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Fractional Hermite-Hadamard type integral inequalities for functions whose modulus of the mixed derivatives are co-ordinated s-preinvex in the second sense

Abstract. In this paper we establish some fractional Hermite-Hadamard type integral inequalities for functions whose modulus of the mixed derivatives are co-ordinated s-preinvex in the second sense.

1. Introduction

Let $f\colon I \subset \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and let $a, b \in I$ with $a < b$. Then

$$f\Big(\frac{a+b}{2}\Big) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

The above inequality is known in the literature as the integral inequality of Hermite-Hadamard (see [14]). It can be considered as a necessary and sufficient condition for a function to be convex. Due to this very close relationship between the theory of inequality and the notion of convexity, researchers have not ceased to explore inequality (1) via different kinds of convexity. Various generalizations, improvements, extensions and variants of these have appeared in the literature, see [1, 3, 6, 7, 8, 9, 10, 11, 16, 17] and references therein.

Also the classical convexity have been generalized by different way. One of the significant one is that introduced by Hanson (see [4]), called invexity or preinvexity, many authors have studied their properties and applications in mathematical programming and optimizations, see for instance [2, 12, 13, 15, 18, 19].

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In [3], Dragomir established the bidimensional analogue of (1) given by

$$\begin{aligned}
f\Big(\frac{a+b}{2},\frac{c+d}{2}\Big)&\\
&\le \frac{1}{2}\Big(\frac{1}{b-a}\int_a^b f\Big(x,\frac{c+d}{2}\Big)\,dx+\frac{1}{d-c}\int_c^d f\Big(\frac{a+b}{2},y\Big)\,dy\Big)\\
&\le \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx\\
&\le \frac{1}{4}\Big(\frac{1}{b-a}\Big(\int_a^b f(x,c)\,dx+\int_a^b f(x,d)\,dx\Big)\\
&\quad+\frac{1}{d-c}\Big(\int_c^d f(a,y)\,dy+\int_c^d f(b,y)\,dy\Big)\Big)\\
&\le \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}.
\end{aligned}$$

In 2014 in [16], Sarikaya gave the following fractional Hermite-Hadamard for co-ordinated convex functions.

Theorem 1
Let $f\colon\Delta\to\mathbb{R}$ be a partial differentiable mapping on $\Delta=[a,b]\times[c,d]\subset\mathbb{R}^2$. If $\big|\frac{\partial^2 f}{\partial s\partial t}\big|$ is a convex function on the co-ordinates on Δ, then one has the inequalities

$$\begin{aligned}
&\Big|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{(b-a)^\alpha(d-c)^\beta}\\
&\qquad\times\big(J^{\alpha,\beta}_{a^+,c^+}f(b,d)+J^{\alpha,\beta}_{a^+,d^-}f(b,c)+J^{\alpha,\beta}_{b^-,c^+}f(a,d)+J^{\alpha,\beta}_{b^-,d^-}f(a,c)-A\big)\Big|\\
&\quad\le\frac{(b-a)(d-c)}{4(\alpha+1)(\beta+1)}\Big(\Big|\frac{\partial^2 f}{\partial s\partial t}(a,c)\Big|+\Big|\frac{\partial^2 f}{\partial s\partial t}(a,d)\Big|+\Big|\frac{\partial^2 f}{\partial s\partial t}(b,c)\Big|+\Big|\frac{\partial^2 f}{\partial s\partial t}(b,d)\Big|\Big),
\end{aligned}$$

where

$$\begin{aligned}
A&=\frac{\Gamma(\beta+1)}{4(d-c)^\beta}\big(J^\beta_{c^+}f(a,d)+J^\beta_{c^+}f(b,d)+J^\beta_{d^-}f(a,c)+J^\beta_{d^-}f(b,c)\big)\\
&\quad+\frac{\Gamma(\alpha+1)}{4(b-a)^\alpha}\big(J^\alpha_{a^+}f(b,c)+J^\alpha_{a^+}f(b,d)+J^\alpha_{b^-}f(a,c)+J^\alpha_{b^-}f(a,d)\big).
\end{aligned}\tag{2}$$

Theorem 2
Let $f\colon\Delta\to\mathbb{R}$ be a partial differentiable mapping on $\Delta=[a,b]\times[c,d]\subset\mathbb{R}^2$. If $\big|\frac{\partial^2 f}{\partial s\partial t}\big|^q$, $q>1$ is a convex function on the co-ordinates on Δ, then one has the inequalities

$$\begin{aligned}
&\Big|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{(b-a)^\alpha(d-c)^\beta}\\
&\qquad\times\big(J^{\alpha,\beta}_{a^+,c^+}f(b,d)+J^{\alpha,\beta}_{a^+,d^-}f(b,c)+J^{\alpha,\beta}_{b^-,c^+}f(a,d)+J^{\alpha,\beta}_{b^-,d^-}f(a,c)-A\big)\Big|
\end{aligned}$$

$$\leq \frac{(b-a)(d-c)}{((\alpha p+1)(\beta p+1))^{\frac{1}{p}}}\Big(\frac{1}{4}\Big)^{\frac{1}{q}}$$
$$\times\Big(\Big|\frac{\partial^2 f}{\partial s\partial t}(a,c)\Big|^q+\Big|\frac{\partial^2 f}{\partial s\partial t}(a,d)\Big|^q+\Big|\frac{\partial^2 f}{\partial s\partial t}(b,c)\Big|^q+\Big|\frac{\partial^2 f}{\partial s\partial t}(b,d)\Big|^q\Big),$$

where A is defined by (2) *and* $\frac{1}{p}+\frac{1}{q}=1$.

Motivated by the above results, in this paper we establish some new fractional Hermite-Hadamard type inequalities for functions whose modulus of the mixed derivatives are co-ordinated s-preinvex in the second sense.

2. Preliminaries

In this section we recall some definitions and lemmas that's well known in the literature. We assume that $\Delta := [a,b]\times[c,d]$ is a bidimensional interval in $\mathbb{R}^2$ with $a<b$ and $c<d$.

Definition 1 ([6])
A function $f\colon \Delta\to\mathbb{R}$ is said to be *co-ordinated convex on* Δ, if the following inequality

$$f(tx+(1-t)u,\lambda y+(1-\lambda)v)\leq t\lambda f(x,y)+t(1-\lambda)f(x,v)$$
$$+(1-t)\lambda f(u,y)+(1-t)(1-\lambda)f(u,v)$$

holds for all $t,\lambda\in[0,1]$ and $(x,y),(x,v),(u,y),(u,v)\in\Delta$.

Definition 2 ([1])
A nonnegative function $f\colon\Delta\subset[0,\infty)^2\to\mathbb{R}$ is said to be *s-convex in the second sense on the co-ordinates on Δ for some fixed $s\in(0,1]$*, if the following inequality

$$f(\lambda x+(1-\lambda)z,ty+(1-t)w)\leq\lambda^s t^s f(x,y)+\lambda^s(1-t)^s f(x,w)$$
$$+(1-\lambda)^s t^s f(z,y)+(1-\lambda)^s(1-t)^s f(z,w),$$

holds for all $(x,y),(z,w),(x,w),(z,y)\in\Delta$ and $\lambda,t\in[0,1]$.

