{\boldsymbol{\mu}}^{\mathbf{q}}(\theta, x) \geq s \text{ for } \theta\text{-a.e. } x \big\}.$$

It is now possible to characterize $b_{\boldsymbol{\mu}}(\mathbf{q})$ in terms of appropriate energy integrals. We easily obtain the following characterization

$$b_{\boldsymbol{\mu}}(\mathbf{q}) = \sup\big\{ s \in \mathbb{R} : \ \exists 0 \neq \theta \in \mathscr{P}(K), \text{ such as } I_{\theta}^{s,\mathbf{q}}(\boldsymbol{\mu}) < +\infty \big\},$$

where

$$I_{\theta}^{s,\mathbf{q}}(\boldsymbol{\mu}) = \iint \big(f_{\boldsymbol{\mu}}^{\mathbf{q},s}(x, |y-x|) \big)^{-1} d\theta(y) d\theta(x).$$

4.2. Proof of Theorem 3.1

Fix $0 < m < n$ and suppose that ν is a Borel probability measure with $\operatorname{supp} \nu \subset B(0,1)$. We begin by investigating the behaviour of the ν_V-measure of a ball in V for $V \in G_{n,m}$ and relate this to local properties of the measure ν. This leads us to introduce a kernel function $\phi_r^m : \mathbb{R}^n \backslash \{0\} \to (0, \infty)$ by setting $\phi_r^m(x) = \min\{1, r^m |x|^{-m}\}$. The convolution product of ϕ_r^m and the measure ν is therefore given by

$$\phi_r^m * \nu(x) = \int \min\big\{1, r^m |x-y|^{-m}\big\} d\nu(y).$$

So, integrating by parts and applying next spherical coordinates (see [13]), we obtain

$$\phi_r^m * \nu(x) = m r^m \int_r^{+\infty} u^{-m-1} \nu(B(x,u)) du$$

and

$$\phi_{2r}^m * \nu(x) \leq 2^m \phi_r^m * \nu(x). \tag{2}$$

We observe for all $V \in G_{n,m}$ that

$$\nu(B(x,r)) \leq \phi_r^m * \nu(x) \leq \phi_r^m * \nu_V(x_V) \tag{3}$$

and

$$\nu(B(x,r)) \leq \nu_V(B(x_V, r)) \leq \phi_r^m * \nu_V(x_V).$$

We present the following technical lemma, which will be used in the proof of our main result.

Lemma 4.1 ([20, Lemma 5.8])
Fix $0 < m \leq n$. Suppose that ν is a compactly supported, finite Borel measure on $\mathbb{R}^n$ with support contained in an s-Ahlfors regular set for some $0 < s \leq m$. Then for all $\varepsilon > 0$ and ν-a.e. x there exist $r_0 > 0$ and $c > 0$ such that for $0 < r \leq r_0$, we have

$$\phi_r^m * \nu(x) \leq c r^{-\varepsilon} \nu(B(x,r)) \tag{4}$$

and

$$\int_{V \in G_{n,m}} \phi_r^m * \nu_V(x_V) d\gamma_{n,m}(V) \leq c r^{-\varepsilon} \nu(B(x,r)).$$

Theorem 3.1 is a consequence of the following propositions.

Proposition 4.1
For compact sets $E \subseteq K$ with $\mu(E) > 0$ and $\nu(E) > 0$, and $\boldsymbol{q} \in \mathbb{R}^2$, we have for all m-dimensional subspaces V,

$$b^{\boldsymbol{q}}_{\boldsymbol{\mu}_V}(\pi_V(E)) = \sup\left\{ b^{\boldsymbol{q}}_{\boldsymbol{\mu}_V}(\theta_V) : \ \theta \in \mathscr{P}(E), \ \theta(E) > 0 \right\}.$$

Proof. This result follows immediately from (1) together with the observation that a finite Borel measure θ on $\pi_V(E)$ may be pulled back to give a finite Borel measure on E.

Proposition 4.2
Let $0 < m < n$, $\boldsymbol{q} \in \mathbb{R}_+ \times \mathbb{R}_+$ and $0 < s \leq m$ such that K is s-Ahlfors regular. For all finite Borel measures θ with support contained in K and for almost every m-dimensional subspaces V we have

$$b^{\boldsymbol{q}}_{\boldsymbol{\mu}_V}(\theta_V) \geq b^{\boldsymbol{q}}_{\boldsymbol{\mu}}(\theta).$$

Proof. Let $s < b^{\mathbf{q}}_{\boldsymbol{\mu}}(\theta)$ to ensure that for θ-a.e. x, $d^{\mathbf{q},s}_{\boldsymbol{\mu}}(\theta, x) = 0$. We will prove for $\gamma_{n,m}$-almost every m-dimensional subspace V the equality

$$d^{\mathbf{q},s}_{\boldsymbol{\mu}_V}(\theta_V, x_V) = \limsup_{r \to 0} \frac{\theta_V(B(x_V, r))}{\mu_V(B(x_V, 3r))^q \nu_V(B(x_V, 3r))^t r^s} = 0$$

for θ-a.e. x which yields the result. For any $\varepsilon > 0$ and $\xi > 0$, let $\gamma = \min(1, 2^{s-2\varepsilon})$, we denote for all $k \in \mathbb{N}$,

