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Almost convergent sequence spaces derived by the domain of quadruple band matrix

Abstract. In this work, we construct the sequence spaces $f(Q(r,s,t,u))$, $f_0(Q(r,s,t,u))$ and $fs(Q(r,s,t,u))$, where $Q(r,s,t,u)$ is quadruple band matrix which generalizes the matrices Δ^3, $B(r,s,t)$, Δ^2, $B(r,s)$ and Δ, where Δ^3, $B(r,s,t)$, Δ^2, $B(r,s)$ and Δ are called third order difference, triple band, second order difference, double band and difference matrix, respectively. Also, we prove that these spaces are BK-spaces and are linearly isomorphic to the sequence spaces f, f_0 and fs, respectively. Moreover, we give the Schauder basis and β, γ-duals of those spaces. Lastly, we characterize some matrix classes related to those spaces.

1. Basic knowledge

By a sequence space, we mean a vector subspace of w, where w is a set of all real (or complex) valued sequences which becomes a vector space under point-wise addition and scalar multiplication. For the spaces of all bounded, null, convergent and absolutely p-summable sequences, we use the symbols ℓ_∞, c_0, c and ℓ_p, respectively, where $1 \leq p < \infty$.

Let X be a Banach sequence space. If each of the maps $p_k \colon X \to \mathbb{C}$, $p_k(x) = x_k$ is continuous for all $k \in \mathbb{N}$, X is called a BK-*space*. The sequence spaces ℓ_∞, c_0 and c are all BK-spaces according to $\|x\|_\infty = \sup_{n\in\mathbb{N}} |x_n|$ and ℓ_p is a BK-space according to

$$\|x\|_p = \left(\sum_{k=0}^{\infty} |x_k|^p\right)^{\frac{1}{p}}, \qquad \text{where } p \in (1,\infty] \text{ (see [22])}.$$

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For a given infinite matrix $A = (a_{nk})$ with $a_{nk} \in \mathbb{C}$ for all $n, k \in \mathbb{N}$ and $x = (x_k) \in w$, the A-*transform* of x is denoted by $y = Ax$ and is defined by

$$y_n = (Ax)_n = \sum_{j=0}^{\infty} a_{nj} x_j$$

for all $n \in \mathbb{N}$ and each of these series being assumed convergent (see [30]). In terms of ease of use, here and in the following, the summation without limits runs from zero to infinity.

By $(X : Y)$, we denote the class of all matrices $A = (a_{nk})$ such that $Ax \in Y$ for every $x \in X$, where X and Y are two arbitrary sequence spaces. The matrix domain X_A of the matrix $A = (a_{nk})$ in a sequence space X is defined by

$$X_A = \{x = (x_k) : \ Ax \in X\}. \tag{1.1}$$

The spaces of all convergent and bounded series are denoted by cs and bs and are defined by $cs = c_S$ and $bs = (\ell_\infty)_S$ where $S = (s_{nk})$ is called the summation matrix defined by

$$s_{nk} = \begin{cases} 1, & 0 \leq k \leq n, \\ 0, & k > n \end{cases}$$

for all $n, k \in \mathbb{N}$.

A matrix $A = (a_{nk})$ is called a triangle provided $a_{nk} = 0$ for $k > n$ and $a_{nn} \neq 0$ for all $n, k \in \mathbb{N}$. Moreover, a triangle matrix uniquely has an inverse.

As an application of the Hahn-Banach theorem to the sequence space ℓ_∞, the concept of Banach Limits was first put forward by the Stefan Banach. Banach first recognized certain non-negative linear functionals on ℓ_∞ which remain invariant under shift operators and which are extension of l, where $l\colon c \longrightarrow \mathbb{R}$, $l(x) = \lim_{n\to\infty} x_n$ is defined and l is linear functional on c. This kind of functionals were later termed "Banach Limits" [13].

A continuous linear functional $L\colon \ell_\infty \to \mathbb{R}$ is called a *Banach Limit* if the following conditions are satisfied:

(i) $L(x_n) \geq 0$ if $x_n \geq 0$, $n = 0, 1, 2, \ldots$;

(ii) $L(P_j(x_n)) = L(x_n)$, $P_j(x_n) = x_{n+j}$, $j = 1, 2, 3, \ldots$;

(iii) $L(e) = 1$, where $e = (1, 1, \ldots)$.

Lorentz continued the study of Banach Limits and brought out a new notion named Almost Convergence. The sequence $x = (x_n) \in \ell_\infty$ is called *almost convergent* and the number $\operatorname{Lim} x_n = \lambda$ is called its F-*limit* if $L(x_n) = \lambda$ holds for every limit L (see [21]).

An approach to the construction of a new sequence space by means of the domains of the difference matrices was used by, $c_0(\Delta)$, $c(\Delta)$ and $\ell_\infty(\Delta)$ in [18], $\Delta c_0(p)$, $\Delta c(p)$ and $\Delta \ell_\infty(p)$ in [1], $c_0(u, \Delta, p)$, $c(u, \Delta, p)$ and $\ell_\infty(u, \Delta, p)$ in [2], $c_0(\Delta^2)$, $c(\Delta^2)$ and $\ell_\infty(\Delta^2)$ in [16], $c_0(u, \Delta^2)$, $c(u, \Delta^2)$ and $\ell_\infty(u, \Delta^2)$ in [23], $c_0(u, \Delta^2, p)$, $c(u, \Delta^2, p)$ and $\ell_\infty(u, \Delta^2, p)$ in [11], $c_0(\Delta^m)$, $c(\Delta^m)$ and $\ell_\infty(\Delta^m)$ in [15], $\hat{\ell_\infty}$, $\hat{c_0}$, $\hat{c}$ and $\hat{\ell_p}$ in [20], $c_0(B)$, $c(B)$, $\ell_\infty(B)$ and $\ell_p(B)$ in [28], $\hat{f}_0$ and $\hat{f}$ in

[3], $f_0(B)$ and $f(B)$ in [29]. For recent developments in this direction, we refer the reader to the textbooks/monographs [8], [9], [24] and [25], and the references therein.