Definition 3 ([9])
Let K_1,K_2 be nonempty subsets of $\mathbb{R}^n$, $(u,v)\in K_1\times K_2$. We say $K_1\times K_2$ *is invex at (u,v) with respect to η_1 and η_2*, if for each $(x,y)\in K_1\times K_2$ and $t,s\in[0,1]$, we have

$$(u+t\eta_1(x,u),v+s\eta_2(y,v))\in K_1\times K_2.$$

Let us recall that the set $K_1\times K_2$ is said to be *an invex set with respect to η_1 and η_2*, if $K_1\times K_2$ is invex at each $(u,v)\in K_1\times K_2$.

In what follows we assume that $K_1\times K_2$ be an invex set with respect to $\eta_1\colon K_1\times K_1\to\mathbb{R}$ and $\eta_2\colon K_2\times K_2\to\mathbb{R}$.

Definition 4 ([8])
A function $f\colon K_1 \times K_2 \to \mathbb{R}$ is said to be *preinvex on the co-ordinates*, if the following inequality

$$\begin{aligned} f(u+\lambda\eta_1(x,u), v+t\eta_2(y,v)) &\le (1-\lambda)(1-t)f(u,v)+(1-\lambda)tf(u,y) \\ &\quad + (1-t)\lambda f(x,v)+\lambda t f(x,y) \end{aligned}$$

holds for all $t,\lambda \in [0,1]$ and $(x,y),(x,v),(u,y),(u,v) \in K_1 \times K_2$.

Definition 5 ([10])
A nonnegative function $f\colon K_1 \times K_2 \subset [0,+\infty) \times [0,+\infty) \to \mathbb{R}$ is said to be *s-preinvex in the second sense on co-ordinates for some fixed $s \in (0,1]$*, if the following inequality

$$\begin{aligned} f(u+\lambda\eta_1(x,u), v+t\eta_2(y,v)) &\le (1-\lambda)^s(1-t)^s f(u,v)+(1-\lambda)^s t^s f(u,y) \\ &\quad + (1-t)^s\lambda^s f(x,v)+\lambda^s t^s f(x,y) \end{aligned}$$

holds for all $t,\lambda \in [0,1]$ and $(x,y),(x,v),(u,y),(u,v) \in K_1 \times K_2$.

Definition 6 ([5])
Let $f \in L[a,b]$. The Riemann-Liouville integrals $J^{\alpha}_{a^+}f$ and $J^{\alpha}_{b^-}f$ of order $\alpha > 0$ with $a \ge 0$ are defined by

$$J^{\alpha}_{a^+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)\,dt, \qquad x > a,$$

$$J^{\alpha}_{b^-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)\,dt, \qquad b > x,$$

respectively, where $\Gamma(\alpha) = \int_0^\infty e^{-t}t^{\alpha-1}dt$, is the Gamma function and $J^{0}_{a^+}f(x) = J^{0}_{b^-}f(x) = f(x)$.

Definition 7 ([7])
Let $f \in L([a,b]\times[c,d])$. The Riemann–Liouville integrals $J^{\alpha,\beta}_{a^+,c^+}$, $J^{\alpha,\beta}_{a^+,d^-}$, $J^{\alpha,\beta}_{b^-,c^+}$ and $J^{\alpha,\beta}_{b^-,d^-}$ of order $\alpha,\beta > 0$ with $a,c \ge 0$, $a < b$ and $c < d$ are defined by

$$J^{\alpha,\beta}_{a^+,c^+}f(b,d) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^b\int_c^d (b-x)^{\alpha-1}(d-y)^{\beta-1}f(x,y)\,dydx,$$

$$J^{\alpha,\beta}_{a^+,d^-}f(b,c) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^b\int_c^d (b-x)^{\alpha-1}(y-c)^{\beta-1}f(x,y)\,dydx,$$

$$J^{\alpha,\beta}_{b^-,c^+}f(a,d) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^b\int_c^d (x-a)^{\alpha-1}(d-y)^{\beta-1}f(x,y)\,dydx,$$

$$J^{\alpha,\beta}_{b^-,d^-}f(a,c) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^b\int_c^d (x-a)^{\alpha-1}(y-c)^{\beta-1}f(x,y)\,dydx,$$

where Γ is the Gamma function and

$$J^{0,0}_{a^+,c^+}f(b,d) = J^{0,0}_{a^+,d^-}f(b,c) = J^{0,0}_{b^-,c^+}f(a,d) = J^{0,0}_{b^-,d^-}f(a,c) = f(x,y).$$

Definition 8 ([16])
Let $f \in L([a,b]\times[c,d])$. The Riemann–Liouville integrals $J^{\alpha}_{b^-}f(a,c)$, $J^{\alpha}_{a^+}f(b,c)$, $J^{\beta}_{d^-}f(a,c)$ and $J^{\alpha}_{c^+}f(a,d)$ of order $\alpha,\beta>0$ with $a,c\geq 0$, $a<b$, and $c<d$ are defined by

$$J^{\alpha}_{b^-}f(a,c)=\frac{1}{\Gamma(\alpha)}\int_a^b (x-a)^{\alpha-1}f(x,c)\,dx,$$

$$J^{\alpha}_{a^+}f(b,c)=\frac{1}{\Gamma(\alpha)}\int_a^b (b-x)^{\alpha-1}f(x,c)\,dx,$$

$$J^{\beta}_{d^-}f(a,c)=\frac{1}{\Gamma(\beta)}\int_c^d (y-c)^{\beta-1}f(a,y)\,dy,$$

$$J^{\alpha}_{c^+}f(a,d)=\frac{1}{\Gamma(\beta)}\int_c^d (d-y)^{\beta-1}f(a,y)\,dy,$$

where Γ is the Gamma function.

Lemma 1 ([11])
Let $f\colon K\to\mathbb{R}$ be a partially differentiable function on K, if $\frac{\partial^2 f}{\partial t\partial\lambda}\in L(K)$, then the following equality holds

$$\begin{aligned}
&\frac{f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))}{4}\\
&\quad -A+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\eta_1(b,a))^{\alpha}(\eta_2(d,c))^{\beta}}\big(J^{\alpha,\beta}_{(a+\eta_1(b,a))^-,(c+\eta_2(d,c))^-}f(a,c)\\
&\quad +J^{\alpha,\beta}_{a^+,(c+\eta_2(d,c))^-}f(a+\eta_1(b,a),c)+J^{\alpha,\beta}_{(a+\eta_1(b,a))^-,c^+}f(a,c+\eta_2(d,c))\\
&\quad +J^{\alpha,\beta}_{a^+,c^+}f(a+\eta_1(b,a),c+\eta_2(d,c))\big)\\
&=\frac{\eta_1(b,a)\eta_2(d,c)}{4}\int_0^1\int_0^1 (t^{\alpha}-(1-t)^{\alpha})(\lambda^{\beta}-(1-\lambda)^{\beta})\\
&\quad\times\frac{\partial^2 f}{\partial t\partial\lambda}(a+t\eta_1(b,a),c+\lambda\eta_2(d,c))\,d\lambda dt,
\end{aligned}$$

where

$$\begin{aligned}
A=&\frac{\Gamma(\alpha+1)}{4(\eta_1(b,a))^{\alpha}}\big(J^{\alpha}_{(a+\eta_1(b,a))^-}f(a,c+\eta_2(d,c))+J^{\alpha}_{(a+\eta_1(b,a))^-}f(a,c)\\
&+J^{\alpha}_{a^+}f(a+\eta_1(b,a),c+\eta_2(d,c))+J^{\alpha}_{a^+}f(a+\eta_1(b,a),c)\big)\\
&+\frac{\Gamma(\beta+1)}{4(\eta_2(d,c))^{\beta}}\big(J^{\beta}_{(c+\eta_2(d,c))^-}f(a+\eta_1(b,a),c)+J^{\beta}_{(c+\eta_2(d,c))^-}f(a,c)\\
&+J^{\alpha}_{c^+}f(a+\eta_1(b,a),c+\eta_2(d,c))+J^{\alpha}_{c^+}f(a,c+\eta_2(d,c))\big).
\end{aligned}\tag{3}$$

