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On the geometric concomitants

Abstract. In this note the necessary and sufficient condition it would the concomitant of the geometric object was the geometric object too is given.

1. Introduction

S. Gołąb introduced the notion of the concomitant of an object in [4]. Let S be the set of elements called the objects and let T be the set of elements called the transformations, for which is defined the binary operation $\circ\colon T \times T \to T$ the superposition of these transformations. The objects in S are said to be geometric if there exists a function $F\colon S \times T \to S$ (called the transformation law) which transforms an object $s \in S$ by a transformation $t \in T$ to the object $F(s,t)$ and such that the diagram

$$\begin{tikzcd}
s \arrow[r, "t_1"] \arrow[dr, "t_1 \circ t_2"'] & F(s,t_1) \arrow[d, "t_2"] \\
& F(s,t_1 \circ t_2) = F(F(s,t_1),t_2)
\end{tikzcd}$$

is commutative and $F(s,I) = s$ for $s \in S$, where I is the identity transformation. From here

$$F(F(s,t_1),t_2) = F(s,t_1 \circ t_2) \qquad \text{for } s \in S,\ t_1,t_2 \in T$$

(the translation equation).

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Let φ be a function from the set S to the set S_1 of objects. The object $\varphi(s)$ for $s \in S$ is said to be the *concomitant* of the object s. For the geometric object s with the transformation law F its concomitant φ is said to be *geometric* if there exists a transformation law $G\colon S_1 \times T \to S_1$ such that the following diagram is commutative

$$\begin{array}{ccc} s & \xrightarrow{\qquad\qquad t \qquad\qquad} & F(s,t) \\ {\scriptstyle\varphi}\big\downarrow & & \big\downarrow{\scriptstyle\varphi} \\ \varphi(s) & \xrightarrow{\;t\;} & G(\varphi(s),t) = \varphi(F(s,t)) \end{array}$$

From here

$$G(\varphi(s),t) = \varphi[F(s,t)] \qquad \text{for } s \in S,\ t \in T$$

(the concomitant equation). If objects in S are geometric their concomitants do not have to be the geometric objects. Hence the general question can be stated: when the concomitant of the geometric object is a geometric object too?

We have not found the answer to this question in the rich literature about the concomitants (see, e.g. the bibliography in [1] and [5]).

2. Main considerations

PROPOSITION 2.1

Let the function $F\colon S \times \mathbb{G} \to S$, where S is an arbitrary set and $(\mathbb{G},+)$ an arbitrary groupoid, be the solution of the translation equation

$$F(F(x,t),s) = F(x,t+s)$$

and let φ be the function from S to an arbitrary set S_1.

(*i*) *There exists a solution $G\colon S_1 \times \mathbb{G} \to S_1$ of the translation equation such that*

$$G(\varphi(x),t) = \varphi[F(x,t)] \qquad \textit{for } (x,t) \in S \times \mathbb{G} \tag{1}$$

if and only if

$$\varphi(x) = \varphi(y) \Rightarrow \varphi[F(x,t)] = \varphi[F(y,t)] \qquad \textit{for } x,y \in S,\ t \in \mathbb{G}. \tag{2}$$

(*ii*) *The functions of the form*

$$G(u,t) = \begin{cases} \varphi[F(x,t)], & \textit{if } u = \varphi(x) \textit{ for some } x \in S,\ t \in \mathbb{G}, \\ f(u) & \textit{for } u \in S_1 \setminus \varphi(S),\ t \in \mathbb{G}, \end{cases} \tag{3}$$

where $f\colon S_1 \setminus \varphi(S) \to S_1 \setminus \varphi(S)$ is an idempotent ($f(f) = f$), are the solutions of (1).

(*iii*) *The elements of the set $\varphi(S)$, where φ is the concomitant of the geometric object with the function F as the transformation law, are the geometric objects if and only if condition* (2) *holds.*

Proof. First we prove (i) and (ii). If $\varphi(x) = \varphi(y)$, then

$$\varphi[F(x,t)] = G(\varphi(x),t) = G(\varphi(y),t) = \varphi[F(y,t)].$$

The function G defined in (3) is well defined by (2). The relation (1) is evident. Moreover,

a) for $u \in \varphi(S)$ there exists $x \in S$ such that $u = \varphi(x)$ and

$$\begin{aligned} G[G(u,t),s] &= G[\varphi(F(x,t)),s] = \varphi[F(F(x,t),s)] \\ &= \varphi[F(x,t+s)] = G(\varphi(x),t+s) = G(u,t+s), \end{aligned}$$

b) for $u \in S_1\ \varphi(S)$ we have

$$G[G(u,t),s] = G(f(u),s) = f(f(u)) = f(u) = G(u,t+s).$$

To show (iii) notice that if the concomitant φ is geometric, then there exists a solution $G\colon S_1 \times T \to S_1$ of (1), thus (2) is evident.

