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Multi-invertible maps and their applications

Abstract. In this article, we define multi-invertible, multivalued maps. These mappings are a natural generalization of r-maps (in particular, the singlevalued invertible maps). They have many interesting properties and applications. In this article, the multi-invertible maps are applied to the construction of morphisms and to the theory of coincidence.

1. Introduction

In mathematical literature multivalued admissible maps are known (see [4]). In [8] we defined multivalued locally admissible maps that are an essentially wider class of maps than admissible maps. In this article we define multivalued, multi-invertible maps in the context of locally admissible maps. These maps are a natural, essential generalization of r-maps (see [1]) in particular, the singlevalued invertible maps. The generalized Vietoris maps considered in the article (see [9]) are particularly multi-invertible. Multi-invertible maps have a lot of interesting properties. The composition of multi-invertible maps is multi-invertible. If a map is multi-invertible then there exists exactly one multi-inverse map. Multi-invertible maps have many applications. In this article they are applied to the construction of morphisms and to the theory of coincidence. The property of coincidence of multi-invertible maps can be applied to solving differential inclusions (see [3]). It is worth mentioning that multi-invertible maps can be also applied to the theory of multi-domination (see [9, 10]).

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2. Preliminaries

Throughout this paper all spaces are assumed to be Hausdorff topological spaces. A continuous mapping $f\colon X \to Y$ is called *perfect* if for each $y \in Y$ the set $f^{-1}(y)$ is non-empty and compact and f is a closed map. Let X and Y be two spaces and assume that for every $x \in X$ a non-empty subset $\varphi(x)$ of Y is given. In such a case we say that $\varphi\colon X \multimap Y$ is a *multivalued mapping*. For a multivalued mapping $\varphi\colon X \multimap Y$ and a subset $A \subset Y$ we let

$$\varphi_s^{-1}(A) = \{x \in X : \ \varphi(x) \subset A\},$$
$$\varphi_b^{-1}(A) = \{x \in X : \ \varphi(x) \cap A \neq \emptyset\}.$$

Let H_* be the *Čech* homology functor with compact carriers and coefficients in the field of rational numbers $\mathbb{Q}$ from the category of Hausdorff topological spaces and continuous maps to the category of a graded vector space and linear maps of degree zero. Thus $H_*(X) = \{H_q(X)\}$ is a graded vector space, $H_q(X)$ being a q-dimensional *Čech* homology group with compact carriers of X. For a continuous map $f\colon X \to Y$, $H_*(f)$ is the induced linear map $f_* = \{f_q\}$, where $f_q\colon H_q(X) \to H_q(Y)$ (see [4]).

A set X is *acyclic* if

(i) X is non-empty,
(ii) $H_q(X) = 0$ for every $q \geq 1$ and
(iii) $H_0(X) \approx \mathbb{Q}$.

A perfect map $p\colon X \to Y$ is called *Vietoris* provided for every $y \in Y$ the set $p^{-1}(y)$ is acyclic. We recall that the composition of two Vietoris mappings is a Vietoris mapping and if $p\colon X \to Y$ is a Vietoris map then $p_*\colon H_*(X) \to H_*(Y)$ is an isomorphism (see [4]). The symbol $D(X,Y)$ will denote the set of all diagrams of the form

$$X \xleftarrow{\ p\ } Z \xrightarrow{\ q\ } Y,$$

where $p\colon Z \to X$ denotes a Vietoris map and $q\colon Z \to Y$ denotes a continuous map. Each such diagram will be denoted by (p,q).

Definition 2.1 (see [4])
Let $(p_1,q_1) \in D(X,Y)$ and $(p_2,q_2) \in D(Y,T)$. The composition of diagrams

$$X \xleftarrow{\ p_1\ } Z_1 \xrightarrow{\ q_1\ } Y \xleftarrow{\ p_2\ } Z_2 \xrightarrow{\ q_2\ } T,$$

is called a diagram $(p,q) \in D(X,T)$,

$$X \xleftarrow{\ p\ } Z_1 \triangle_{q_1 p_2} Z_2 \xrightarrow{\ q\ } T,$$

where

$$Z_1 \triangle_{q_1 p_2} Z_2 = \{(z_1,z_2) \in Z_1 \times Z_2 : \ q_1(z_1) = p_2(z_2)\},$$
$$p = p_1 \circ \pi_1, \quad q = q_2 \circ \pi_2,$$
$$Z_1 \xleftarrow{\ \pi_1\ } Z_1 \triangle_{q_1 p_2} Z_2 \xrightarrow{\ \pi_2\ } Z_2,$$

$$\pi_1(z_1,z_2)=z_1 \text{ (Vietoris map)}, \quad \pi_2(z_1,z_2)=z_2 \qquad \text{for each } (z_1,z_2)\in Z.$$

It shall be written

$$(p,q)=(p_2,q_2)\circ(p_1,q_1).$$

Let $\varphi\colon X \multimap Y$ be a multivalued map. We recall that the map φ is *admissible* (resp. s-admissible) (see [4]) if there exist a Vietoris map $p\colon Z\to X$ and a continuous map $q\colon Z\to Y$ such that for each $x\in X$,

$$q(p^{-1}(x))\subset\varphi(x) \quad (\text{resp. } q(p^{-1}(x))=\varphi(x)),$$

to simplify the notation we will write $(p,q)\subset\varphi$ (resp. $(p,q)=\varphi$)).
Let $\varphi\colon X\multimap Y$ be a map and let $A\subset X$ be a non-empty set. We denote by $\varphi_A\colon A\multimap X$ a map given by the formula $\varphi_A(x)=\varphi(x)$ for each $x\in A$.

Definition 2.2
A multivalued map $\varphi\colon X\multimap Y$ is called *locally admissible* provided for any compact and non-empty set $K\subset X$ there exists an open set $V\subset X$ such that $K\subset V$ and $\varphi_V\colon V\multimap X$ is admissible.

Proposition 2.3 ([8])
Let $\varphi\colon X\multimap Y$ and $\psi\colon Y\multimap Z$ be locally admissible maps. Then $\psi\circ\varphi\colon X\multimap Z$ is locally admissible.

It is obvious that if a space X is compact then $\varphi\colon X\multimap Y$ is locally admissible if and only if $\varphi\colon X\multimap Y$ is admissible.

Definition 2.4
A topological vector space is called *Klee admissible* provided for every compact $K\subset E$ and for every open neighborhood of zero V in E there exists a continuous map $\pi_V\colon K\to E$ such that

(2.4.1) $(x-\pi_V(x))\in V$ for every $x\in K$,
(2.4.2) there exists a natural number $n=n_K$ such that $\pi_V(K)\subset E^n$, where E^n is an n-dimensional subspace of E.

It is well known that any locally convex space is Klee admissible. We will write that a space $X\in A_C$ ($X\in NA_C$) if there exists a Klee admissible space E and a closed embedding $h\colon X\to E$ such that $h(X)$ is a retract of E ($h(X)$ is a retract of some open set $U\subset E$ such that $h(X)\subset U$).