3. Main results

In what follows we assume that $K = [a, a + \eta_1(b, a)] \times [c, c + \eta_2(d, c)]$ be an invex subset of $[0, +\infty) \times [0, +\infty)$ with respect to η_1, η_2 where $\eta_1, \eta_2 \colon \mathbb{R}^2 \to \mathbb{R}$ are two bifunctions such that $\eta_1(b, a) > 0$ and $\eta_2(d, c) > 0$, and K° is the interior of K.

Theorem 3
Let $f \colon K \to \mathbb{R}$ be a partially differentiable function on K°. If $|\frac{\partial^2 f}{\partial t \partial \lambda}|$ is a co-ordinated s-preinvex function on K with respect to η_1 and η_2, then the following fractional inequality holds

$$\begin{aligned}
&\Big|\frac{f(a,c) + f(a, c+\eta_2(d,c)) + f(a+\eta_1(b,a), c) + f(a+\eta_1(b,a), c+\eta_2(d,c))}{4} \\
&\quad - A + \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\eta_1(b,a))^\alpha(\eta_2(d,c))^\beta}\big(J^{\alpha,\beta}_{(a+\eta_1(b,a))^-,(c+\eta_2(d,c))^-} f(a,c) \\
&\quad + J^{\alpha,\beta}_{a^+,(c+\eta_2(d,c))^-} f(a+\eta_1(b,a), c) + J^{\alpha,\beta}_{(a+\eta_1(b,a))^-,c^+} f(a, c+\eta_2(d,c)) \\
&\quad + J^{\alpha,\beta}_{a^+,c^+} f(a+\eta_1(b,a), c+\eta_2(d,c))\big)\Big| \\
&\le \frac{\eta_1(b,a)\eta_2(d,c)}{4}\Big(\frac{1}{\alpha+s+1} + B(\alpha+1, s+1)\Big)\Big(\frac{1}{\beta+s+1} + B(\beta+1, s+1)\Big) \\
&\times \Big(\Big|\frac{\partial^2 f}{\partial t \partial \lambda}(a,c)\Big| + \Big|\frac{\partial^2 f}{\partial t \partial \lambda}(a,d)\Big| + \Big|\frac{\partial^2 f}{\partial t \partial \lambda}(b,c)\Big| + \Big|\frac{\partial^2 f}{\partial t \partial \lambda}(b,d)\Big|\Big),
\end{aligned}$$

where A is defined as in (3).

Proof. From Lemma 1, and properties of modulus we have

$$\begin{aligned}
&\Big|\frac{f(a,c) + f(a, c+\eta_2(d,c)) + f(a+\eta_1(b,a), c) + f(a+\eta_1(b,a), c+\eta_2(d,c))}{4} \\
&\quad - A + \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\eta_1(b,a))^\alpha(\eta_2(d,c))^\beta}\big(J^{\alpha,\beta}_{(a+\eta_1(b,a))^-,(c+\eta_2(d,c))^-} f(a,c) \\
&\quad + J^{\alpha,\beta}_{a^+,(c+\eta_2(d,c))^-} f(a+\eta_1(b,a), c) + J^{\alpha,\beta}_{(a+\eta_1(b,a))^-,c^+} f(a, c+\eta_2(d,c)) \\
&\quad + J^{\alpha,\beta}_{a^+,c^+} f(a+\eta_1(b,a), c+\eta_2(d,c))\big)\Big| \\
&\le \frac{\eta_1(b,a)\eta_2(d,c)}{4}\int_0^1\int_0^1 \Big(|t^\alpha - (1-t)^\alpha| \times |\lambda^\beta - (1-\lambda)^\beta| \\
&\qquad \times \Big|\frac{\partial^2 f}{\partial t \partial \lambda}(a + t\eta_1(b,a), c + \lambda\eta_2(d,c))\Big|\Big) d\lambda dt \\
&\le \frac{\eta_1(b,a)\eta_2(d,c)}{4}\int_0^1\int_0^1 \Big((t^\alpha + (1-t)^\alpha)(\lambda^\beta + (1-\lambda)^\beta) \\
&\qquad \times \Big|\frac{\partial^2 f}{\partial t \partial \lambda}(a + t\eta_1(b,a), c + \lambda\eta_2(d,c))\Big|\Big) d\lambda dt.
\end{aligned}$$

Using s-preinvexity on the co-ordinates of $|\frac{\partial^2 f}{\partial t\partial\lambda}|$, we get

$$\begin{aligned}
&\Big|\frac{f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))}{4}\\
&\quad - A + \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\eta_1(b,a))^\alpha(\eta_2(d,c))^\beta}\big(J^{\alpha,\beta}_{(a+\eta_1(b,a))^-,(c+\eta_2(d,c))^-}f(a,c)\\
&\quad + J^{\alpha,\beta}_{a^+,(c+\eta_2(d,c))^-}f(a+\eta_1(b,a),c) + J^{\alpha,\beta}_{(a+\eta_1(b,a))^-,c^+}f(a,c+\eta_2(d,c))\\
&\quad + J^{\alpha,\beta}_{a^+,c^+}f(a+\eta_1(b,a),c+\eta_2(d,c))\big)\Big|\\
&\quad \le \frac{\eta_1(b,a)\eta_2(d,c)}{4}\bigg(\Big|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\Big|\int_0^1\int_0^1\big((t^\alpha+(1-t)^\alpha)\\
&\qquad\qquad \times(\lambda^\beta+(1-\lambda)^\beta)(1-t)^s(1-\lambda)^s\big)\,d\lambda dt\\
&\quad + \Big|\frac{\partial^2 f}{\partial t\partial\lambda}(a,d)\Big|\int_0^1\int_0^1(t^\alpha+(1-t)^\alpha)(\lambda^\beta+(1-\lambda)^\beta)(1-t)^s\lambda^s\,d\lambda dt\\
&\quad + \Big|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\Big|\int_0^1\int_0^1(t^\alpha+(1-t)^\alpha)(\lambda^\beta+(1-\lambda)^\beta)t^s(1-\lambda)^s\,d\lambda dt\\
&\quad + \Big|\frac{\partial^2 f}{\partial t\partial\lambda}(b,d)\Big|\int_0^1\int_0^1(t^\alpha+(1-t)^\alpha)(\lambda^\beta+(1-\lambda)^\beta)t^s\lambda^s\,d\lambda dt\bigg)\\
&\quad = \frac{\eta_1(b,a)\eta_2(d,c)}{4}\Big(\Big|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\Big|+\Big|\frac{\partial^2 f}{\partial t\partial\lambda}(a,d)\Big|+\Big|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\Big|+\Big|\frac{\partial^2 f}{\partial t\partial\lambda}(b,d)\Big|\Big)\\
&\quad \times\bigg(\int_0^1(t^\alpha(1-t)^s+(1-t)^{\alpha+s})\,dt\bigg)\bigg(\int_0^1(\lambda^\beta(1-\lambda)^s+(1-\lambda)^{\beta+s})\,d\lambda\bigg)\\
&\quad = \frac{\eta_1(b,a)\eta_2(d,c)}{4}\Big(\Big|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\Big|+\Big|\frac{\partial^2 f}{\partial t\partial\lambda}(a,d)\Big|+\Big|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\Big|+\Big|\frac{\partial^2 f}{\partial t\partial\lambda}(b,d)\Big|\Big)\\
&\quad \times\Big(\frac{1}{\alpha+s+1}+B(\alpha+1,s+1)\Big)\Big(\frac{1}{\beta+s+1}+B(\beta+1,s+1)\Big),
\end{aligned}$$