2. Almost convergence quadruple band matrix

In this section, we mention some old works and construct the sequence spaces $f_0(Q(r,s,t,u))$, $f(Q(r,s,t,u))$ and $fs(Q(r,s,t,u))$. Moreover, we prove that these spaces are BK-spaces and are linearly isomorphic to the sequence spaces f_0, f and fs, respectively.

Lorentz characterized the almost convergent sequences by giving the next theorem.

Theorem 1 (see [21])
In order that F-limit, $\operatorname{Lim} x_n = \lambda$ *exists for the sequence $x = (x_n)$, it is necessary and sufficient that*

$$\lim_{k\to\infty} \frac{x_n + x_{n+1} + \cdots + x_{n+k}}{k+1} = \lambda$$

holds uniformly in n.

By connecting the notion of almost convergence and Theorem 1, the spaces f, f_0 and fs of all almost convergent sequences, almost null sequences and almost convergent series are defined by

$$f = \left\{ x = (x_k) \in w : \ \exists \lambda \in \mathbb{C} \ \lim_{i\to\infty} \sum_{k=0}^{i} \frac{x_{n+k}}{i+1} = \lambda \text{ uniformly in } n \right\},$$

$$f_0 = \left\{ x = (x_k) \in w : \ \lim_{i\to\infty} \sum_{k=0}^{i} \frac{x_{n+k}}{i+1} = 0 \text{ uniformly in } n \right\}$$

and

$$fs = \left\{ x = (x_k) \in w : \ \exists \lambda \in \mathbb{C} \ \lim_{i\to\infty} \sum_{k=0}^{i} \sum_{j=0}^{n+k} \frac{x_j}{i+1} = \lambda \text{ uniformly in } n \right\},$$

respectively. By using the relation (1.1), the sequence space fs can be redefined as follows

$$fs = f_S.$$

Theorem 2 (see [12])
The inclusions $c \subset f \subset \ell_\infty$ strictly hold.

Theorem 3 (see [12])
The sequence spaces f and f_0 are BK-spaces with the norm

$$\|x\|_f = \sup_{i,n\in\mathbb{N}} \left| \sum_{k=0}^{i} \frac{x_{n+k}}{i+1} \right|$$

and fs is a BK-space with the norm $\|x\|_{fs} = \|Sx\|_f$.

In order to define sequence spaces, the difference matrix was first used by Kızmaz in [18]. He constructed the difference sequence spaces $c_0(\Delta)$, $c(\Delta)$ and $\ell_\infty(\Delta)$, as follows

$$c_0(\Delta) = \Big\{x = (x_k) \in w : \lim_{k\to\infty} (x_k - x_{k+1}) = 0\Big\},$$

$$c(\Delta) = \Big\{x = (x_k) \in w : \lim_{k\to\infty} (x_k - x_{k+1}) \text{ exists}\Big\}$$

and

$$\ell_\infty(\Delta) = \Big\{x = (x_k) \in w : \sup_{k\in\mathbb{N}} |x_k - x_{k+1}| < \infty\Big\}$$

and the difference matrix $\Delta = (e_{nk})$ is defined by

$$e_{nk} = \begin{cases} 1, & k = n, \\ -1, & k = n+1, \\ 0, & \text{otherwise} \end{cases}$$

for all $n, k \in \mathbb{N}$.

Afterward, Başar and Kirişçi used the generalized difference matrix in [3] for defining the generalized difference sequence spaces $\hat{f}_0$ and $\hat{f}$ which are defined by

$$\hat{f}_0 = \left\{x = (x_k) \in w : \lim_{m\to\infty} \sum_{j=0}^{m} \frac{rx_{n+j} + sx_{n+j-1}}{m+1} = 0 \text{ uniformly in } n\right\}$$

and

$$\hat{f} = \left\{x = (x_k) \in w : \exists\lambda \in \mathbb{C} \lim_{m\to\infty} \sum_{j=0}^{m} \frac{rx_{n+j} + sx_{n+j-1}}{m+1} = \lambda \text{ uniformly in } n\right\}.$$

Moreover, Sönmez used the triple band matrix in [29] for defining almost convergent sequence spaces derived by the domain of triple band matrix. These spaces are defined by

$$f_0(B(r,s,t) = \left\{x = (x_k) \in w : \lim_{m\to\infty} \sum_{j=0}^{m} \frac{rx_{n+j} + sx_{n+j-1} + tx_{n+j-2}}{m+1} = 0 \text{ uniformly in } n\right\}$$

and

$$f(B(r,s,t)) = \left\{x = (x_k) \in w : \exists\lambda \in \mathbb{C} \lim_{m\to\infty} \sum_{j=0}^{m} \frac{rx_{n+j} + sx_{n+j-1} + tx_{n+j-2}}{m+1} = \lambda \text{ uniformly in } n\right\}.$$