Proposition 2.5 ([4, 6])
Let $X\in NA_C$ and let $U\subset X$ be an open set. Then $U\in NA_C$.

Theorem 2.6 ([4, 6])
Let $X\in NA_C$. Consider a diagram

$$X \xleftarrow{\;p\;} Z \xrightarrow{\;q\;} X,$$

where p is Vietoris and q is compact ($\overline{q(Z)}\subset X$ is compact). Then $q_*\circ p_*^{-1}$ is a Leray endomorphism and $\Lambda(q_*\circ p_*^{-1})\neq 0$ implies that p and q have a coincidence point, that is, there is a point $z\in Z$ such that $p(z)=q(z)$.

3. Multi-invertible maps

Definition 3.1

We say that a multivalued map $\varphi\colon X \multimap Y$ is *multi-invertible* if there exists a multivalued map $\overleftarrow{\varphi}\colon Y \multimap X$ (multi-inverse map) such that the following conditions are satisfied

(3.1.1) for each $x \in X$ and for each $y \in Y$ $(y \in \varphi(x) \Rightarrow x \in \overleftarrow{\varphi}(y))$,

(3.1.2) for each $x \in X$ and for each $y \in Y$ $(x \in \overleftarrow{\varphi}(y) \Rightarrow y \in \varphi(x))$.

We observe that if $\varphi\colon X \multimap Y$ is multi-invertible then there exists exactly one multi-inverse map $\overleftarrow{\varphi}\colon Y \multimap X$ given by the formula

$$\overleftarrow{\varphi}(y) = \varphi_b^{-1}(y) = \{x \in X :\ y \in \varphi(x)\}.$$

It is clear that a multivalued map $\varphi\colon X \multimap Y$ is multi-invertible if and only if $\varphi(X) = Y$ and if a singlevalued map $f\colon X \to Y$ is invertible then $\overleftarrow{f} = f^{-1}$. Let $\varphi\colon X \multimap Y$ be a multivalued map. If φ has compact images and for every open $U \subset Y$ the set $\varphi_s^{-1}(U)$ $(\varphi_b^{-1}(Y \setminus U))$ is open (is closed), then φ is called an *upper semi-continuous mapping*; we shall write that φ is u.s.c. We will say that an u.s.c. multivalued map $\varphi\colon X \multimap Y$ is *perfect*, if for each non-empty and compact set $A \subset Y$ the set $\varphi_b^{-1}(A)$ is non-empty and compact and φ is a closed map.

Proposition 3.2

A map $\varphi\colon X \multimap Y$ is perfect if and only if φ is multi-invertible and $\overleftarrow{\varphi}$ is a perfect map.

Proof. It is obvious that

$$\varphi(A) = \overleftarrow{\varphi}_b^{-1}(A) \qquad \text{for each non-empty set } A \subset X$$

and

$$\overleftarrow{\varphi}(B) = \varphi_b^{-1}(B) \qquad \text{for each non-empty set } B \subset Y.$$

From Proposition 3.2 we get

Proposition 3.3

Let X and Y be compact spaces and let $\varphi\colon X \multimap Y$ such that $\varphi(X) = Y$. A multivalued map $\varphi\colon X \multimap Y$ is u.s.c. if and only if $\overleftarrow{\varphi}\colon Y \multimap X$ is u.s.c..

We will give a few examples. Let $\mathbb{K}^n$ be a closed ball in euclidean space $\mathbb{R}^n$ with the center of 0 and radius 1 and let $\mathbb{S}^n \subset \mathbb{K}^{n+1}$ be a sphere. We denote by $\odot$ a scalar product in $\mathbb{R}^n$ and let $I = [0, 1]$.

Example 3.4

Let $\varphi\colon \mathbb{S}^n \multimap \mathbb{S}^n$ be a multivalued map given by the formula

$$\varphi(x) = \{y \in \mathbb{S}^n :\ x \odot y = 0\} \qquad \text{for each } x \in \mathbb{S}^n.$$

We observe that φ is multi-invertible and $\overleftarrow{\varphi} = \varphi$. For $n = 2k$, φ is not admissible. Indeed, assume the contrary, i.e. that φ is admissible. Then there exist a Vietoris map $p\colon Z\to\mathbb{S}^n$ and a continuous map $q\colon Z\to\mathbb{S}^n$ such that $(p,q)\subset\varphi$. Hence, for each $z\in Z$,

$$p(z)\odot q(z)=0,\qquad \text{so } p \text{ and } q \text{ are homotopic.}$$

We have

$$\Lambda(q_*\circ p_*^{-1})=\Lambda(p_*\circ p_*^{-1})=\Lambda(Id_{H_*(\mathbb{S}^n)})=2.$$

From Theorem 2.6 there exists a point $z\in Z$ such that $p(z)=q(z)$, but it is a contradiction. From the mathematical literature we know that for $n=2k-1$ there exists a continuous map $f\colon\mathbb{S}^n\to\mathbb{S}^n$ such that, for each $x\in\mathbb{S}^n$, $x\odot f(x)=0$. Hence, φ is admissible, because $(Id_{\mathbb{S}^n},f)\subset\varphi$.

Example 3.5
Let $\varphi\colon\mathbb{S}^n\multimap\mathbb{K}^{n+1}$ be a multivalued map given by the formula

$$\varphi(x)=\{t\cdot x:\ t\in I\}.$$

The map φ is u.s.c. and for each $x\in\mathbb{S}^n$ the set $\varphi(x)$ is compact and convex, so φ is s-admissible (in particular, admissible). A multivalued map $\psi\colon\mathbb{K}^{n+1}\multimap\mathbb{S}^n$ given by the formula

$$\psi(x)=\begin{cases} x/\|x\| & \text{for } x\neq 0,\\ \mathbb{S}^n & \text{for } x=0\end{cases}$$

is multi-inverse to φ. We observe that ψ it is not an admissible map. Indeed, assume the contrary, i.e. that φ is admissible. Then there exist a Vietoris map $p\colon Z\to\mathbb{K}^{n+1}$ and a continuous map $q\colon Z\to\mathbb{S}^n$ such that $(p,q)\subset\psi$. Let $j\colon\mathbb{S}^n\to\mathbb{K}^{n+1}$ be an inclusion. We have the following diagram

$$H_*(\mathbb{S}^n)\xrightarrow{\ j_*\ }H_*(\mathbb{K}^{n+1})\xleftarrow{\ p_*\ }H_*(Z)\xrightarrow{\ q_*\ }H_*(\mathbb{S}^n).$$

Hence, it result that

$$(q_*\circ p_*^{-1})\circ j_*=Id_{H_*(\mathbb{S}^n)},$$

but it is not possible.