where we have used the fact that

$$\begin{aligned}
&\int_0^1\int_0^1(t^\alpha+(1-t)^\alpha)(\lambda^\beta+(1-\lambda)^\beta)(1-t)^s(1-\lambda)^s\,d\lambda dt\\
&\qquad = \int_0^1\int_0^1(t^\alpha+(1-t)^\alpha)(\lambda^\beta+(1-\lambda)^\beta)(1-t)^s\lambda^s\,d\lambda dt\\
&\qquad = \int_0^1\int_0^1(t^\alpha+(1-t)^\alpha)(\lambda^\beta+(1-\lambda)^\beta)t^s(1-\lambda)^s\,d\lambda dt\\
&\qquad = \int_0^1\int_0^1(t^\alpha+(1-t)^\alpha)(\lambda^\beta+(1-\lambda)^\beta)t^s\lambda^s\,d\lambda dt\\
&\qquad = \Big(\frac{1}{\alpha+s+1}+B(\alpha+1,s+1)\Big)\Big(\frac{1}{\beta+s+1}+B(\beta+1,s+1)\Big).
\end{aligned}$$

The proof is completed.

Corollary 1

In Theorem 3 if we put $s=1$, we obtain the following fractional inequality

$$\begin{aligned}&\Big|\frac{f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))}{4}\\ &\quad -A+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\eta_1(b,a))^{\alpha}(\eta_2(d,c))^{\beta}}\big(J^{\alpha,\beta}_{(a+\eta_1(b,a))^-,(c+\eta_2(d,c))^-}f(a,c)\\ &\quad +J^{\alpha,\beta}_{a^+,(c+\eta_2(d,c))^-}f(a+\eta_1(b,a),c)+J^{\alpha,\beta}_{(a+\eta_1(b,a))^-,c^+}f(a,c+\eta_2(d,c))\\ &\quad +J^{\alpha,\beta}_{a^+,c^+}f(a+\eta_1(b,a),c+\eta_2(d,c))\big)\Big|\\ &\le \frac{\eta_1(b,a)\eta_2(d,c)}{4(\alpha+1)(\beta+1)}\Big(\Big|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\Big|+\Big|\frac{\partial^2 f}{\partial t\partial\lambda}(a,d)\Big|+\Big|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\Big|+\Big|\frac{\partial^2 f}{\partial t\partial\lambda}(b,d)\Big|\Big).\end{aligned}$$

Corollary 2

In Theorem 3 if we choose $\eta_1(b,a)=\eta_2(b,a)=b-a$, we obtain the following fractional inequality

$$\begin{aligned}&\Big|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}-A+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^{\alpha}(d-c)^{\beta}}\\ &\quad\times\big(J^{\alpha,\beta}_{b^-,d^-}f(a,c)+J^{\alpha,\beta}_{a^+,d^-}f(b,c)+J^{\alpha,\beta}_{b^-,c^+}f(a,d)+J^{\alpha,\beta}_{a^+,c^+}f(b,d)\big)\Big|\\ &\le\frac{(b-a)(d-c)}{4}\Big(\frac{1}{\alpha+s+1}+B(\alpha+1,s+1)\Big)\Big(\frac{1}{\beta+s+1}+B(\beta+1,s+1)\Big)\\ &\quad\times\Big(\Big|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\Big|+\Big|\frac{\partial^2 f}{\partial t\partial\lambda}(a,d)\Big|+\Big|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\Big|+\Big|\frac{\partial^2 f}{\partial t\partial\lambda}(b,d)\Big|\Big).\end{aligned}$$

Moreover if we take $s=1$, we get Theorem 5 from [16].

Theorem 4

Let $f\colon K\to\mathbb{R}$ be a partially differentiable function on K°. If $|\frac{\partial^2 f}{\partial t\partial\lambda}|^q$, where $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ is a co-ordinated s-preinvex function on K with respect to η_1 and η_2, then the following fractional inequality holds

$$\begin{aligned}&\Big|\frac{f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))}{4}\\ &\quad -A+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\eta_1(b,a))^{\alpha}(\eta_2(d,c))^{\beta}}\big(J^{\alpha,\beta}_{(a+\eta_1(b,a))^-,(c+\eta_2(d,c))^-}f(a,c)\\ &\quad +J^{\alpha,\beta}_{a^+,(c+\eta_2(d,c))^-}f(a+\eta_1(b,a),c)+J^{\alpha,\beta}_{(a+\eta_1(b,a))^-,c^+}f(a,c+\eta_2(d,c))\\ &\quad +J^{\alpha,\beta}_{a^+,c^+}f(a+\eta_1(b,a),c+\eta_2(d,c))\big)\Big|\\ &\le\frac{\eta_1(b,a)\eta_2(d,c)}{(\alpha p+1)^{\frac{1}{p}}(\beta p+1)^{\frac{1}{p}}(s+1)^{\frac{2}{q}}}\end{aligned}$$

$$\times\Big(\Big|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\Big|^q+\Big|\frac{\partial^2 f}{\partial t\partial\lambda}(a,d)\Big|^q+\Big|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)|^q+\Big|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\Big|^q\Big)^{\frac{1}{q}},$$

where A *is defined as in* (3).