Recently, Bişgin has defined the sequence spaces $c_0(Q)$, $c(Q)$, $\ell_\infty(Q)$ and $\ell_p(Q)$ as follows

$$c_0(Q) = \Big\{x = (x_k) \in w : \lim_{k\to\infty}(rx_k + sx_{k-1} + tx_{k-2} + ux_{k-3}) = 0\Big\},$$

$$c(Q) = \Big\{x = (x_k) \in w : \lim_{k\to\infty}(rx_k + sx_{k-1} + tx_{k-2} + ux_{k-3}) \text{ exists}\Big\},$$

$$\ell_\infty(Q) = \Big\{x = (x_k) \in w : \sup_{k\in\mathbb{N}}|rx_k + sx_{k-1} + tx_{k-2} + ux_{k-3}| < \infty\Big\}$$

and

$$\ell_p(Q) = \Big\{x = (x_k) \in w : \sum_k |rx_k + sx_{k-1} + tx_{k-2} + ux_{k-3}|^p < \infty\Big\},$$

where $1 \le p < \infty$ and the quadruple band matrix $Q = Q(r, s, t, u) = (q_{nk}(r, s, t, u))$ is defined by

$$q_{nk}(r,s,t,u) = \begin{cases} r, & k = n, \\ s, & k = n-1, \\ t, & k = n-2, \\ u, & k = n-3, \\ 0, & \text{otherwise} \end{cases}$$

for all $n, k \in \mathbb{N}$, $r, s, t, u \in \mathbb{R} \setminus \{0\}$. Here, we would like to bring attention that $Q(1, -3, 3, -1) = \Delta^3$, $Q(r, s, t, 0) = B(r, s, t)$, $Q(1, -2, 1, 0) = \Delta^2$, $Q(r, s, 0, 0) = B(r, s)$ and $Q(1, -1) = \Delta$, where Δ^3, $B(r, s, t)$, Δ^2, $B(r, s)$ and Δ are called third order difference, triple band, second order difference, double band (generalized difference) and difference matrices, respectively. Therefore, our results derived from the matrix domain of the quadruple band matrix are more general and more comprehensive than the results on the matrix domain of the others mentioned above.

Now, we define the spaces $f_0(Q(r, s, t, u))$, $f(Q(r, s, t, u))$ and $fs(Q(r, s, t, u))$ by

$$f_0(Q(r,s,t,u)) = \Big\{x = (x_k) \in w : \\ \lim_{i\to\infty}\sum_{j=0}^{i}\frac{rx_{n+j} + sx_{n+j-1} + tx_{n+j-2} + ux_{n+j-3}}{i+1} = 0 \text{ uniformly in } n\Big\},$$

$$f(Q(r,s,t,u)) = \Big\{x = (x_k) \in w : \\ \exists\lambda \in \mathbb{C}\ \lim_{i\to\infty}\sum_{j=0}^{i}\frac{rx_{n+j} + sx_{n+j-1} + tx_{n+j-2} + ux_{n+j-3}}{i+1} = \lambda \text{ uniformly in } n\Big\}$$

and

$$fs(Q(r,s,t,u)) = \Bigg\{x = (x_k) \in w :$$
$$\exists \lambda \in \mathbb{C} \ \lim_{i\to\infty} \sum_{j=0}^{i} \sum_{v=0}^{n+j} \frac{rx_v + sx_{v-1} + tx_{v-2} + ux_{v-3}}{i+1} = \lambda \text{ uniformly in } n\Bigg\},$$

respectively. By taking into account the notation (1.1), the sequence spaces $f_0(Q(r,s,t,u))$, $f(Q(r,s,t,u))$ and $fs(Q(r,s,t,u))$ can be redefined by means of the domain of the quadruple band matrix $Q = Q(r,s,t,u)$ as follows

$$f_0(Q) = (f_0)_Q, \quad f(Q) = f_Q \quad \text{and} \quad fs(Q) = fs_Q. \tag{2.1}$$

Also, for given an arbitrary sequence $x = (x_k) \in w$, the $Q(r,s,t,u)$-transform of $x = (x_k)$ is defined by

$$y_k = (Q(r,s,t,u)x)_k = rx_k + sx_{k-1} + tx_{k-2} + ux_{k-3}$$

for all $k \in \mathbb{N}$.

Theorem 4
The sequence spaces $f_0(Q(r,s,t,u))$, $f(Q(r,s,t,u))$ and $fs(Q(r,s,t,u))$ equipped with the norms

$$\|x\|_{f(Q(r,s,t,u))} = \|x\|_{f_0(Q(r,s,t,u))} = \|Q(r,s,t,u)x\|_f$$

and

$$\|x\|_{fs(Q(r,s,t,u))} = \|Q(r,s,t,u)x\|_{fs},$$

respectively, are BK-spaces.

Proof. Notice that f, f_0 and fs are known to be BK-spaces. Moreover, quadruple band matrix is a triangle matrix and the condition (2.1) holds. If we connect these results with Theorem 4.3.12 of Wilansky [30], we conclude that $f(Q(r,s,t,u))$, $f_0(Q(r,s,t,u))$ and $fs(Q(r,s,t,u))$ are BK-spaces. This completes the proof of theorem.

Now, let us pay attention to the equation

$$rz^3 + sz^2 + tz + u = 0,$$

where $r,s,t,u \in \mathbb{R} \setminus \{0\}$. We know that this equation has three roots such that $z_1 = \frac{1}{3r}[a-b-s]$, $z_2 = -\frac{1}{6r}[(1-i\sqrt{3})a - (1+i\sqrt{3})b + 2s]$ and $z_3 = -\frac{1}{6r}[(1+i\sqrt{3})a - (1-i\sqrt{3})b + 2s]$, where

$$a = \sqrt[3]{\frac{\sqrt{(-27r^2u + 9rst - 2s^3)^2 + 4(3rt - s^2)^3} - 27r^2u + 9rst - 2s^3}{2}}$$

and

$$b = \sqrt[3]{\frac{\sqrt{(-27r^2u + 9rst - 2s^3)^2 + 4(3rt - s^2)^3} + 27r^2u - 9rst + 2s^3}{2}}.$$