Definition 3.6
A locally admissible map $\varphi\colon X\multimap Y$ is *multi-invertible* if there exists a multi-inverse and locally admissible map $\overleftarrow{\varphi}\colon Y\multimap X$.

Proposition 3.7
Let $f\colon X\to Y$ be a continuous map. The map f is multi-invertible if and only if for each compact set $K\subset Y$ there exist an open neighborhood U of K and a continuous map $g\colon U\to X$ such that $f\circ g\colon U\to Y$ is a Vietoris map.

Proof. Let $f\colon X\to Y$ be a continuous and multi-invertible map. From Definition 3.6 the multi-inverse map $\overleftarrow{f}\colon Y\multimap X$ is locally admissible. Let $K\subset Y$ be a compact set. There exists an open neighborhood $U\subset Y$ of K such that $\overleftarrow{f}_U\colon U\multimap X$ is an admissible map, that is, there exists a diagram $(p,q)\in D(U,X)$ such that $(p,q)\subset\overleftarrow{f}$. Hence we have

$$q(p^{-1}(y))\subset f^{-1}(y)\qquad\text{for each } y\in U,$$

so $f\circ q=p$ is a Vietoris map. The proof in the opposite direction is obvious.

In particular, if a continuous map $f\colon X \to Y$ is an r-map, that is, there exists $g\colon Y \to X$ such that $f \circ g = Id_Y$, then it is multi-invertible.

PROPOSITION 3.8
Let $\varphi\colon X \multimap Y$ be an admissible map. Assume that there exist Vietoris maps $p_1\colon Z \to X$ and $p_2\colon Z \to Y$ such that $(p_1, p_2) \subset \varphi$. Then φ is multi-invertible.

Proof. We observe that $\varphi(X) = Y$ and $((p_1, p_2) \subset \varphi) \Leftrightarrow ((p_2, p_1) \subset \overleftarrow{\varphi})$.

We will say that $\varphi\colon X \multimap Y$ is relatively proper if for each non-empty and compact set $K \subset Y$ the set $\overline{\varphi_b^{-1}(K)}$ is non-empty and compact.

PROPOSITION 3.9
Let $\varphi\colon X \multimap Y$ be relatively proper. Assume that for each compact set $K \subset X$ there exist an open neighborhood $U \subset X$ of K, an open neighborhood $V \subset Y$ of $\varphi(K)$ and Vietoris maps $p_1\colon Z \to U$ and $p_2\colon Z \to V$ such that $(p_1, p_2) \subset \varphi_U$. Then φ is multi-invertible and $\overleftarrow{\varphi}$ is locally admissible.

Proof. It is easy to observe that φ is locally admissible. We show that the multi-inverse map $\overleftarrow{\varphi}\colon Y \multimap X$ is locally admissible. Let $K \subset Y$ be a compact set. We denote by $K_1 = \overline{\varphi_b^{-1}(K)} = \overline{\overleftarrow{\varphi}(K)}$. From the assumption the set K_1 is compact, so there exist an open neighborhood $U \subset X$ of K_1, an open neighborhood $V \subset Y$ of $\varphi(K_1)$ and Vietoris maps $p_1\colon Z \to U$ and $p_2\colon Z \to V$ such that $(p_1, p_2) \subset \varphi_U$. We have

$$K \subset \varphi(\overleftarrow{\varphi}(K)) \subset \varphi(\overline{\overleftarrow{\varphi}(K)}) \subset V.$$

Hence $(p_2, p_1) \subset \overleftarrow{\varphi}_V$ and the proof is complete.

A few obvious properties of multi-invertible mappings will follow, which do not require proof.

PROPOSITION 3.10
Let $\varphi, \psi\colon X \multimap Y$, $\eta\colon Y \multimap Z$ and $\theta\colon T \multimap S$ be multi-invertible maps. Then we have

(3.10.1) *for each $x \in X$ and $y \in Y$ $y \in \varphi(\overleftarrow{\varphi}(y))$ and $x \in \overleftarrow{\varphi}(\varphi(x))$,*
(3.10.2) $\overleftarrow{\eta \circ \varphi} = \overleftarrow{\varphi} \circ \overleftarrow{\eta}$,
(3.10.3) $\overleftarrow{(\overleftarrow{\varphi})} = \varphi$,
(3.10.4) $\overleftarrow{\varphi \times \theta} = \overleftarrow{\varphi} \times \overleftarrow{\theta}$, *where the map $\varphi \times \theta\colon X \times T \multimap Y \times S$ given by the formula*

$$(\varphi \times \theta)(x, t) = \varphi(x) \times \theta(t) \qquad \text{for each } (x, t) \in (X \times T),$$

(3.10.5) *if $\varphi(x) \cap \psi(x) \neq \emptyset$ for each $x \in X$ and if $\Theta\colon X \multimap Y$ is a map given by the formula*

$$\Theta(x) = \varphi(x) \cap \psi(x) \qquad \text{for each } x \in X,$$

then Θ is multi-invertible and $\overleftarrow{\Theta}\colon Y \multimap X$ is given by the formula

$$\overleftarrow{\Theta}(y) = \overleftarrow{\varphi}(y) \cap \overleftarrow{\psi}(y) \qquad \text{for each } y \in Y.$$

Remark 3.11
Let

$$\begin{array}{ccc} X & \xrightarrow{f_1} & T \\ \big\downarrow{\scriptstyle \Delta_f} & & \big\downarrow{\scriptstyle Id_T} \\ Y & \xrightarrow{f_2} & T \end{array}$$

be a commutative diagram, where f_1 and f_2 are continuous maps such that $f_1(X) = f_2(Y)$ and $\Delta_f \colon X \multimap Y$ is a multivalued map given by the formula

$$\Delta_f(x) = f_2^{-1}(f_1(x)) \qquad \text{for each } x \in X. \tag{1}$$

We observe that the map Δ_f has a closed graph. Furthermore the map Δ_f is multi-invertible and $\overleftarrow{\Delta}_f \colon Y \multimap X$ is given by the formula

$$\overleftarrow{\Delta}_f(y) = f_1^{-1}(f_2(y)) \qquad \text{for each } y \in Y. \tag{2}$$

Moreover, if f_1 and f_2 are perfect maps then Δ_f is u.s.c.. Let $(p_1, q_1), (p_2, q_2) \in D(X, Y)$ be such that $q_1 \circ \overleftarrow{p_1} = q_2 \circ \overleftarrow{p_2}$ and let

$$\begin{array}{ccccc} X & \xleftarrow{p_1} & Z_1 & \xrightarrow{q_1} & Y \\ \big\downarrow{\scriptstyle Id_X} & & \big\downarrow{\scriptstyle \Delta_{pq}} & & \big\downarrow{\scriptstyle Id_Y} \\ X & \xleftarrow{p_2} & Z_2 & \xrightarrow{q_2} & Y \end{array}$$

be commutative diagrams, where $\Delta_{pq} \colon Z_1 \multimap Z_2$ is a multivalued map given by the formula

$$\Delta_{pq}(x) = \Delta_p(x) \cap \Delta_q(x) \qquad \text{for each } x \in X$$

(see (1)). It is easy to show that Δ_{pq} is well defined. From Proposition 3.10 (see (3.10.5)) the map Δ_{pq} is multi-invertible (see (2)) and u.s.c (see [4, 2]). It is clear that

$$p_2 \circ \Delta_{pq} = p_1 \quad \text{and} \quad q_2 \circ \Delta_{pq} = q_1.$$

We observe that if $\varphi \colon Z_1 \multimap Z_2$ is a multivalued map such that

$$p_2 \circ \varphi = p_1 \quad \text{and} \quad q_2 \circ \varphi = q_1,$$

then $\varphi(x) \subset \Delta_{pq}(x)$ for each $x \in X$.