If the condition (2) is satisfied, then the function G defined by (3) with $f(u) = u$ is the solution of (1). Furthermore, for every $u \in \varphi(S)$ there exists $x \in S$ such that $u = \varphi(x)$, thus

$$G(u,I) = G(\varphi(x),I) = \varphi(F(x,I)) = \varphi(x) = u.$$

For $u \in S_1 \setminus \varphi(S)$ we have $G(u,I) = f(u) = u$, which completes the proof.

Interpretation of the implication (2). The condition (2) means that the family of the levels of the function φ, i.e. the family $\{\varphi^{-1}(a) :\ a \in S_1\}$, is such that the image of the level by the function from the family $\{F(.,t) :\ t \in \mathbb{G}\}$ is included in some level.

3. Examples

Example 3.1
An injection and a constant function are evidently the solutions of the conditional equation (2) for every function F. If φ is an injection, then $G(u,t) = \varphi[F(\varphi^{-1}(u),t)]$ for $u \in \varphi(S)$ and $G(u,t) = u$ for $u \in S_1 \setminus \varphi(S)$, thus it can happen that $G \neq F$.

Example 3.2
The function $\varphi(x) = h^{-1}[h(x) + a]$, where h is a bijection of $\mathbb{R}$ and $a \in \mathbb{R}$, is a solution of (2) for the function $F(x,t) = h^{-1}[h(x) + t]\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. We have $G = F$ in this case.

Example 3.3
The function

$$\varphi(x) = \begin{cases} 0 & \text{for } x = 0,\ t \in \mathbb{R}, \\ \exp[\ln x + a] & \text{for } x > 0,\ t \in \mathbb{R},\ a \in \mathbb{R}, \\ -\exp[\ln(-x) + b] & \text{for } x < 0,\ t \in \mathbb{R},\ b \in \mathbb{R}, \end{cases}$$

is the solution of (2) for the function $F(x,t) = x \exp t$. We have $G = F$ in this case.

Example 3.4
Let $(\mathbb{G}, +)$ be a group and let $F(x,t) = x + t$ for $x, t \in \mathbb{G}$. A function $\varphi\colon \mathbb{G} \to \mathbb{G}$ is the solution of (2) if and only if $\varphi(x) = \Phi([x])$ for $x \in \mathbb{G}$, where $[x] \in F$, $F = \{[x]\}_{x \in \mathbb{G}}$ is the family of the right cosets of the group $\mathbb{G}$ for a subgroup $\mathbb{G}^*$ and Φ is an injection from F to $\mathbb{G}$. If φ is of this form, then (2) has the form

$$\Phi([x]) = \Phi([y]) \Rightarrow \Phi([x+t]) = \Phi([y+t]).$$

If $\Phi([x]) = \Phi([y])$, then $[x] = [y]$, thus $y - x \in \mathbb{G}^*$. From here $(y+t) - (x+t) = y + t - t - x = y - x \in \mathbb{G}^*$ and this yields that $[x+t] = [y+t]$.

Assume now that φ is a solution of (2). We define a relation R on $\mathbb{G}$ as follows

$$xRy \Leftrightarrow \varphi(x) = \varphi(y).$$

It is an equivalence relation and xRy implies $(x+t)R(y+t)$. From here the family of the equivalence classes of R is the same as the family F of the right cosets of the group $\mathbb{G}$ for a subgroup [3]. Let S be a selector of this family. Put $\Phi([x]) = \varphi(x_0)$, where $x_0 \in S \cap [x]$. Since $x_0 \in [x]$, thus $[x] = [x_0]$. For $x \in [x] = [x_0]$ we have $\varphi(x) = \varphi(x_0)$, thus $\varphi(x) = \Phi([x])$.

Example 3.5
The injection from $\mathbb{R}$ to $\mathbb{R}$ and a constant function are the only solutions of (2) for the geometric object $x = \frac{v_2}{v_1}$, where v_1, v_2 are the coordinates of the 2-dimensional contravariant vector. Indeed, this object has the transformation law of the form

$$\frac{A_1^1 + A_2^1 x}{A_1^2 + A_2^2 x}$$

(see [1]), thus (2) is of the form

$$\varphi(x) = \varphi(y) \Rightarrow \varphi\Big(\frac{A_1^1 + A_2^1 x}{A_1^2 + A_2^2 x}\Big) = \varphi\Big(\frac{A_1^1 + A_2^1 y}{A_1^2 + A_2^2 y}\Big). \tag{4}$$