We henceforth prefer that μ_1, μ_2 and μ_3 are random three roots of the equation $rz^3 + sz^2 + tz + u = 0$. Also, by using a simple calculation, we have

$$\mu_1 + \mu_2 + \mu_3 = -\frac{s}{r}, \quad \mu_1\mu_2 + \mu_1\mu_3 + \mu_2\mu_3 = \frac{t}{r} \quad \text{and} \quad \mu_1\mu_2\mu_3 = -\frac{u}{r},$$

$$\mu_1^3 + \frac{s}{r}\mu_1^2 + \frac{t}{r}\mu_1 + \frac{u}{r} = 0, \tag{2.2}$$

$$\mu_1^2 + \mu_2^2 + \frac{s}{r}(\mu_1 + \mu_2) + \mu_1\mu_2 + \frac{t}{r} = 0, \tag{2.3}$$

$$\mu_1^2 + \mu_2^2 + \mu_3^2 + \mu_1\mu_2 + \mu_1\mu_3 + \mu_2\mu_3 + \frac{s}{r}(\mu_1 + \mu_2 + \mu_3) + \frac{t}{r} = 0, \tag{2.4}$$

$$\mu_1 + \mu_2 + \mu_3 + \frac{s}{r} = 0. \tag{2.5}$$

Theorem 5

The sequence spaces $f_0(Q(r,s,t,u))$, $f(Q(r,s,t,u))$ and $fs(Q(r,s,t,u))$ are linearly isomorphic to the sequence spaces f_0, f and fs, respectively.

Proof. Because of the equivalence $f_0(Q(r,s,t,u)) \cong f_0$ and $fs(Q(r,s,t,u)) \cong fs$ can be proved by using similar operations, the proof is given for only the sequence space $f(Q(r,s,t,u))$. For this purpose, the presence of a linear bijection between the sequence spaces $f(Q(r,s,t,u))$ and f should be shown.

In that case, let us define a transformation $L\colon f(Q(r,s,t,u)) \to f$ so that $L(x) = Q(r,s,t,u)x$. It is clear that $L(x) = Q(r,s,t,u)x \in f$ whenever $x = (x_k) \in f(Q(r,s,t,u))$. Also, it is obviously provided that L is a linear transformation and $x = 0$ in case of $L(x) = 0$. Thus, L is injective.

Now, through the medium of $y = (y_k) \in f$, let us define a sequence $x = (x_k)$ such that

$$x_k = \frac{1}{r}\sum_{j=0}^{k}\sum_{i=0}^{k-j}\sum_{v=0}^{k-j-i} \mu_1^{k-j-i-v}\mu_2^v\mu_3^i y_j$$

for all $k \in \mathbb{N}$. Then, by considering (2.2)–(2.5), we obtain

$$\begin{aligned}
(Qx)_k &= rx_k + sx_{k-1} + tx_{k-2} + ux_{k-3} \\
&= \sum_{j=0}^{k}\sum_{i=0}^{k-j}\sum_{v=0}^{k-j-i} \mu_1^{k-j-i-v}\mu_2^v\mu_3^i y_j \\
&\quad + \frac{s}{r}\sum_{j=0}^{k-1}\sum_{i=0}^{k-j-1}\sum_{v=0}^{k-j-i-1} \mu_1^{k-j-i-v-1}\mu_2^v\mu_3^i y_j \\
&\quad + \frac{t}{r}\sum_{j=0}^{k-2}\sum_{i=0}^{k-j-2}\sum_{v=0}^{k-j-i-2} \mu_1^{k-j-i-v-2}\mu_2^v\mu_3^i y_j \\
&\quad + \frac{u}{r}\sum_{j=0}^{k-3}\sum_{i=0}^{k-j-3}\sum_{v=0}^{k-j-i-3} \mu_1^{k-j-i-v-3}\mu_2^v\mu_3^i y_j
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=0}^{k-3}\Bigg[\sum_{i=0}^{k-j-3}\Bigg[\sum_{v=0}^{k-j-i-3}\mu_3^i\mu_2^v\mu_1^{k-j-i-v-3}\Big(\mu_1^3+\frac{s}{r}\mu_1^2+\frac{t}{r}\mu_1+\frac{u}{r}\Big)\\
&\quad+\mu_3^i\mu_2^{k-j-i-2}\Big(\mu_1^2+\mu_2^2+\frac{s}{r}(\mu_1+\mu_2)+\mu_1\mu_2+\frac{t}{r}\Big)\\
&\quad+\mu_3^{k-j-2}\Big(\mu_1^2+\mu_2^2+\mu_3^2+\mu_1\mu_2+\mu_1\mu_3+\mu_2\mu_3\\
&\quad+\frac{s}{r}(\mu_1+\mu_2+\mu_3)+\frac{t}{r}\Big)\Bigg]\Bigg]y_j\\
&\quad+\Big[y_{k-2}\Big(\mu_1^2+\mu_2^2+\mu_3^2+\mu_1\mu_2+\mu_1\mu_3+\mu_2\mu_3\\
&\quad+\frac{s}{r}(\mu_1+\mu_2+\mu_3)+\frac{t}{r}\Big)\\
&\quad+y_{k-1}\Big(\mu_1+\mu_2+\mu_3+\frac{s}{r}\Big)+y_k\Big]\\
&= y_k
\end{aligned}
$$

for all $k \in \mathbb{N}$, which yields that

$$\lim_{i\to\infty}\sum_{j=0}^{i}\frac{rx_{n+j}+sx_{n+j-1}+tx_{n+j-2}+ux_{n+j-3}}{i+1}=\lim_{i\to\infty}\sum_{j=0}^{i}\frac{y_{n+j}}{i+1}=F-\lim y_n.$$

Namely, $x=(x_k)\in f(Q(r,s,t,u))$ and $L(x)=y$. Hence, L is surjective. Also, for all $x=(x_k)\in f(Q(r,s,t,u))$, we have

$$\|L(x)\|_f=\|Q(r,s,t,u)x\|_f=\|x\|_{f(Q(r,s,t,u))}.$$

That's why, L is norm preserving. As a consequences of these, L is a linear bijection. Therefore, we obtain $f(Q(r,s,t,u))\cong f$ as desired. This completes the proof of theorem.