(i) If there exists a homeomorphism $h \colon Z_1 \to Z_2$ such that

$$p_2 \circ h = p_1 \quad \text{and} \quad q_2 \circ h = q_1,$$

then $(p_1, q_1) \approx_{K1} (p_2, q_2)$ (in the sense of Kryszewski (see [7])) and $(Id_{Z_1}, h) \subset \Delta_{pq}$ and $(Id_{Z_2}, h^{-1}) \subset \overleftarrow{\Delta}_{pq}$, where $h^{-1} \colon Z_2 \to Z_1$ is an inverse homeomorphism to h.

(ii) If there exist continuous maps $f\colon Z_1 \to Z_2$ and $g\colon Z_2 \to Z_1$ such that

$$p_2 \circ f = p_1, \quad q_2 \circ f = q_1, \quad p_1 \circ g = p_2 \quad \text{and} \quad q_1 \circ g = q_2,$$

then $(p_1, q_1) \approx_G (p_2, q_2)$ (in the sense of Górniewicz (see [5])) and $(Id_{Z_1}, f) \subset \Delta_{pq}$ and $(Id_{Z_2}, g) \subset \overleftarrow{\Delta}_{pq}$.

(iii) If there exist Vietoris maps $v_1\colon Z \to Z_1$ and $v_2\colon Z \to Z_2$ such that

$$p_2 \circ v_2 = p_1 \circ v_1 \quad \text{and} \quad q_2 \circ v_2 = q_1 \circ v_1, \tag{3}$$

then $(p_1, q_1) \approx_{K2} (p_2, q_2)$ (in the sense of Kryszewski (see [7, 11])) and $(v_1, v_2) \subset \Delta_{pq}$ and $(v_2, v_1) \subset \overleftarrow{\Delta}_{pq}$.

4. Admissible morphisms

Remark 3.11 justify the following definition.

Definition 4.1
Let $(p_1, q_1), (p_2, q_2) \in D(X, Y)$ and let

$$X \xleftarrow{\;p_1\;} Z_1 \xrightarrow{\;q_1\;} Y, \quad X \xleftarrow{\;p_2\;} Z_2 \xrightarrow{\;q_2\;} Y.$$

We will say that the diagrams (p_1, q_1) and (p_2, q_2) are in relation in the set $D(X, Y)$ (we will write $(p_1, q_1) \approx_{ad} (p_2, q_2)$) if there exists an admissible and multi-invertible map $\varphi\colon Z_1 \multimap Z_2$ such that $\overleftarrow{\varphi}$ is admissible and the following diagram is commutative

$$\begin{array}{ccccc} X & \xleftarrow{\;p_1\;} & Z_1 & \xrightarrow{\;q_1\;} & Y \\ \big\downarrow{\scriptstyle Id_X} & & \big\downarrow{\scriptstyle \varphi} & & \big\downarrow{\scriptstyle Id_Y} \\ X & \xleftarrow{\;p_2\;} & Z_2 & \xrightarrow{\;q_2\;} & Y \end{array}$$

that is

$$p_2 \circ \varphi = p_1 \quad \text{and} \quad q_2 \circ \varphi = q_1.$$

Proposition 4.2
The relation in the set $D(X, Y)$ introduced in Definition 4.1 is an equivalence relation.

Proof. In the proof of reflexivity of the relation, it is enough to assume that $Z_1 = Z_2$ and $\varphi = Id_{Z_1}$. We observe that if

$$p_2 \circ \varphi = p_1 \quad \text{and} \quad q_2 \circ \varphi = q_1$$

then

$$p_1 \circ \overleftarrow{\varphi} = p_2 \quad \text{and} \quad q_1 \circ \overleftarrow{\varphi} = q_2, \tag{4}$$

where $\varphi\colon Z_1 \multimap Z_2$ is a multi-invertible and admissible map and $\overleftarrow{\varphi}$ is admissible. Hence, the relation is symmetric.

It shall be now proven that the relation is transitive. Suppose that $(p_1, q_1) \approx_{ad} (p_2, q_2)$ and $(p_2, q_2) \approx_{ad} (p_3, q_3)$. Then from the assumption we have the following commutative diagram

$$\begin{array}{ccccc} X & \xleftarrow{p_1} & Z_1 & \xrightarrow{q_1} & Y \\ \downarrow{\scriptstyle Id} & & \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle Id} \\ X & \xleftarrow{p_2} & Z_2 & \xrightarrow{q_2} & Y \\ \downarrow{\scriptstyle Id} & & \downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle Id} \\ X & \xleftarrow{p_3} & Z_3 & \xrightarrow{q_3} & Y \end{array}$$

that is

$$p_2 \circ \varphi_1 = p_1, \quad q_2 \circ \varphi_1 = q_1, \quad p_3 \circ \varphi_2 = p_2, \quad q_3 \circ \varphi_2 = q_2,$$

where φ_1 and φ_2 are admissible and multi-invertible maps. Let $\varphi = \varphi_2 \circ \varphi_1$. By Proposition 3.10 (see (3.10.2)) φ is an admissible and multi-invertible map. We have

$$p_3 \circ \varphi = p_3 \circ (\varphi_2 \circ \varphi_1) = p_1 \quad \text{and} \quad q_3 \circ \varphi = q_3 \circ (\varphi_2 \circ \varphi_1) = q_1$$

and the proof is complete.