Proof. From Lemma 1, properties of modulus, and Hölder inequality, we have

$$\begin{aligned}
&\Big|\frac{f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))}{4}\\
&\quad-A+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\eta_1(b,a))^\alpha(\eta_2(d,c))^\beta}\big(J^{\alpha,\beta}_{(a+\eta_1(b,a))^-,(c+\eta_2(d,c))^-}f(a,c)\\
&\quad+J^{\alpha,\beta}_{a^+,(c+\eta_2(d,c))^-}f(a+\eta_1(b,a),c)+J^{\alpha,\beta}_{(a+\eta_1(b,a))^-,c^+}f(a,c+\eta_2(d,c))\\
&\quad+J^{\alpha,\beta}_{a^+,c^+}f(a+\eta_1(b,a),c+\eta_2(d,c))\big)\Big|\\
&\le\frac{\eta_1(b,a)\eta_2(d,c)}{4}\Bigg(\Big(\int_0^1\int_0^1 t^{\alpha p}\lambda^{\beta p}\,d\lambda dt\Big)^{\frac{1}{p}}+\Big(\int_0^1\int_0^1 t^{\alpha p}(1-\lambda)^{\beta p}\,d\lambda dt\Big)^{\frac{1}{p}}\\
&\quad+\Big(\int_0^1\int_0^1(1-t)^{p\alpha}\lambda^{p\beta}\,d\lambda dt\Big)^{\frac{1}{p}}+\Big(\int_0^1\int_0^1(1-t)^{p\alpha}(1-\lambda)^{p\beta}\,d\lambda dt\Big)^{\frac{1}{p}}\Bigg)\\
&\quad\times\Big(\int_0^1\int_0^1\Big|\frac{\partial^2 f}{\partial t\partial\lambda}(a+t\eta_1(b,a),c+\lambda\eta_2(d,c))\Big|^q\,d\lambda dt\Big)^{\frac{1}{q}}\\
&=\frac{\eta_1(b,a)\eta_2(d,c)}{(\alpha p+1)^{\frac{1}{p}}(\beta p+1)^{\frac{1}{p}}}\Big(\int_0^1\int_0^1\Big|\frac{\partial^2 f}{\partial t\partial\lambda}(a+t\eta_1(b,a),c+\lambda\eta_2(d,c))\Big|^q\,d\lambda dt\Big)^{\frac{1}{q}}.
\end{aligned}$$

Since $|\frac{\partial^2 f}{\partial t\partial\lambda}|^q$ is co-ordinated s-preinvex, we deduce

$$\begin{aligned}
&\Big|\frac{f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))}{4}\\
&\quad-A+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\eta_1(b,a))^\alpha(\eta_2(d,c))^\beta}\big(J^{\alpha,\beta}_{(a+\eta_1(b,a))^-,(c+\eta_2(d,c))^-}f(a,c)\\
&\quad+J^{\alpha,\beta}_{a^+,(c+\eta_2(d,c))^-}f(a+\eta_1(b,a),c)+J^{\alpha,\beta}_{(a+\eta_1(b,a))^-,c^+}f(a,c+\eta_2(d,c))\\
&\quad+J^{\alpha,\beta}_{a^+,c^+}f(a+\eta_1(b,a),c+\eta_2(d,c))\big)\Big|\\
&\le\frac{\eta_1(b,a)\eta_2(d,c)}{(\alpha p+1)^{\frac{1}{p}}(\beta p+1)^{\frac{1}{p}}}\Big(\Big|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\Big|^q\int_0^1\int_0^1(1-t)^s(1-\lambda)^s\,d\lambda dt\\
&\quad+\Big|\frac{\partial^2 f}{\partial t\partial\lambda}(a,d)\Big|^q\int_0^1\int_0^1(1-t)^s\lambda^s\,d\lambda dt+\Big|\frac{\partial^2 f}{\partial t\partial\lambda}(b,d)\Big|^q\int_0^1\int_0^1 t^s\lambda^s\,d\lambda dt\\
&\quad+\Big|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\Big|^q\int_0^1\int_0^1 t^s(1-\lambda)^s\,d\lambda dt\Big)^{\frac{1}{q}}\\
&=\frac{\eta_1(b,a)\eta_2(d,c)}{(\alpha p+1)^{\frac{1}{p}}(\beta p+1)^{\frac{1}{p}}}
\end{aligned}$$

$$\times\left(\frac{\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,d)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|^q}{(s+1)^2}\right)^{\frac{1}{q}},$$

which is the desired result.

Corollary 3
In Theorem 4 if we put $s=1$, we obtain the following fractional inequality

$$\begin{aligned}&\Big|\frac{f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))}{4}\\&\quad-A+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\eta_1(b,a))^\alpha(\eta_2(d,c))^\beta}\big(J^{\alpha,\beta}_{(a+\eta_1(b,a))^-,(c+\eta_2(d,c))^-}f(a,c)\\&\quad+J^{\alpha,\beta}_{a^+,(c+\eta_2(d,c))^-}f(a+\eta_1(b,a),c)+J^{\alpha,\beta}_{(a+\eta_1(b,a))^-,c^+}f(a,c+\eta_2(d,c))\\&\quad+J^{\alpha,\beta}_{a^+,c^+}f(a+\eta_1(b,a),c+\eta_2(d,c))\big)\Big|\\&\le\frac{\eta_1(b,a)\eta_2(d,c)}{4^{\frac{1}{q}}(\alpha p+1)^{\frac{1}{p}}(\beta p+1)^{\frac{1}{p}}}\\&\quad\times\left(\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,d)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|^q\right)^{\frac{1}{q}}.\end{aligned}$$

Corollary 4
In Theorem 4 if we choose $\eta_1(b,a)=\eta_2(b,a)=b-a$, we obtain the following fractional inequality

$$\begin{aligned}&\Big|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}-A+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^\alpha(d-c)^\beta}\\&\quad\times\big(J^{\alpha,\beta}_{b^-,d^-}f(a,c)+J^{\alpha,\beta}_{a^+,d^-}f(b,c)+J^{\alpha,\beta}_{b^-,c^+}f(a,d)+J^{\alpha,\beta}_{a^+,c^+}f(b,d)\big)\Big|\\&\le\frac{(b-a)(d-c)}{(s+1)^{\frac{2}{q}}(\alpha p+1)^{\frac{1}{p}}(\beta p+1)^{\frac{1}{p}}}\\&\quad\times\left(\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,d)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|^q\right)^{\frac{1}{q}}.\end{aligned}$$

Moreover if we take $s=1$, we get Theorem 6 from [16].

Theorem 5
Under the assumptions of Theorem 4, we have the following fractional inequality

$$\begin{aligned}&\Big|\frac{f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))}{4}\\&\quad A+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\eta_1(b,a))^\alpha(\eta_2(d,c))^\beta}\big(J^{\alpha,\beta}_{(a+\eta_1(b,a))^-,(c+\eta_2(d,c))^-}f(a,c)\\&\quad+J^{\alpha,\beta}_{a^+,(c+\eta_2(d,c))^-}f(a+\eta_1(b,a),c)+J^{\alpha,\beta}_{(a+\eta_1(b,a))^-,c^+}f(a,c+\eta_2(d,c))\end{aligned}$$

$$
\begin{aligned}
&+ J^{\alpha,\beta}_{a^+,c^+} f(a+\eta_1(b,a), c+\eta_2(d,c))\big)\Big| \qquad (4)\\
&\leq \frac{\eta_1(b,a)\eta_2(d,c)}{2^{\frac{1}{q}}(\alpha p+1)^{\frac{1}{p}}(1+s)^{\frac{1}{q}}}\Big(B(\beta q+1, s+1)+\frac{1}{\beta q+s+1}\Big)^{\frac{1}{q}}\\
&\times\Big(\Big|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\Big|^q+\Big|\frac{\partial^2 f}{\partial t\partial\lambda}(a,d)\Big|^q+\Big|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\Big|^q+\Big|\frac{\partial^2 f}{\partial t\partial\lambda}(b,d)\Big|^q\Big).
\end{aligned}
$$