3. The Schauder basis and β, γ-duals

In this section, we mention the Schauder basis and determine β- and γ-duals of the spaces $f(Q(r,s,t,u))$ and $fs(Q(r,s,t,u))$.

If a normed space X contains a sequence $b=(b_k)$ with the property that, for all $x=(x_k)\in X$, there is a unique sequence of scalars (α_n) such that

$$\Big\|x-\sum_{k=0}^{n}\alpha_k b_k\Big\|_X\to 0$$

as $n\to\infty$, then $b=(b_k)$ is called a *Schauder basis* for X.

Corollary 1 (see [3])
Almost convergent sequence space f has no Schauder basis.

Remark 1
For an arbitrary sequence space X and a triangle matrix $A=(a_{nk})$, it is known that X_A has a basis if and only if X has a basis, [17].

Remark 1 and Corollary 1 give us the next corollary.

Corollary 2
The sequence spaces $f(Q(r,s,t,u))$ and $fs(Q(r,s,t,u))$ have no Schauder basis.

Let X and Y be two arbitrary sequence spaces. Define the multiplier space $M(X,Y)$ by

$$M(X,Y) = \big\{a = (a_k) \in w : \ xa = (x_k a_k) \in Y \ \text{ for all } x = (x_k) \in X\big\}.$$

Then, the β- and γ-duals of a sequence space X are defined by

$$X^{\beta} = M(X, cs) \quad \text{and} \quad X^{\gamma} = M(X, bs),$$

respectively.

For a given infinite matrix $A = (a_{nk})$ of complex numbers, we write the followings

$$\sup_{n\in\mathbb{N}} \sum_k |a_{nk}| < \infty, \tag{3.1}$$

$$\lim_{n\to\infty} a_{nk} = \xi_k \quad \text{for each fixed } k \in \mathbb{N}, \tag{3.2}$$

$$\lim_{n\to\infty} \sum_k a_{nk} = \xi, \tag{3.3}$$

$$\lim_{n\to\infty} \sum_k |\Delta(a_{nk} - \xi_k)| = 0, \tag{3.4}$$

$$\sup_{n\in\mathbb{N}} \sum_k |\Delta a_{nk}| < \infty, \tag{3.5}$$

$$\lim_{k\to\infty} a_{nk} = 0 \quad \text{for each fixed } n \in \mathbb{N}, \tag{3.6}$$

$$\lim_{n\to\infty} \sum_k |\Delta^2 a_{nk}| = \alpha, \tag{3.7}$$

where $\Delta a_{nk} = a_{nk} - a_{n,k+1}$ and $\Delta^2 a_{nk} = \Delta(\Delta a_{nk})$. Then, we can give the next lemma.

Lemma 1
Let $A = (a_{nk})$ be an infinite matrix of complex numbers. Then, the followings hold:

(i) $A = (a_{nk}) \in (f : \ell_\infty) \Leftrightarrow$ (3.1) *holds (see [27]);*
(ii) $A = (a_{nk}) \in (f : c) \Leftrightarrow$ (3.1), (3.2), (3.3) *and* (3.4) *hold (see [27]);*
(iii) $A = (a_{nk}) \in (fs : \ell_\infty) \Leftrightarrow$ (3.5) *and* (3.6) *hold (see [3]);*
(iv) $A = (a_{nk}) \in (fs : c) \Leftrightarrow$ (3.2), (3.5), (3.6) *and* (3.7) *hold (see [26]).*

Theorem 6
Given the sets t_1, t_2, t_3, t_4, t_5, t_6 and t_7 as follows

$$t_1 = \left\{ a = (a_k) \in w : \sup_{n\in\mathbb{N}} \sum_{k=0}^{n} \left| \frac{1}{r} \sum_{j=k}^{n} \sum_{i=0}^{j-k} \sum_{v=0}^{j-k-i} \mu_1^{j-k-i-v} \mu_2^v \mu_3^i a_j \right| < \infty \right\},$$

$$t_2 = \left\{ a = (a_k) \in w : \lim_{n\to\infty} \frac{1}{r} \sum_{j=k}^{n} \sum_{i=0}^{j-k} \sum_{v=0}^{j-k-i} \mu_1^{j-k-i-v} \mu_2^v \mu_3^i a_j \text{ exists for each } k \in \mathbb{N} \right\},$$

$$t_3 = \left\{ a = (a_k) \in w : \lim_{n\to\infty} \sum_{k=0}^{n} \left[\frac{1}{r} \sum_{j=0}^{k} \sum_{i=0}^{k-j} \sum_{v=0}^{k-j-i} \mu_1^{k-j-i-v} \mu_2^v \mu_3^i \right] a_k \text{ exists} \right\},$$

$$t_4 = \left\{ a = (a_k) \in w : \lim_{n\to\infty} \sum_{k} \left| \Delta \left[\frac{1}{r} \sum_{j=k}^{n} \sum_{i=0}^{j-k} \sum_{v=0}^{j-k-i} \mu_1^{j-k-i-v} \mu_2^v \mu_3^i a_j - \xi_k \right] \right| = 0 \right\},$$

$$t_5 = \left\{ a = (a_k) \in w : \sup_{n\in\mathbb{N}} \sum_{k} \left| \Delta \left[\frac{1}{r} \sum_{j=k}^{n} \sum_{i=0}^{j-k} \sum_{v=0}^{j-k-i} \mu_1^{j-k-i-v} \mu_2^v \mu_3^i a_j \right] \right| < \infty \right\},$$