Proposition 4.3

Let $(p_1, q_1), (p_2, q_2) \in D(X, Y)$. The relation $\approx_{ad}$ in the set $D(X, Y)$ satisfies the following conditions

(4.3.1) $((p_1, q_1) \approx_{ad} (p_2, q_2)) \Rightarrow (q_1 \circ \overleftarrow{p_1} = q_2 \circ \overleftarrow{p_2})$,

(4.3.2) $((p_1, q_1) \approx_{ad} (p_2, q_2)) \Rightarrow (q_{1*} \circ p_{1*}^{-1} = q_{2*} \circ p_{2*}^{-1})$,

(4.3.3) *let $(p_3, q_3), (p_4, q_4) \in D(Y, T)$, then*

$$((p_1, q_1) \approx_{ad} (p_2, q_2) \text{ and } (p_3, q_3) \approx_{ad} (p_4, q_4)) \\ \Rightarrow (((p_3, q_3) \circ (p_1, q_1)) \approx_{ad} ((p_4, q_4) \circ (p_2, q_2))).$$

Proof. (4.3.1). Let $(p_1, q_1) \approx_{ad} (p_2, q_2)$. Then there exists a multi-invertible and admissible map φ such that $p_2 \circ \varphi = p_1$ and $q_2 \circ \varphi = q_1$. We observe that if $p_2 \circ \varphi = p_1$ then $\overleftarrow{\varphi} \circ \overleftarrow{p_2} = \overleftarrow{p_1}$. Hence we have

$$q_1 \circ \overleftarrow{p_1} = q_1 \circ (\overleftarrow{\varphi} \circ \overleftarrow{p_2}) = (q_1 \circ \overleftarrow{\varphi}) \circ \overleftarrow{p_2} = q_2 \circ \overleftarrow{p_2} \qquad \text{(see (4))}.$$

(4.3.2). Let $(p_1, q_1) \approx_{ad} (p_2, q_2)$. Then there exists a multi-invertible and admissible map φ such that $p_2 \circ \varphi = p_1$ and $q_2 \circ \varphi = q_1$. Let $(r, s) \subset \overleftarrow{\varphi}$ then (see (4)),

$$q_1 \circ s = q_2 \circ r \quad \text{and} \quad p_1 \circ s = p_2 \circ r.$$

Hence

$$q_{1*} \circ s_* = q_{2*} \circ r_* \quad \text{and} \quad p_{1*} \circ s_* = p_{2*} \circ r_*.$$

We observe that s_* is an isomorphism, so

$$p_{1*} = p_{2*} \circ r_* \circ s_*^{-1}.$$

We have

$$\begin{aligned} q_{1*} \circ p_{1*}^{-1} &= q_{1*} \circ (p_{2*} \circ r_* \circ s_*^{-1})^{-1} = q_{1*} \circ (s_* \circ r_*^{-1} \circ p_{2*}^{-1}) \\ &= (q_{1*} \circ s_*) \circ r_*^{-1} \circ p_{2*}^{-1} = q_{2*} \circ p_{2*}^{-1}. \end{aligned}$$

(4.3.3). We have the following commutative diagrams

$$\begin{array}{ccccccccc} X & \xleftarrow{p_1} & Z_1 & \xrightarrow{q_1} & Y & \xleftarrow{p_3} & Z_3 & \xrightarrow{q_3} & T \\ \downarrow{\scriptstyle Id_X} & & \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle Id_Y} & & \downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle Id_T} \\ X & \xleftarrow{p_2} & Z_2 & \xrightarrow{q_2} & Y & \xleftarrow{p_4} & Z_4 & \xrightarrow{q_4} & T \end{array}$$

that is

$$p_2 \circ \varphi_1 = p_1, \quad q_2 \circ \varphi_1 = q_1, \quad p_4 \circ \varphi_2 = p_3, \quad q_4 \circ \varphi_2 = q_3.$$

We make the following diagram (see Definition 2.1)

$$\begin{array}{ccccc} X & \xleftarrow{p} & Z_1 \triangle_{q_1 p_3} Z_3 & \xrightarrow{q} & T \\ \downarrow{\scriptstyle Id_X} & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle Id_T}, \\ X & \xleftarrow{p'} & Z_2 \triangle_{q_2 p_4} Z_4 & \xrightarrow{q'} & T \end{array}$$

where $(p, q) = (p_3, q_3) \circ (p_1, q_1)$, $(p', q') = (p_4, q_4) \circ (p_2, q_2)$ and $\psi(z_1, z_3) = \varphi_1(z_1) \times \varphi_2(z_3)$ for each $(z_1, z_3) \in Z_1 \triangle_{q_1 p_3} Z_3$. First we need to prove that the map ψ is well defined. Let $(z_1, z_3) \in Z_1 \triangle_{q_1 p_3} Z_3$ and $(z_2, z_4) \in \varphi_1(z_1) \times \varphi_2(z_3)$. We have

$$q_2(z_2) = q_1(z_1) = p_3(z_3) = p_4(z_4).$$

It is clear that (see (3.10.4)) the map ψ is multi-invertible, admissible and $\overleftarrow{\psi}$ given by the formula

$$\overleftarrow{\psi}(z_2, z_4) = \overleftarrow{\varphi_2}(z_2) \times \overleftarrow{\varphi_4}(z_4)$$

is admissible. We will show now that the above diagram is commutative. Let $f_1\colon Z_1 \triangle_{q_1 p_3} Z_3 \to Z_1$, $f_3\colon Z_1 \triangle_{q_1 p_3} Z_3 \to Z_3$, $f_2\colon Z_2 \triangle_{q_2 p_4} Z_4 \to Z_2$, $f_4\colon Z_2 \triangle_{q_2 p_4} Z_4 \to Z_4$ be projections (see Definition 2.1). Note that f_1 and f_2 are Vietoris mappings. We recall that by Definition 2.1 we have

$$p = p_1 \circ f_1 \quad q = q_3 \circ f_3, \quad p' = p_2 \circ f_2, \quad q' = q_4 \circ f_4.$$

Let $(z_1, z_3) \in Z_1 \triangle_{q_1 p_3} Z_3$ and $(z_2, z_4) \in \varphi_1(z_1) \times \varphi_2(z_3)$. Thus

$$p'(z_2, z_4) = p_2(f_2(z_2, z_4)) = p_2(z_2) = p_1(z_1) = p_1(f_1(z_1, z_3)) = p(z_1, z_3)$$

and similarly

$$q'(z_2, z_4) = q_4(f_4(z_2, z_4)) = q_4(z_4) = q_3(z_3) = q_3(f_3(z_1, z_3)) = q(z_1, z_3)$$

and the proof is complete.

The set of the class of the abstraction of the relation $\approx_{ad}$ will be denoted by the symbol
$$M_{ad}(X,Y) = D(X,Y)_{/\approx_{ad}}.$$
The elements of the set $M_{ad}(X,Y)$ will be called admissible morphisms and denoted by $\varphi_{ad}, \psi_{ad}, \ldots$. The following denotation is assumed

$$\varphi_{ad} = [(p,q)]_{ad} \qquad \text{(we write } (p,q) \in \varphi_{ad}\text{)},$$

where the diagram (p,q) is representative of the class of the abstraction $[(p,q)]_{ad}$ in the relation $\approx_{ad}$. We recall that a multivalued u.s.c. map $\varphi\colon X \multimap Y$ is *acyclic* if for each $x \in X$ the set $\varphi(x)$ is acyclic. The acyclic map φ is determined by an admissible morphism $\varphi_{ad} = [(p_\varphi, q_\varphi)]_{ad} \in M_{ad}(X,Y)$, where

$$X \xleftarrow{\;p_\varphi\;} \Gamma_\varphi \xrightarrow{\;q_\varphi\;} Y,$$

$\Gamma_\varphi = \{(x,y) \in X \times Y :\ y \in \varphi(x)\}$, $p_\varphi(x,y) = x$ (Vietoris map), $q_\varphi(x,y) = y$ for each $(x,y) \in \Gamma_\varphi$ such that for each $x \in X$,

$$q_\varphi(p_\varphi^{-1}(x)) = \varphi(x).$$

For singlevalued mappings, there is the following fact (see [11]).