Proof. From Lemma 1, properties of modulus, and Hölder inequality, we have

$$
\begin{aligned}
&\Big|\frac{f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))}{4}\\
&- A+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\eta_1(b,a))^{\alpha}(\eta_2(d,c))^{\beta}}\big(J^{\alpha,\beta}_{(a+\eta_1(b,a))^-,(c+\eta_2(d,c))^-} f(a,c)\\
&+ J^{\alpha,\beta}_{a^+,(c+\eta_2(d,c))^-} f(a+\eta_1(b,a),c)+J^{\alpha,\beta}_{(a+\eta_1(b,a))^-,c^+} f(a,c+\eta_2(d,c))\\
&+ J^{\alpha,\beta}_{a^+,c^+} f(a+\eta_1(b,a), c+\eta_2(d,c))\big)\Big|\\
&\leq \frac{\eta_1(b,a)\eta_2(d,c)}{4}\Big(\Big(\int_0^1\!\!\int_0^1 t^{\alpha p}\,d\lambda dt\Big)^{\frac{1}{p}}+\Big(\int_0^1\!\!\int_0^1 (1-t)^{\alpha p}\,d\lambda dt\Big)^{\frac{1}{p}}\Big) \qquad (5)\\
&\times\Big(\int_0^1\!\!\int_0^1 (\lambda^{\beta}+(1-\lambda)^{\beta})^q\Big|\frac{\partial^2 f}{\partial t\partial\lambda}(a+t\eta_1(b,a), c+\lambda\eta_2(d,c))\Big|^q d\lambda dt\Big)^{\frac{1}{q}}\\
&\leq \frac{\eta_1(b,a)\eta_2(d,c)}{2(\alpha p+1)^{\frac{1}{p}}}\\
&\times\Big(\int_0^1\!\!\int_0^1 (\lambda^{\beta}+(1-\lambda)^{\beta})^q\Big|\frac{\partial^2 f}{\partial t\partial\lambda}(a+t\eta_1(b,a), c+\lambda\eta_2(d,c))\Big|^q d\lambda dt\Big)^{\frac{1}{q}}.
\end{aligned}
$$

Using the following algebraic inequality $(a+b)^p \leq 2^{p-1}(a^p+b^p)$ for $a,b>0$ and $p\geq 1$, (4) becomes

$$
\begin{aligned}
&\Big|\frac{f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))}{4}\\
&- A+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\eta_1(b,a))^{\alpha}(\eta_2(d,c))^{\beta}}\big(J^{\alpha,\beta}_{(a+\eta_1(b,a))^-,(c+\eta_2(d,c))^-} f(a,c)\\
&+ J^{\alpha,\beta}_{a^+,(c+\eta_2(d,c))^-} f(a+\eta_1(b,a),c)+J^{\alpha,\beta}_{(a+\eta_1(b,a))^-,c^+} f(a,c+\eta_2(d,c))\\
&+ J^{\alpha,\beta}_{a^+,c^+} f(a+\eta_1(b,a), c+\eta_2(d,c))\big)\Big|\\
&\leq \frac{\eta_1(b,a)\eta_2(d,c)}{2^{\frac{1}{q}}(\alpha p+1)^{\frac{1}{p}}}\\
&\times\Big(\int_0^1\!\!\int_0^1 (\lambda^{\beta q}+(1-\lambda)^{\beta q})\Big|\frac{\partial^2 f}{\partial t\partial\lambda}(a+t\eta_1(b,a), c+\lambda\eta_2(d,c))\Big|^q d\lambda dt\Big)^{\frac{1}{q}}.
\end{aligned}
$$

Now, using the s-preinvexity on the co-ordinates of $\left|\frac{\partial^2 f}{\partial t\partial\lambda}\right|^q$, (5) gives

$$\begin{aligned}
&\Big|\frac{f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))}{4}\\
&\quad - A + \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\eta_1(b,a))^{\alpha}(\eta_2(d,c))^{\beta}}\big(J^{\alpha,\beta}_{(a+\eta_1(b,a))^-,(c+\eta_2(d,c))^-}f(a,c)\\
&\quad + J^{\alpha,\beta}_{a^+,(c+\eta_2(d,c))^-}f(a+\eta_1(b,a),c) + J^{\alpha,\beta}_{(a+\eta_1(b,a))^-,c^+}f(a,c+\eta_2(d,c))\\
&\quad + J^{\alpha,\beta}_{a^+,c^+}f(a+\eta_1(b,a),c+\eta_2(d,c))\big)\Big|\\
&\le \frac{\eta_1(b,a)\eta_2(d,c)}{2^{\frac{1}{q}}(\alpha p+1)^{\frac{1}{p}}}\\
&\quad\times\bigg(\Big|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\Big|^q\int_0^1\int_0^1(\lambda^{\beta q}(1-\lambda)^s+(1-\lambda)^{\beta q+s})(1-t)^s\,d\lambda dt\\
&\quad + \Big|\frac{\partial^2 f}{\partial t\partial\lambda}(a,d)\Big|^q\int_0^1\int_0^1(\lambda^{\beta q}(1-\lambda)^s+(1-\lambda)^{\beta q+s})t^s\,d\lambda dt\\
&\quad + \Big|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\Big|^q\int_0^1\int_0^1(\lambda^{\beta q+s}+\lambda^s(1-\lambda)^{\beta q})(1-t)^s\,d\lambda dt\\
&\quad + \Big|\frac{\partial^2 f}{\partial t\partial\lambda}(b,d)|^q\int_0^1\int_0^1(\lambda^{\beta q+s}+\lambda^s(1-\lambda)^{\beta q})\lambda^s t^s\,d\lambda dt\bigg)\\
&= \frac{\eta_1(b,a)\eta_2(d,c)}{2^{\frac{1}{q}}(\alpha p+1)^{\frac{1}{p}}(1+s)^{\frac{1}{q}}}\Big(B(\beta q+1,s+1)+\frac{1}{\beta q+s+1}\Big)^{\frac{1}{q}}\\
&\quad\times\Big(\Big|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\Big|^q+\Big|\frac{\partial^2 f}{\partial t\partial\lambda}(a,d)\Big|^q+\Big|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\Big|^q+\Big|\frac{\partial^2 f}{\partial t\partial\lambda}(b,d)\Big|^q\Big).
\end{aligned}$$

The proof is completed.