For $A_2^1 = A_1^2 = 1$ and $A_2^2 = 0$ we have

$$\varphi(x) = \varphi(y) \Rightarrow \varphi(A_1^1 + x) = \varphi(A_1^1 + y)$$

for every $x, y, A_1^1 \in \mathbb{R}$. From here the family F of the levels of φ is invariant by the arbitrary translations. This yields that this family $F = \mathbb{R}/\mathbb{R}^*$, where $\mathbb{R}/\mathbb{R}^*$ is the quotient group of $\mathbb{R}$ for some subgroup $\mathbb{R}^*$ (see the previous example). Assume that $\{0\} \neq \mathbb{R}^* \neq \mathbb{R}$. This implies that there exist x_1, x_2 such that $x_1 \neq 0$ and $\varphi(0) = \varphi(x_1) \neq \varphi(x_2)$. We have by (4) that

$$\varphi(x) = \varphi(y) \Rightarrow \varphi(ax) = \varphi(ay)$$

for every $x, y, a \in \mathbb{R}$. From here

$$\varphi(0) = \varphi\Big(\frac{x_2}{x_1} \cdot 0\Big) = \varphi\Big(\frac{x_2}{x_1} \cdot x_1\Big) = \varphi(x_2),$$

thus the contradiction. If $\mathbb{R}^* = \{0\}$, then φ is an injection, if $\mathbb{R}^* = \mathbb{R}$, then φ is a constant function.

Example 3.6
Let the solution of translation equation $F\colon \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be of the form

$$F(x,t)=\begin{cases} h_n^{-1}[h_n(x)+t] & \text{for } x\in I_n,\ t\in\mathbb{R},\\ x & \text{for } x\in\mathbb{R}\setminus\bigcup I_n,\ t\in\mathbb{R},\end{cases} \tag{5}$$

where I_n for $n\in K_1\subset\mathbb{N}$ are open and disjoint non-empty intervals and $h_n\colon I_n\to\mathbb{R}$ are bijections.

Put $K_2=\mathbb{R}\setminus\bigcup I_n$ and $K=K_1\cup K_2$. The function φ of the form

$$\varphi(x)=\begin{cases} \alpha(x) & \text{for } x\in K_2,\\ \alpha(n) & \text{for } x\in I_n,\ \alpha(n)\in K_2,\\ h_{\alpha(n)}^{-1}[h_n(x)+\beta(n)] & \text{for } x\in I_n,\ \alpha(n)\in K_1,\end{cases} \tag{6}$$

where $\alpha\colon K\to K$ is such that $\alpha(K_2)\subset K_2$ and β is an arbitrary function from K_1 to $\mathbb{R}$, is the solution of (2), since $\varphi[F(x,t)]=F(\varphi(x),t)$. Every solution φ of the last equation is of the form (6) (see [6]).

If $K_1\neq\emptyset$, then the function $\varphi(x)=x_0\in I_k$ for some $k\in K_1$ and $x\in\mathbb{R}$ is the solution of (2) and it is not the solution of the equation

$$\varphi[F(x,t)]=F(\varphi(x),t) \tag{7}$$

for the continuous function F. Indeed, if the constant function $\varphi(x)=c$ is a solution of (7), then $F(c,t)=c$ for $t\in\mathbb{G}$ and $x_0\notin\mathbb{R}\setminus\bigcup I_n$. This yields that the implication (2) and the equation (7) are not equivalent.

The solution $F\colon\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ of the translation equation is of the form (5) if it is continuous with respect to the second variable for every $x\in S$ and for which at least one of the functions $F(.,t)\colon\mathbb{R}\to\mathbb{R}$ is continuous and $F(x,0)=x$ for $x\in\mathbb{R}$ ([7], this form of F is proved in [8] if it is continuous). Notice that F is of this form too if it is Carathéodory, i.e. the function $F(x,.)\colon\mathbb{R}\to\mathbb{R}$ is measurable for every $x\in\mathbb{R}$ and the function $F(.,t)\colon\mathbb{R}\to\mathbb{R}$ is continuous for every $t\in\mathbb{R}$, since it is continuous in this case (see [2]).

4. Remarks

Remark 4.1
The operation "+" in $\mathbb{G}$ occurs not explicitly in the implication (2), it is "hidden" in the function F. From here the solution of (2) depends on „+” (see the above Examples 3.2–3.4).

Remark 4.2
If $\operatorname{card}[S_1\setminus\varphi(S)]>1$, then a function G which satisfies (1) is not unique since we have in this case many idempotents from $S_1\setminus\varphi(S)$ to $S_1\setminus\varphi(S)$.

Remark 4.3
Let S_1, S_2 be sets, $\mathbb{G}_1$, $\mathbb{G}_2$ be groups, $F\colon S_1\times\mathbb{G}_1\to S_1$ and $G\colon S_2\times\mathbb{G}_2\to S_2$

be a solutions of the translation equation. Moreover, let $\lambda\colon \mathbb{G}_1 \to \mathbb{G}_2$ be a homomorphism and let φ be the function from S_1 to S_2. The generalized concomitant equation

$$G(\varphi(x), \lambda(t)) = \varphi[F(x,t)], \tag{8}$$

in which F and G are the given solutions of the translation equation, is solved in [6] (notice that the typing errors are in this paper!). In [6], among others, it is proved that the multiplication by scalar is the only geometric concomitant of the vector.

Remark 4.4
The equation (8) occurs in the theory of abstract automata in the concept of their homomorphism [9].

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