$$\begin{aligned} G^{\mathbf{q},s}_k(x) = \Big\{ V \in G_{n,m} : \ \phi^m_{2^{-(k+1)}} * \theta_V(x_V) > \chi\, \mu_V\Big(B\Big(x_V, \frac{3}{2^{(k+1)}}\Big)\Big)^q \\ \times \nu_V\Big(B\Big(x_V, \frac{3}{2^{(k+1)}}\Big)\Big)^t \Big\}, \end{aligned}$$

where $\chi = \frac{2^m \gamma \xi}{2^{(k+1)(s-2\varepsilon)}}$. By recalling (2) we deduce that for all $k \in \mathbb{N}$,

$$\Big\{V\in G_{n,m}:\ \exists r\in]2^{-(k+1)},2^{-k}],\ \phi_r^m*\theta_V(x_V)>\\ \frac{\xi}{r^{2\varepsilon-s}}\mu_V(B(x_V,3r))^q\nu_V(B(x_V,3r))^t\Big\}\subseteq G_k^{\mathbf{q},s}(x).$$

Whenever

$$\sum_{k=1}^{\infty}\gamma_{n,m}\big(G_k^{\mathbf{q},s}(x)\big)<+\infty,$$

then Borel-Cantelli lemma yields that with probability 1 only a finite number of the events $G_k^{\mathbf{q},s}(x)$ can occur, i.e.

$$\limsup_{r\to 0}\frac{\phi_r^m*\theta_V(x_V)}{\mu_V(B(x_V,3r))^q\nu_V(B(x_V,3r))^t r^{s-2\varepsilon}}=0$$

for almost every m-dimensional subspaces V. In view of the monotonicity of the $(\mathbf{q},\boldsymbol{\mu})$-upper density in s, the exceptional set may be chosen the same for all s under consideration. Then Fubini's theorem with respect to the measure $\theta\times\gamma_{n,m}$ and the inequalities (3) and (4) yield, for $\gamma_{n,m}$-almost every m-dimensional subspace V and θ-a.e. x that

$$d_{\boldsymbol{\mu}_V}^{\mathbf{q},s-2\varepsilon}(\theta_V,x_V)=\limsup_{r\to 0}\frac{\phi_r^m*\theta_V(x_V)}{\mu_V(B(x_V,3r))^q\nu_V(B(x_V,3r))^t r^{s-2\varepsilon}}=0.$$

Choosing a sequence $\varepsilon_i\to 0$, we conclude that for $\gamma_{n,m}$-almost every m-dimensional subspace V and θ-a.e. x,

$$b_{\boldsymbol{\mu}_V}^{\mathbf{q}}(\theta_V,x_V)\geq s.$$

Consequently, we have for $\gamma_{n,m}$-almost every m-dimensional subspace V,

$$b_{\boldsymbol{\mu}_V}^{\mathbf{q}}(\theta_V)\geq s.$$

In order to prove the convergence of the above series, observe that from Lemma 4.1, for all $\varepsilon>0$ and θ-a.e. x there is a constant $c>0$ and $\delta>0$ such that for all $0<r\leq\delta$,

$$\int_{V\in G_{n,m}}\phi_r^m*\theta_V(x_V)d\gamma_{n,m}\leq c\ r^{-\varepsilon}\theta(B(x,r)). \tag{5}$$

For $0<r\leq\delta$ and since $q,t\geq 0$, we have

$$\int_{V\in G_{n,m}}\frac{\phi_r^m*\theta_V(x_V)}{\mu_V(B(x_V,3r))^q\nu_V(B(x_V,3r))^t r^{s-2\varepsilon}}d\gamma_{n,m}\\ \leq c\frac{\theta(B(x,r))}{\mu(B(x,3r))^q\nu(B(x,3r))^t r^{s-\varepsilon}}.$$

Thus by choosing $0<\delta_0<\delta$ such that for $0<r\leq\delta_0$,

$$\frac{\theta(B(x,r))}{\mu(B(x,3r))^q\nu(B(x,3r))^t r^{s}}\leq 1,$$

we may estimate, from Markov's inequality and (5), that for any $r \leq \delta_0$ and $\xi > 0$,

$$\gamma_{n,m}\big(\big\{V \in G_{n,m}:\ \phi_r^m * \theta_V(x_V) > \xi\mu_V(B(x_V,3r))^q\nu_V(B(x_V,3r))^t r^{s-2\varepsilon}\big\}\big) \\ \leq \frac{r^{2\varepsilon-s}}{\xi\mu(B(x,3r))^q\nu(B(x,3r))^t}\int_{V\in G_{n,m}} \phi_r^m * \theta_V(x_V)d\gamma_{n,m} \leq \frac{c}{\xi}r^{\varepsilon}.$$

Choosing r in $]2^{-(k+1)}, 2^{-k}]$, we get the result.

Remark 4.2
Fix $0 < m < n$, $\mathbf{q} \in \mathbb{R}_+ \times \mathbb{R}_+$ and $0 < s \leq m$ such that K is s-Ahlfors regular. We have for $\gamma_{n,m}$-almost every m-dimensional subspace V,

$$b_{\boldsymbol{\mu}_V}(\mathbf{q}) \geq b_{\boldsymbol{\mu}}(\mathbf{q}).$$

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