Proposition 4.4
Let $f\colon X \to Y$ be a continuous mapping and let $(p,q) \in D(X,Y)$, where

$$X \xleftarrow{\;p\;} Z \xrightarrow{\;q\;} Y.$$

Then the following conditions are equivalent

(4.4.1) $q = f \circ p$,
(4.4.2) $(p,q) \approx_{ad} (Id, f)$,
(4.4.3) $q(p^{-1}(x)) = f(x)$ *for each* $x \in X$.

Proof. (4.4.1) $\Rightarrow$ (4.4.2). There is the following commutative diagram

$$\begin{array}{ccccc}
X & \xleftarrow{\;p\;} & Z & \xrightarrow{\;q\;} & Y \\
\big\downarrow{\scriptstyle Id_X} & & \big\downarrow{\scriptstyle p} & & \big\downarrow{\scriptstyle Id_Y}. \\
X & \xleftarrow{\;Id_X\;} & X & \xrightarrow{\;f\;} & Y
\end{array}$$

Let $\varphi = p$. Then φ is a multi-invertible and admissible map and $\overleftarrow{\varphi}$ is admissible. Hence $(p,q) \approx_{ad} (Id, f)$.

(4.4.2) $\Rightarrow$ (4.4.3). This implication is the result of Proposition 4.3 (see 4.3.1).

(4.4.3) $\Rightarrow$ (4.4.1). Let $(p,q) \in D(X,Y)$ such that for each $x \in X$ $q(p^{-1}(x)) = f(x)$ and let $z \in Z$. Then there exists a point $x_1 \in X$ such that $z \in p^{-1}(x_1)$. Hence we get
$$q(z) = f(x_1) = f(p(z)),$$

and the proof is complete.

Let $(p_1, q_1), (p_2, q_2) \in D(X, Y)$ and let $(p_1, q_1) \approx_{K2} (p_2, q_2)$ (see (3)). If $\varphi = v_2 \circ \overleftarrow{v_1}$ then $\overleftarrow{\varphi} = v_1 \circ \overleftarrow{v_2}$ and

$$p_2 \circ \varphi = p_1 \quad \text{and} \quad q_2 \circ \varphi = q_1.$$

Hence $(p_1, q_1) \approx_{ad} (p_2, q_2)$.

Example 4.5
Let $J = [-1, 1]$.

$$\begin{array}{ccccc} I & \xleftarrow{p_1} & I \times J & \xrightarrow{q_1} & I \\ \big\downarrow{\scriptstyle Id_I} & & \big\downarrow{\scriptstyle \varphi} & & \big\downarrow{\scriptstyle Id_I}, \\ I & \xleftarrow{p_2} & I \times I & \xrightarrow{q_2} & I \end{array}$$

where $p_1(x, y) = x$, $q_1(x, y) = y^2$ for each $(x, y) \in I \times J$, $p_2(x, y) = x$, $q_2(x, y) = y$ for each $(x, y) \in I \times I$ and $\varphi(x, y) = (x, y^2)$ for each $(x, y) \in I \times J$. It is clear that

$$(p_1, q_1) \approx_{ad} (p_2, q_2).$$

Assume that $(p_1, q_1) \approx_{K2} (p_2, q_2)$. Then there exist Vietoris maps $v_1 \colon Z \to I \times J$ and $v_2 \colon Z \to I \times I$ such that $q_1 \circ v_1 = q_2 \circ v_2$. Let $y \in I$. We have

$$q_1^{-1}(y) \xleftarrow{v_1} v_1^{-1}(q_1^{-1}(y)) = v_2^{-1}(q_2^{-1}(y)) \xrightarrow{v_2} q_2^{-1}(y).$$

Hence, $H_*(q_1^{-1}(y)) \cong H_*(q_2^{-1}(y))$ for each $y \in I$, but this is not possible.

Definition 4.6
For any $\varphi_{ad} \in M_{ad}(X, Y)$, the set $\varphi(x) = q(p^{-1}(x))$ where $\varphi_{ad} = [(p, q)]_{ad}$ is called an *image of point x in an admissible morphism φ_{ad}*.

We denote by

$$\varphi \colon X \to_{ad} Y$$

a multivalued map (see, Definition 4.6) determined by an admissible morphism $\varphi_{ad} = [(p, q)]_{ad} \in M_{ad}(X, Y)$.

Let **TOP** denote categories in which Hausdorff topological spaces are objects and continuous mappings are category mappings. Let $\mathbf{TOP_{ad}}$ denote categories in which Hausdorff topological spaces are objects and multivalued maps determined by admissible morphisms are category mappings. According to Proposition 4.3, (4.4.3) the category of $\mathbf{TOP_{ad}}$ is well defined and $\mathbf{TOP} \subset \mathbf{TOP_{ad}}$. Let $\mathbf{VECT_G}$ denote categories in which linear graded vector spaces are objects and linear mappings of degree zero are category mappings.

Theorem 4.7 (see [12])
The mapping $\widetilde{\mathbf{H}}_* \colon \mathbf{TOP_{ad}} \to \mathbf{VECT_G}$ given by the formula

$$\widetilde{\mathbf{H}}_*(\varphi) = q_* \circ p_*^{-1},$$

where φ is a multivalued map determined by $\varphi_{ad} = [(p, q)]_{ad}$ is a functor and the extension of the functor of the Čech homology $\mathbf{H}_* \colon \mathbf{TOP} \to \mathbf{VECT_G}$.

5. The strongly acyclic spaces

We will say that a space X is strongly acyclic if for each compact set $K \subset X$ there exists a compact and acyclic set $A \subset X$ such that $K \subset A$. We observe that from the construction of the Čech homology with compact carriers, we get:

Proposition 5.1
If X is a strongly acyclic space then X is an acyclic space.

Proof. Let

$$C(X) = \{K \subset X : \ K \text{ is compact}\}$$

and let

$$CA(X) = \{A \subset X : \ A \text{ is compact and acyclic}\}.$$

From the assumption the set $CA(X)$ is cofinal in the set $C(X)$ and the proof is complete.

The following fact results from the Mazur's Lemma.

Proposition 5.2
If E is a Banach space then it is a strongly acyclic space.