Corollary 5
In Theorem 5 if we put $s=1$, we obtain the following fractional inequality

$$\begin{aligned}
&\Big|\frac{f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))}{4}\\
&\quad - A + \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\eta_1(b,a))^{\alpha}(\eta_2(d,c))^{\beta}}\big(J^{\alpha,\beta}_{(a+\eta_1(b,a))^-,(c+\eta_2(d,c))^-}f(a,c)\\
&\quad + J^{\alpha,\beta}_{a^+,(c+\eta_2(d,c))^-}f(a+\eta_1(b,a),c) + J^{\alpha,\beta}_{(a+\eta_1(b,a))^-,c^+}f(a,c+\eta_2(d,c))\\
&\quad + J^{\alpha,\beta}_{a^+,c^+}f(a+\eta_1(b,a),c+\eta_2(d,c))\big)\Big|\\
&\le \frac{\eta_1(b,a)\eta_2(d,c)}{2^{\frac{2}{q}}(\alpha p+1)^{\frac{1}{p}}(\beta q+1)^{\frac{1}{q}}}\\
&\quad\times\Big(\Big|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\Big|^q+\Big|\frac{\partial^2 f}{\partial t\partial\lambda}(a,d)\Big|^q+\Big|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\Big|^q+\Big|\frac{\partial^2 f}{\partial t\partial\lambda}(b,d)\Big|^q\Big).
\end{aligned}$$

Corollary 6

In Theorem 5 if we choose $\eta_1(b,a) = \eta_2(b,a) = b - a$, we obtain the following fractional inequality

$$\begin{aligned}&\Big|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} - A + \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^{\alpha}(d-c)^{\beta}}\\&\quad\times\big(J^{\alpha,\beta}_{b^-,d^-}f(a,c)+J^{\alpha,\beta}_{a^+,d^-}f(b,c)+J^{\alpha,\beta}_{b^-,c^+}f(a,d)+J^{\alpha,\beta}_{a^+,c^+}f(b,d)\big)\Big|\\&\quad\le \frac{(b-a)(d-c)}{2^{\frac{1}{q}}(\alpha p+1)^{\frac{1}{p}}(1+s)^{\frac{1}{q}}}\Big(B(\beta q+1,s+1)+\frac{1}{\beta q+s+1}\Big)^{\frac{1}{q}}\\&\quad\times\Big(\Big|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\Big|^q+\Big|\frac{\partial^2 f}{\partial t\partial\lambda}(a,d)\Big|^q+\Big|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\Big|^q+\Big|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\Big|^q\Big)^{\frac{1}{q}}.\end{aligned}$$

Moreover if we take $s = 1$, we get Theorem 6 from [16].

Theorem 6

Under the assumptions of Theorem 4, we have the following fractional inequality

$$\begin{aligned}&\Big|\frac{f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))}{4}\\&\quad - A + \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\eta_1(b,a))^{\alpha}(\eta_2(d,c))^{\beta}}\big(J^{\alpha,\beta}_{(a+\eta_1(b,a))^-,(c+\eta_2(d,c))^-}f(a,c)\\&\quad + J^{\alpha,\beta}_{a^+,(c+\eta_2(d,c))^-}f(a+\eta_1(b,a),c)+J^{\alpha,\beta}_{(a+\eta_1(b,a))^-,c^+}f(a,c+\eta_2(d,c))\\&\quad + J^{\alpha,\beta}_{a^+,c^+}f(a+\eta_1(b,a),c+\eta_2(d,c))\big)\Big|\\&\le \frac{2^{\frac{2}{p}}\eta_1(b,a)\eta_2(d,c)}{(\alpha p+1)^{\frac{1}{p}}(\beta p+1)^{\frac{1}{p}}(s+1)^{\frac{2}{q}}}\Big(1-\Big(\frac{1}{2}\Big)^{\alpha p+1}\Big)^{\frac{1}{p}}\Big(1-\Big(\frac{1}{2}\Big)^{\beta p+1}\Big)^{\frac{1}{p}}\\&\quad\times\Big(\Big|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\Big|^q+\Big|\frac{\partial^2 f}{\partial t\partial\lambda}(a,d)\Big|^q+\Big|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\Big|^q+\Big|\frac{\partial^2 f}{\partial t\partial\lambda}(b,d)\Big|^q\Big).\end{aligned}$$

Proof. From Lemma 1, properties of modulus, and Hölder inequality, we have

$$\begin{aligned}&\Big|\frac{f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))}{4}\\&\quad - A + \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\eta_1(b,a))^{\alpha}(\eta_2(d,c))^{\beta}}\big(J^{\alpha,\beta}_{(a+\eta_1(b,a))^-,(c+\eta_2(d,c))^-}f(a,c)\\&\quad + J^{\alpha,\beta}_{a^+,(c+\eta_2(d,c))^-}f(a+\eta_1(b,a),c)+J^{\alpha,\beta}_{(a+\eta_1(b,a))^-,c^+}f(a,c+\eta_2(d,c))\\&\quad + J^{\alpha,\beta}_{a^+,c^+}f(a+\eta_1(b,a),c+\eta_2(d,c))\big)\Big|\\&\le \frac{\eta_1(b,a)\eta_2(d,c)}{4}\Big(\int_0^1\int_0^1\Big|\frac{\partial^2 f}{\partial t\partial\lambda}(a+t\eta_1(b,a),c+\lambda\eta_2(d,c))\Big|^q d\lambda dt\Big)^{\frac{1}{q}}\\&\quad\times\Big(\int_0^1(t^{\alpha}+(1-t)^{\alpha})^p\,dt\Big)^{\frac{1}{p}}\Big(\int_0^1(\lambda^{\beta}+(1-\lambda)^{\beta})^p\,d\lambda\Big)^{\frac{1}{p}}\end{aligned}$$

$$= \frac{\eta_1(b,a)\eta_2(d,c)}{4}\Big(\int_0^1\int_0^1\Big|\frac{\partial^2 f}{\partial t\partial\lambda}(a+t\eta_1(b,a),c+\lambda\eta_2(d,c))\Big|^q d\lambda dt\Big)^{\frac{1}{q}}$$
$$\times\Big(\int_0^{\frac{1}{2}}(t^\alpha+(1-t)^\alpha)^p\,dt+\int_{\frac{1}{2}}^1(t^\alpha+(1-t)^\alpha)^p\,dt\Big)^{\frac{1}{p}}$$
$$\times\Big(\int_0^{\frac{1}{2}}(\lambda^\beta+(1-\lambda)^\beta)^p\,d\lambda+\int_{\frac{1}{2}}^1(\lambda^\beta+(1-\lambda)^\beta)^p\,d\lambda\Big)^{\frac{1}{p}}$$
$$\le \frac{\eta_1(b,a)\eta_2(d,c)}{4}\Big(\int_0^1\int_0^1\Big|\frac{\partial^2 f}{\partial t\partial\lambda}(a+t\eta_1(b,a),c+\lambda\eta_2(d,c))\Big|^q d\lambda dt\Big)^{\frac{1}{q}}$$
$$\times 4\Big(\int_0^{\frac{1}{2}}(1-t)^{\alpha p}\,dt+\int_{\frac{1}{2}}^1 t^{\alpha p}\,dt\Big)^{\frac{1}{p}}\Big(\int_0^{\frac{1}{2}}(1-\lambda)^{\beta p}\,d\lambda+\int_{\frac{1}{2}}^1\lambda^{\beta p}\,d\lambda\Big)^{\frac{1}{p}}$$
$$\le \frac{2^{\frac{2}{p}}\eta_1(b,a)\eta_2(d,c)}{(\alpha p+1)^{\frac{1}{p}}(\beta p+1)^{\frac{1}{p}}}\Big(1-\Big(\frac{1}{2}\Big)^{\alpha p+1}\Big)^{\frac{1}{p}}\Big(1-\Big(\frac{1}{2}\Big)^{\beta p+1}\Big)^{\frac{1}{p}}$$
$$\times\Big(\int_0^1\int_0^1\Big|\frac{\partial^2 f}{\partial t\partial\lambda}(a+t\eta_1(b,a),c+\lambda\eta_2(d,c))\Big|^q d\lambda dt\Big)^{\frac{1}{q}}.$$