$$t_6 = \left\{ a = (a_k) \in w : \lim_{k\to\infty} \frac{1}{r} \sum_{j=k}^{n} \sum_{i=0}^{j-k} \sum_{v=0}^{j-k-i} \mu_1^{j-k-i-v} \mu_2^v \mu_3^i a_j = 0 \text{ for each } n \in \mathbb{N} \right\},$$

$$t_7 = \left\{ a = (a_k) \in w : \lim_{n\to\infty} \sum_{k} \left| \Delta^2 \left[\frac{1}{r} \sum_{j=k}^{n} \sum_{i=0}^{j-k} \sum_{v=0}^{j-k-i} \mu_1^{j-k-i-v} \mu_2^v \mu_3^i a_j \right] \right| \text{ exists} \right\},$$

where

$$\lim_{n\to\infty} \frac{1}{r} \sum_{j=k}^{n} \sum_{i=0}^{j-k} \sum_{v=0}^{j-k-i} \mu_1^{j-k-i-v} \mu_2^v \mu_3^i a_j = \xi_k$$

for all $k \in \mathbb{N}$. Then the following statements hold:

(i) $\{f(Q(r,s,t,u))\}^\beta = t_1 \cap t_2 \cap t_3 \cap t_4$;

(ii) $\{f(Q(r,s,t,u))\}^\gamma = t_1$;

(iii) $\{fs(Q(r,s,t,u))\}^\beta = t_2 \cap t_5 \cap t_6 \cap t_7$;

(iv) $\{fs(Q(r,s,t,u))\}^\gamma = t_5 \cap t_6$.

Proof. To avoid the repetition of the similar statements, we give the proof for only part (i). Let us take $a = (a_k) \in w$ and consider a sequence $x = (x_k)$ such that

$$x_k = \frac{1}{r} \sum_{j=0}^{k} \sum_{i=0}^{k-j} \sum_{v=0}^{k-j-i} \mu_1^{k-j-i-v} \mu_2^v \mu_3^i y_j$$

for all $k \in \mathbb{N}$.

Then, we have

$$\begin{aligned}\sum_{k=0}^{n} a_k x_k &= \sum_{k=0}^{n}\left[\frac{1}{r}\sum_{j=0}^{k}\sum_{i=0}^{k-j}\sum_{v=0}^{k-j-i}\mu_1^{k-j-i-v}\mu_2^{v}\mu_3^{i}y_j\right]a_k\\ &= \sum_{k=0}^{n}\left[\frac{1}{r}\sum_{j=k}^{n}\sum_{i=0}^{j-k}\sum_{v=0}^{j-k-i}\mu_1^{j-k-i-v}\mu_2^{v}\mu_3^{i}a_j\right]y_k = (Dy)_n\end{aligned}$$

for all $n \in \mathbb{N}$, where $D = (d_{nk})$ is defined by

$$d_{nk} = \begin{cases}\frac{1}{r}\sum\limits_{j=k}^{n}\sum\limits_{i=0}^{j-k}\sum\limits_{v=0}^{j-k-i}\mu_1^{j-k-i-v}\mu_2^{v}\mu_3^{i}a_j, & 0 \le k \le n,\\ 0, & k > n\end{cases}$$

for all $k, n \in \mathbb{N}$. Therefore, $ax = (a_k x_k) \in cs$ in case of $x = (x_k) \in f(Q(r,s,t,u))$ if and only if $Dy \in c$ in case of $y = (y_k) \in f$, namely $a = (a_k) \in \{f(Q(r,s,t,u))\}^{\beta}$ if and only if $D \in (f : c)$. If we combine this fact with (ii) of Lemma 1, we can see that $a = (a_k) \in \{f(Q(r,s,t,u))\}^{\beta}$ if and only if

$$\sup_{n\in\mathbb{N}}\sum_k |d_{nk}| < \infty,$$

$$\lim_{n\to\infty} d_{nk} = \xi_k \quad \text{for each fixed } k \in \mathbb{N},$$

$$\lim_{n\to\infty}\sum_k d_{nk} = \xi$$

and

$$\lim_{n\to\infty}\sum_k |\Delta(d_{nk} - \xi_k)| = 0.$$

This result gives us that $\{f(Q(r,s,t,u))\}^{\beta} = t_1 \cap t_2 \cap t_3 \cap t_4$. This completes the proof of theorem.

4. Matrix Classes

In this section, we characterize some matrix classes related to the sequence spaces $f(Q(r,s,t,u))$ and $fs(Q(r,s,t,u))$. Here and after, we use the matrices $G = (g_{nk})$ and $V = (v_{nk})$ defined by

$$g_{nk} = \frac{1}{r}\sum_{j=k}^{\infty}\sum_{i=0}^{j-k}\sum_{v=0}^{j-k-i}\mu_1^{j-k-i-v}\mu_2^{v}\mu_3^{i}a_{nj} \tag{4.1}$$

$$v_{nk} = ra_{nk} + sa_{n-1,k} + ta_{n-2,k} + ua_{n-3,k} \tag{4.2}$$

for all $n, k \in \mathbb{N}$, respectively.

Theorem 7
Let X be an arbitrary sequence space and $A = (a_{nk})$ be an infinite matrix whose entires provide the relations (4.1) and (4.2). Then, the followings hold:

(i) $A \in (f(Q(r,s,t,u)) : X) \Leftrightarrow G \in (f : X)$ *and* $\{a_{nk}\}_{k\in\mathbb{N}} \in \{f(Q(r,s,t,u))\}^{\beta}$ *for all* $n \in \mathbb{N}$*;*

(ii) $A \in (X : f(Q(r,s,t,u))) \Leftrightarrow V \in (X : f)$.