Proposition 5.3
Let $\{X_t, \pi_s^t, \Sigma\}$ be an inverse system, where Σ is a directed set and for each $t \in \Sigma$ a space X_t is strongly acyclic. Assume that for each $t \in \Sigma$ and for any compact set $K_t \subset X_t$ there exists an acyclic set $A_t \subset X_t$ such that $K_t \subset A_t$ and $\{A_t, (\pi_A)_s^t, \Sigma\}$ is an inverse system, where for $s \leq t$ the map $(\pi_A)_s^t$ is a restriction of π_s^t. Then a space

$$Y = \varprojlim \{X_t, \pi_s^t, \Sigma\}$$

is strongly acyclic.

Proof. Let $K \subset Y$ be a compact set and let $t \in \Sigma$. We denote by $f_t \colon Y \to X_t$ a restriction of projection map. The set $K_t = f_t(K) \subset X_t$ is compact. From the mathematical literature (see [2]) we know that

$$K = \varprojlim \{K_t, (\pi_K)_s^t, \Sigma\},$$

where for $s \leq t$ the map $(\pi_K)_s^t$ is a restriction of π_s^t. In turn, from the assumption there exists an acyclic and compact set $A_t \subset X_t$ such that $K_t \subset A_t$ and $\{A_t, (\pi_A)_s^t, \Sigma\}$ is an inverse system, where for $s \leq t$ the map $(\pi_A)_s^t$ is a restriction of π_s^t. The set

$$A = \varprojlim \{A_t, (\pi_A)_s^t, \Sigma\} \subset Y$$

is compact and from the continuity of the Čech homology it results that A is acyclic. It is clear that $K \subset A$ and the proof is complete.

Proposition 5.4
If $X_1, X_2, \ldots, X_n$ are strongly acyclic spaces then $X_1 \times X_2 \times \cdots \times X_n$ is a strongly acyclic space.

Proof. Let X_1, X_2, ..., X_n be strongly acyclic spaces and let $K \subset X_1 \times X_2 \times \cdots \times X_n$ be a compact set. We denote by $K_i = \pi_i(K)$, where $\pi_i \colon X_1 \times X_2 \times \cdots \times X_n \to X_i$ is a projection, $i = 1, 2, \ldots, n$. The set K_i is compact, so there exists an acyclic and compact set $A_i \subset X_i$ such that $K_i \subset A_i$, $i = 1, 2, \ldots, n$. Let $A = A_1 \times A_2 \times \cdots \times A_n$. Then A is compact and acyclic (see [4]), $K \subset A$ and the proof is complete.

From Proposition 5.3 and Proposition 5.4 we get the following fact.

Proposition 5.5
Let S be a non-empty set and let for each $s \in S$ a space X_s be strongly acyclic. Then the cartesian product

$$X = \prod_{s \in S} X_s$$

is a strongly acyclic space.

Proof. Let $\Sigma = \{\xi \subset S : \ \xi \text{ is a finite set}\}$. Then $(\Sigma, \leq)$ is a directed set, where $\leq$ is an inclusion. From the mathematical literature we know that

$$X = \lim_{\leftarrow} \{Y_\xi, \pi^\xi_\zeta, \Sigma\},$$

where $Y_\xi = X_{s_1} \times X_{s_2} \times \cdots \times X_{s_n}$, $\xi = \{s_1, s_2, \ldots, s_n\} \subset S$ and for each $\zeta \leq \xi$, $\pi^\xi_\zeta \colon Y_\xi \to Y_\zeta$ is a projection. From Proposition 5.4 the space Y_ξ for each $\xi \in \Sigma$ is strongly acyclic. We observe that the inverse system satisfies the assumption of Proposition 5.3 (see proof of Proposition 5.4) and the proof is complete.

Proposition 5.6
Let Σ be a non-empty, directed set and let E_t be a Banach space for each $t \in \Sigma$. Let $\{E_t, \pi^t_s, \Sigma\}$ be an inverse system. Assume that

$$E = \lim_{\leftarrow} \{E_t, \pi^t_s, \Sigma\}$$

is a linear space. Then E is a strongly acyclic space.

Proof. Let $K \subset E$ be a compact set and let $t \in \Sigma$. We denote by $f_t \colon E \to E_t$ a restriction of a projection map. The set $K_t = f_t(K) \subset E_t$ is compact. We have (see proof of Proposition 5.3),

$$K = \lim_{\leftarrow} \{K_t, (\pi_K)^t_s, \Sigma\},$$

where for $s \leq t$ the map $(\pi_K)^t_s$ is a restriction of π^t_s. From the assumption E_t is a Banach space, so the set $\overline{\text{conv}}(K_t) \subset E_t$ is compact and convex. Let

$$A = \prod_{t \in \Sigma} \overline{\text{conv}}(K_t).$$

Then A is compact and convex. The space E is a closed subset in the cartesian product $\prod_{t \in \Sigma} E_t$, so the set $A \cap E \subset E$ is compact and, from the assumption, is convex. Hence the set $f_t(A \cap E) = A_t \subset E_t$ is compact and convex (in particular, acyclic), $K_t \subset A_t$ and $\{A_t, (\pi_A)^t_s, \Sigma\}$ is an inverse system, where for $s \leq t$ the map $(\pi_A)^t_s$ is a restriction of π^t_s. From Proposition 5.3 it results that E is strongly acyclic and the proof is complete.

The next fact is obvious.

Proposition 5.7
Let X and Y be homeomorphic spaces. The space X is strongly acyclic if and only if the space Y is strongly acyclic.

We will give the following important example.

Example 5.8
By $C^k([0,m],\mathbb{R}^n)$, where $m\in\mathbb{N}$, $k=0,1,\ldots$, we denote the Banach space of all C^k-functions with the usual maximum norm

$$\|x\|_m=\sum_{i=0}^{k}\max\{\|x^{(i)}(t)\|,\ t\in[0,m]\}.$$

Here $x^{(k)}$ denotes the k-th derivative of x and we also put $x^{(1)}=x'$, $x^{(0)}=x$. Let $C^k([0,\infty],\mathbb{R}^n)$ be a Fréchet space of all C^k-functions with the metric

$$d(x,y)=\sum_{m=1}^{\infty}2^{-m}\frac{\|x-y\|_m}{1+\|x-y\|_m}.$$

Let $\{C^k_m,\pi^p_m,\mathbb{N}\}$ be an inverse system, where $\pi^p_m=x|_{[0,m]}$ for every $x\in C^k_p$. One can easily check that

$$E=\lim_{\leftarrow}\{C^k_m,\pi^p_m,\mathbb{N}\}\text{ is homeomorphic to }C^k([0,\infty],\mathbb{R}^n)$$

and E is a linear space. From Proposition 5.6 and Proposition 5.7 the space $C^k([0,\infty],\mathbb{R}^n)$ is strongly acyclic.