Using the s-preinvexity on the co-ordinates of $\big|\frac{\partial^2 f}{\partial t\partial\lambda}\big|^q$, (5) gives

$$\Big|\frac{f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))}{4}$$
$$-A+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\eta_1(b,a))^\alpha(\eta_2(d,c))^\beta}\big(J^{\alpha,\beta}_{(a+\eta_1(b,a))^-,(c+\eta_2(d,c))^-}f(a,c)$$
$$+J^{\alpha,\beta}_{a^+,(c+\eta_2(d,c))^-}f(a+\eta_1(b,a),c)+J^{\alpha,\beta}_{(a+\eta_1(b,a))^-,c^+}f(a,c+\eta_2(d,c))$$
$$+J^{\alpha,\beta}_{a^+,c^+}f(a+\eta_1(b,a),c+\eta_2(d,c))\big)\Big|$$
$$\le \frac{2^{\frac{2}{p}}\eta_1(b,a)\eta_2(d,c)}{(\alpha p+1)^{\frac{1}{p}}(\beta p+1)^{\frac{1}{p}}}\Big(1-\Big(\frac{1}{2}\Big)^{\alpha p+1}\Big)^{\frac{1}{p}}\Big(1-\Big(\frac{1}{2}\Big)^{\beta p+1}\Big)^{\frac{1}{p}}$$
$$\times\Big(\Big|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\Big|^q\int_0^1\int_0^1(1-t)^s(1-\lambda)^s\,d\lambda dt$$
$$+\Big|\frac{\partial^2 f}{\partial t\partial\lambda}(a,d)\Big|^q\int_0^1\int_0^1(1-t)^s\lambda^s\,d\lambda dt+\Big|\frac{\partial^2 f}{\partial t\partial\lambda}(b,d)\Big|^q\int_0^1\int_0^1 t^s\lambda^s\,d\lambda dt$$
$$+\Big|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\Big|^q\int_0^1\int_0^1 t^s(1-\lambda)^s\,d\lambda dt\Big)^{\frac{1}{q}}$$
$$=\frac{2^{\frac{2}{p}}\eta_1(b,a)\eta_2(d,c)}{(\alpha p+1)^{\frac{1}{p}}(\beta p+1)^{\frac{1}{p}}(s+1)^{\frac{2}{q}}}\Big(1-\Big(\frac{1}{2}\Big)^{\alpha p+1}\Big)^{\frac{1}{p}}\Big(1-\Big(\frac{1}{2}\Big)^{\beta p+1}\Big)^{\frac{1}{p}}$$
$$\times\Big(\Big|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\Big|^q+\Big|\frac{\partial^2 f}{\partial t\partial\lambda}(a,d)\Big|^q+\Big|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\Big|^q+\Big|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\Big|^q\Big)^{\frac{1}{q}}.$$

The proof is completed.

Corollary 7
In Theorem 6 if we put $s=1$, we obtain the following fractional inequality

$$\begin{aligned}&\Big|\frac{f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))}{4}\\&\quad-A+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(\eta_1(b,a))^{\alpha}(\eta_2(d,c))^{\beta}}\big(J^{\alpha,\beta}_{(a+\eta_1(b,a))^-,(c+\eta_2(d,c))^-}f(a,c)\\&\quad+J^{\alpha,\beta}_{a^+,(c+\eta_2(d,c))^-}f(a+\eta_1(b,a),c)+J^{\alpha,\beta}_{(a+\eta_1(b,a))^-,c^+}f(a,c+\eta_2(d,c))\\&\quad+J^{\alpha,\beta}_{a^+,c^+}f(a+\eta_1(b,a),c+\eta_2(d,c))\big)\Big|\\&\leq\frac{\eta_1(b,a)\eta_2(d,c)}{2^{\frac{2}{q}-\frac{2}{p}}(\alpha p+1)^{\frac{1}{p}}(\beta p+1)^{\frac{1}{p}}}\Big(1-\Big(\frac{1}{2}\Big)^{\alpha p+1}\Big)^{\frac{1}{p}}\Big(1-\Big(\frac{1}{2}\Big)^{\beta p+1}\Big)^{\frac{1}{p}}\\&\quad\times\Big(\Big|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\Big|^q+\Big|\frac{\partial^2 f}{\partial t\partial\lambda}(a,d)\Big|^q+\Big|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\Big|^q+\Big|\frac{\partial^2 f}{\partial t\partial\lambda}(b,d)\Big|^q\Big).\end{aligned}$$

Corollary 8
In Theorem 6 if we choose $\eta_1(b,a)=\eta_2(b,a)=b-a$, we obtain the following fractional inequality

$$\begin{aligned}&\Big|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}-A+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^{\alpha}(d-c)^{\beta}}\\&\quad\times\big(J^{\alpha,\beta}_{b^-,d^-}f(a,c)+J^{\alpha,\beta}_{a^+,d^-}f(b,c)+J^{\alpha,\beta}_{b^-,c^+}f(a,d)+J^{\alpha,\beta}_{a^+,c^+}f(b,d)\big)\Big|\\&\leq\frac{2^{\frac{2}{p}}(b-a)(d-c)}{(\alpha p+1)^{\frac{1}{p}}(\beta p+1)^{\frac{1}{p}}(s+1)^{\frac{2}{q}}}\Big(1-\Big(\frac{1}{2}\Big)^{\alpha p+1}\Big)^{\frac{1}{p}}\Big(1-\Big(\frac{1}{2}\Big)^{\beta p+1}\Big)^{\frac{1}{p}}\\&\quad\times\Big(\Big|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\Big|^q+\Big|\frac{\partial^2 f}{\partial t\partial\lambda}(a,d)\Big|^q+\Big|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\Big|^q+\Big|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\Big|^q\Big)^{\frac{1}{q}}.\end{aligned}$$

Moreover if we take $s=1$, we get

$$\begin{aligned}&\Big|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}-A+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^{\alpha}(d-c)^{\beta}}\\&\quad\times\big(J^{\alpha,\beta}_{b^-,d^-}f(a,c)+J^{\alpha,\beta}_{a^+,d^-}f(b,c)+J^{\alpha,\beta}_{b^-,c^+}f(a,d)+J^{\alpha,\beta}_{a^+,c^+}f(b,d)\big)\Big|\\&\leq\frac{(b-a)(d-c)}{(\alpha p+1)^{\frac{1}{p}}(\beta p+1)^{\frac{1}{p}}2^{\frac{2}{q}-\frac{2}{p}}}\Big(1-\Big(\frac{1}{2}\Big)^{\alpha p+1}\Big)^{\frac{1}{p}}\Big(1-\Big(\frac{1}{2}\Big)^{\beta p+1}\Big)^{\frac{1}{p}}\\&\quad\times\Big(\Big|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\Big|^q+\Big|\frac{\partial^2 f}{\partial t\partial\lambda}(a,d)\Big|^q+\Big|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\Big|^q+\Big|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\Big|^q\Big)^{\frac{1}{q}}.\end{aligned}$$

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