Proof. (i). Assume that $A \in (f(Q(r,s,t,u)) : X)$. Let us take an arbitrary sequence $y = (y_k) \in f$ and consider $f(Q(r,s,t,u)) \cong f$, where $y = Q(r,s,t,u)x$. Then, $GQ(r,s,t,u)$ exists and $\{a_{nk}\}_{k\in\mathbb{N}} \in \{f(Q(r,s,t,u))\}^{\beta}$ for all $n \in \mathbb{N}$, which yields that $\{g_{nk}\}_{k\in\mathbb{N}} \in \ell_1$ for all $n \in \mathbb{N}$. Therefore, Gy exists and

$$\sum_k g_{nk} y_k = \sum_k a_{nk} x_k$$

for all $n \in \mathbb{N}$, that is $Gy = Ax$. Thus, $G \in (f : X)$.

For the converse, assume that $G \in (f : X)$ and $\{a_{nk}\}_{k\in\mathbb{N}} \in \{f(Q(r,s,t,u))\}^{\beta}$ for all $n \in \mathbb{N}$. Now, we take an arbitrary sequence $x = (x_k) \in f(Q(r,s,t,u))$. Then, it is obvious that Ax exists. Moreover, we get

$$\begin{aligned}\sum_{k=0}^{\sigma} a_{nk} x_k &= \sum_{k=0}^{\sigma}\left[\frac{1}{r}\sum_{j=0}^{k}\sum_{i=0}^{k-j}\sum_{v=0}^{k-j-i}\mu_1^{k-j-i-v}\mu_2^{v}\mu_3^{i} y_j\right] a_{nk}\\ &= \sum_{k=0}^{\sigma}\left[\frac{1}{r}\sum_{j=k}^{\sigma}\sum_{i=0}^{j-k}\sum_{v=0}^{j-k-i}\mu_1^{j-k-i-v}\mu_2^{v}\mu_3^{i} a_{nj}\right] y_k\end{aligned}$$

for all $n \in \mathbb{N}$. If we pass to limit as $\sigma \to \infty$, we obtain that $Ax = Gy$. This gives us that $A \in (f(Q(r,s,t,u)) : X)$.

(ii). For any $x = (x_k) \in X$, let us consider the next equality

$$\begin{aligned}\{Q(Ax)\}_n &= r(Ax)_n + s(Ax)_{n-1} + t(Ax)_{n-2} + u(Ax)_{n-3}\\ &= \sum_k (ra_{nk} + sa_{n-1,k} + ta_{n-2,k} + ua_{n-3,k}) x_k\\ &= (Vx)_n\end{aligned}$$

for all $n \in \mathbb{N}$. By passing to the generalized limit, we get that $Ax \in f(Q(r,s,t,u))$ if and only if $Vx \in f$. This completes the proof of theorem.

Now, for the next lemma, let us write the followings by considering an infinite matrix $A = (a_{nk})$.

$$F - \lim_{n\to\infty} a_{nk} = \xi_k \quad \text{for all fixed } k \in \mathbb{N}, \tag{4.3}$$

$$F - \lim_{n\to\infty} \sum_k a_{nk} = \xi, \tag{4.4}$$

$$F-\lim_{n\to\infty}\sum_{j=0}^{n}a_{jk}=\xi_k \quad \text{for all fixed } k\in\mathbb{N}, \tag{4.5}$$

$$\sup_{n\in\mathbb{N}}\sum_k\left|\Delta\left(\sum_{j=0}^{n}a_{jk}\right)\right|<\infty, \tag{4.6}$$

$$\sup_{n\in\mathbb{N}}\sum_k\left|\sum_{j=0}^{n}a_{jk}\right|<\infty, \tag{4.7}$$

$$\sum_n a_{nk}=\xi_k \quad \text{for all fixed } k\in\mathbb{N}, \tag{4.8}$$

$$\sum_n\sum_k a_{nk}=\xi, \tag{4.9}$$

$$\lim_{n\to\infty}\sum_k\left|\Delta\left[\sum_{j=0}^{n}a_{jk}-\xi_k\right]\right|=0, \tag{4.10}$$

$$\lim_{\vartheta\to\infty}\sum_k\left|\frac{1}{\vartheta+1}\sum_{j=0}^{\vartheta}a_{n+j,k}-\xi_k\right|=0 \quad \text{uniformly in } n, \tag{4.11}$$

$$\lim_{\vartheta\to\infty}\sum_k\left|\Delta\left[\frac{1}{\vartheta+1}\sum_{j=0}^{\vartheta}a_{n+j,k}-\xi_k\right]\right|=0 \quad \text{uniformly in } n, \tag{4.12}$$

$$\lim_{\vartheta\to\infty}\sum_k\frac{1}{\vartheta+1}\left|\sum_{i=0}^{\vartheta}\Delta\left[\sum_{j=0}^{n+i}a_{jk}-\xi_k\right]\right|=0 \quad \text{uniformly in } n, \tag{4.13}$$

$$\lim_{\vartheta\to\infty}\sum_k\frac{1}{\vartheta+1}\left|\sum_{i=0}^{\vartheta}\Delta^2\left[\sum_{j=0}^{n+i}a_{jk}-\xi_k\right]\right|=0 \quad \text{uniformly in } n. \tag{4.14}$$

Lemma 2
For a given infinite matrix $A=(a_{nk})$, the followings hold:

(i) $A=(a_{nk})\in(c:f)\Leftrightarrow$ (3.1), (4.3) *and* (4.4) *hold (see [19]);*
(ii) $A=(a_{nk})\in(\ell_\infty:f)\Leftrightarrow$ (3.1), (4.3) *and* (4.11) *hold (see [14]);*
(iii) $A=(a_{nk})\in(f:f)\Leftrightarrow$ (3.1), (4.3), (4.4) *and* (4.12) *hold (see [14]);*
(iv) $A=(a_{nk})\in(f:cs)\Leftrightarrow$ (4.7), (4.8), (4.9) *and* (4.10) *hold (see [5]);*
(v) $A=(a_{nk})\in(cs:f)\Leftrightarrow$ (3.5) *and* (4.3) *hold (see [4]);*
(vi) $A=(a_{nk})\in(cs:fs)\Leftrightarrow$ (4.5) *and* (4.6) *hold (see [4]);*
(vii) $A=(a_{nk})\in(bs:f)\Leftrightarrow$ (3.5), (3.6), (4.3) *and* (4.13) *hold (see [6]);*
(viii) $A=(a_{nk})\in(bs:fs)\Leftrightarrow$ (3.6), (4.5), (4.6) *and* (4.13) *hold (see [6]);*

(ix) $A = (a_{nk}) \in (fs : f) \Leftrightarrow$ (3.6), (4.3), (4.12) *and* (4.13) *hold (see [7]);*

(x) $A = (a_{nk}) \in (fs : fs) \Leftrightarrow$ (4.5), (4.6), (4.13) *and* (4.14) *hold (see [7]).*

If we combine Lemma 1, relations (4.1), (4.2), Theorem 7 and Lemma 2, we can give two more results.

Corollary 3
Let us take the entries of the matrix $G = (g_{nk})$ instead of the entries of the matrix $A = (a_{nk})$ in (3.1)–(3.7) *and* (4.3)–(4.14). *Then the following statements hold:*

(i) $A = (a_{nk}) \in (f(Q(r,s,t,u)) : c) \Leftrightarrow \{a_{nk}\}_{k\in\mathbb{N}} \in \{f(Q(r,s,t,u))\}^{\beta}$ *for all* $n \in \mathbb{N}$ *and* (3.1), (3.2), (3.3) *and* (3.7) *hold;*

(ii) $A = (a_{nk}) \in (f(Q(r,s,t,u)) : \ell_\infty) \Leftrightarrow \{a_{nk}\}_{k\in\mathbb{N}} \in \{f(Q(r,s,t,u))\}^{\beta}$ *for all* $n \in \mathbb{N}$ *and* (3.1) *holds;*

(iii) $A = (a_{nk}) \in (f(Q(r,s,t,u)) : cs) \Leftrightarrow \{a_{nk}\}_{k\in\mathbb{N}} \in \{f(Q(r,s,t,u))\}^{\beta}$ *for all* $n \in \mathbb{N}$ *and* (4.7), (4.8), (4.9) *and* (4.10) *hold;*

(iv) $A = (a_{nk}) \in (f(Q(r,s,t,u)) : bs) \Leftrightarrow \{a_{nk}\}_{k\in\mathbb{N}} \in \{f(Q(r,s,t,u))\}^{\beta}$ *for all* $n \in \mathbb{N}$ *and* (4.8) *holds.*

Corollary 4
Let us take the entries of the matrix $V = (v_{nk})$ instead of the entries of the matrix $A = (a_{nk})$ in (3.1)–(3.7) *and* (4.3)–(4.14). *Then the following statements hold:*

(i) $A = (a_{nk}) \in (c : f(Q(r,s,t,u))) \Leftrightarrow$ (3.1), (4.3) *and* (4.4) *hold;*

(ii) $A = (a_{nk}) \in (\ell_\infty : f(Q(r,s,t,u))) \Leftrightarrow$ (3.1), (4.3) *and* (4.11) *hold;*

(iii) $A = (a_{nk}) \in (f : f(Q(r,s,t,u))) \Leftrightarrow$ (3.1), (4.3), (4.4) *and* (4.12) *hold;*

(iv) $A = (a_{nk}) \in (cs : f(Q(r,s,t,u))) \Leftrightarrow$ (3.5) *and* (4.3) *hold;*

(v) $A = (a_{nk}) \in (bs : f(Q(r,s,t,u))) \Leftrightarrow$ (3.5), (3.6), (4.3) *and* (4.13) *hold;*

(vi) $A = (a_{nk}) \in (fs : f(Q(r,s,t,u))) \Leftrightarrow$ (3.6), (4.3), (4.12) *and* (4.13) *hold;*

(vii) $A = (a_{nk}) \in (cs : fs(Q(r,s,t,u))) \Leftrightarrow$ (4.5) *and* (4.6) *hold;*

(viii) $A = (a_{nk}) \in (bs : fs(Q(r,s,t,u))) \Leftrightarrow$ (3.6), (4.5), (4.6) *and* (4.13) *hold;*

(ix) $A = (a_{nk}) \in (fs : fs(Q(r,s,t,u))) \Leftrightarrow$ (4.5), (4.6), (4.13) *and* (4.14) *hold.*

5. Conclusion

By remembering the definition of Quadruple band matrix, one can easily see that $Q(1,-3,3,-1) = \Delta^3$, $Q(r,s,t,0) = B(r,s,t)$, $Q(1,-2,1,0) = \Delta^2$, $Q(r,s,0,0) = B(r,s)$ and $Q(1,-1) = \Delta$, where Δ^3, $B(r,s,t)$, Δ^2, $B(r,s)$ and Δ are called third order difference, triple band, second order difference, double band (generalized difference) and difference matrices, in turn. Also, Quadruple band matrix is not a special case of m-th order generalized difference matrix B^m defined in [10] and is not a special case of the weighed mean matrices. Therefore, this work fills up a gap in the known literature.

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