6. The points of coincidence

We observe that a map $\varphi\colon X\multimap Y$ is admissible if and only if there exists a map $\Delta_\varphi\colon X\to_{ad}Y$ such that $\Delta_\varphi(x)\subset\varphi(x)$ (we write $\Delta_\varphi\subset\varphi$) for each $x\in X$. We say that a map φ is compact if $\overline{\varphi(X)}\subset Y$ is a compact set. Let $\varphi\colon X\to_{ad}X$. By the symbol $\Lambda(\varphi)$ we will denote a generalized Lefschetz number of φ (see [4]), that is

$$\Lambda(\varphi)=\Lambda(\varphi_*)=\Lambda(q_*\circ p_*^{-1})\quad\text{(provided that it is well defined),}$$

where $(p,q)\in\varphi_{ad}$ (see Proposition 4.3). Let $\varphi,\psi\colon X\multimap Y$ be multivalued maps. We recall that the maps φ and ψ have a coincidence point if there exists a point $x\in X$ such that

$$\varphi(x)\cap\psi(x)\neq\emptyset.$$

Let $\varphi\colon X\to_{ad}X$ be a multivalued map given by $\varphi_{ad}=[(p,q)]_{ad}\in M_{ad}(X,Y)$. It is easy to see that $p,q\colon Z\to X$ have a coincidence point if and only if there exists a fixed point of φ, that is, there exists $x_0\in X$ such that $x_0\in\varphi(x_0)$.

Theorem 6.1
Let $\varphi\colon X \multimap Y$ be a multi-invertible and locally admissible map and let $\psi\colon X \multimap Y$ be a compact and locally admissible map. Let $X \in NA_C$ ($Y \in NA_C$). Then there exists an open set $U \subset X$ ($U \subset Y$) and $\Delta\colon U \to_{ad} U$, $\Delta \subset (\overleftarrow{\varphi} \circ \psi)_U$ ($\Delta \subset (\psi \circ \overleftarrow{\varphi})_U$) such that $\Lambda(\Delta)$ is well defined and if $\Lambda(\Delta) \neq 0$ then φ and ψ have a coincidence point.

Proof. Let $X \in NA_C$. From the assumption the map $\overleftarrow{\varphi}\colon Y \multimap X$ is locally admissible. Let $\psi\colon X \multimap Y$ be a compact and locally admissible map. The set $K_1 = \overline{\psi(X)} \subset Y$ is compact, so there exists an open neighborhood $V \subset Y$ of K_1 such that $\overleftarrow{\varphi}_V\colon V \multimap X$ is admissible. Hence, there exists a multivauled map $\Phi_V\colon V \to_{ad} X$ such that $\Phi_V \subset \overleftarrow{\varphi}_V$. Let $K = \Phi_V(K_1) \subset X$. It is clear that K is compact. There exists an open neighborhood $U \subset X$ of K such that $\psi_U\colon U \multimap V \subset Y$ is admissible. Let $\Psi_U\colon U \to_{ad} V$ be a map such that $\Psi_U \subset \psi_U$. We have the following diagram

$$U \xrightarrow{\Psi_U} V \xrightarrow{\Phi_V} X.$$

Let $\Delta = \Phi_V \circ \Psi_U$. We observe that Δ is compact, $\overline{\Delta(U)} \subset U$ and $\Delta \subset (\overleftarrow{\varphi} \circ \psi)_U$. Hence and from Proposition 2.5 and Theorem 2.6 $\Lambda(\Delta)$ is well defined. Assume that $\Lambda(\Delta) \neq 0$ then there exists a point $x \in U$ such that

$$x \in \Delta(x) \subset \overleftarrow{\varphi}(\psi(x)).$$

There exists a point $y \in \psi(x)$ such that $x \in \overleftarrow{\varphi}(y)$. Hence, $y \in \varphi(x)$, so

$$\varphi(x) \cap \psi(x) \neq \emptyset.$$

Let $Y \in NA_C$ and let $\psi\colon X \multimap Y$ be a compact and locally admissible map. From Proposition 2.3 the map $\psi \circ \overleftarrow{\varphi}\colon Y \multimap Y$ is locally admissible. From the assumption the set $K = \overline{\psi(\overleftarrow{\varphi}(Y))}$ is compact, so there exists an open neighborhood $U \subset Y$ of K such that $(\psi \circ \overleftarrow{\varphi})_U\colon U \multimap U \subset Y$ is admissible. Hence, there exists $\Delta\colon U \to_{ad} U$ such that $\Delta \subset (\psi \circ \overleftarrow{\varphi})_U$. It is obvious that Δ is compact and $\Lambda(\Delta)$ is well defined. Assume that $\Lambda(\Delta) \neq 0$ then there exists a point $y \in U$ such that

$$y \in \Delta(y) \subset \psi(\overleftarrow{\varphi}(y)).$$

There exists a point $x \in \overleftarrow{\varphi}(y)$ such that $y \in \psi(x)$. Hence, $y \in \varphi(x)$, so

$$\varphi(x) \cap \psi(x) \neq \emptyset$$

and the proof is complete.

The following fact results from Theorem 6.1.

Proposition 6.2
Let $\varphi\colon X \multimap Y$ be a multi-invertible and locally admissible map and let $\psi\colon X \multimap Y$ be a compact and locally admissible map. Assume that $X \in NA_C$ is strongly acyclic or $Y \in NA_C$ is strongly acyclic, then φ and ψ have a coincidence point.

Proof. In the proof of Theorem 6.1 it is enough to replace the set K with the set of acyclic and compact A such that $K \subset A$. Then $\Lambda(\Delta) = 1$ and the proof is complete.

From the last fact and Proposition 5.2, we get

Proposition 6.3
Let E be a Banach space. Let $\varphi\colon X \multimap E$ be a multi-invertible and locally admissible map and let $\psi\colon X \multimap E$ be a compact and locally admissible map. Then φ and ψ have a coincidence point.

The next fact is the simple conclusion of Proposition 6.2.

Proposition 6.4
Let $\varphi\colon X \multimap Y$ be a multi-invertible and admissible map and let $\psi\colon X \multimap Y$ be a compact and admissible map. Furthermore, assume that an inverse map $\overleftarrow{\varphi}$ is admissible. If $X \in A_C$ or $Y \in A_C$ then the maps φ and ψ have a coincidence point.

7. Conclusion

In the third paragraph we have proposed the definition of a multivalued invertible mapping. In the context of such a definition if a mapping is multi-invertible then there exists exactly one multi-inverse mapping. Moreover, if a singlevalued mapping $f\colon X \to Y$ is invertible, then it is multi-invertible and $\overleftarrow{f} = f^{-1}$. Multi-invertible mappings constitute a wide class of mappings and have many interesting applications. In paragraph four we have applied multi-invertible mappings for the construction of morphisms. Then, in paragraph six, it was proven that multi-invertible mappings have coincidence properties.

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