Annales Universitatis Paedagogicae Cracoviensis
Studia Mathematica XVIII (2019)

Ahmet Daşdemir

Gelin-Cesáro identities for Fibonacci and Lucas quaternions

Abstract. To date, many identities of different quaternions, including the Fibonacci and Lucas quaternions, have been investigated. In this study, we present Gelin-Cesáro identities for Fibonacci and Lucas quaternions. The identities are a worthy addition to the literature. Moreover, we give Catalan's identity for the Lucas quaternions.

1. Introduction

Sir W. R. Hamilton introduced quaternions as an expansion of complex numbers into higher spatial dimensions. The set of real quaternions is denoted by $\mathbb{H}$ in honour of its discoverer and is defined as

$$\mathbb{H} = \{q = q_0 + \tilde{q} : \ \tilde{q} = q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} \ \text{ and } \ q_0, q_1, q_2, q_3 \in \mathbb{R}\},$$

where $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are basis vectors with the multiplication rule

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1. \tag{1}$$

Note that q_0 is called the scalar part of q, whereas $\tilde{q}$ is its vector part.

Quaternions have been extensively investigated, because they have very important features. Horadam [1] presented one of the most interesting investigations and defined the Fibonacci and Lucas quaternions, respectively, by

$$Q_n := F_n + F_{n+1}\mathbf{i} + F_{n+2}\mathbf{j} + F_{n+3}\mathbf{k} \tag{2}$$

and

$$K_n := L_n + L_{n+1}\mathbf{i} + L_{n+2}\mathbf{j} + L_{n+3}\mathbf{k}. \tag{3}$$

Here, F_n and L_n are the n-th Fibonacci and Lucas numbers, respectively, in the following forms

$$F_0 = 0, \quad F_1 = 1 \quad \text{and} \quad F_{n+1} = F_n + F_{n-1} \qquad \text{for } n > 0 \tag{4}$$

and

$$L_0 = 2, \quad L_1 = 1 \quad \text{and} \quad L_{n+1} = L_n + L_{n-1} \qquad \text{for } n > 0. \tag{5}$$

Note that equations (2) and (3) satisfy the recurrence relations

$$Q_{n+1} = Q_n + Q_{n-1} \qquad \text{for } n > 0$$

and

$$K_{n+1} = K_n + K_{n-1} \qquad \text{for } n > 0.$$

In addition, Binet's formulae of the Fibonacci and Lucas quaternions are respectively [5],

$$Q_n = \frac{\tilde{\alpha}\alpha^n - \tilde{\beta}\beta^n}{\alpha - \beta}$$

and

$$K_n = \tilde{\alpha}\alpha^n + \tilde{\beta}\beta^n,$$

where α is the golden ratio, $\beta = -\alpha^{-1}$, $\tilde{\alpha} = 1 + \alpha\mathbf{i} + \alpha^2\mathbf{j} + \alpha^3\mathbf{k}$ and $\tilde{\beta} = 1 + \beta\mathbf{i} + \beta^2\mathbf{j} + \beta^3\mathbf{k}$. In [7], Iyer presented some remarkable results regarding the Fibonacci quaternions.

Inspired by the definition given by Horadam [1], many important generalizations of the Fibonacci quaternions have been defined by employing different generalizations of the usual Fibonacci numbers. For example, Ramírez [3] gave a new generalization of (2) associated with the k-Fibonacci numbers introduced by Falcón and Plaza [2] as follows

$$D_{k,n} = F_{k,n} + F_{k,n+1}\mathbf{i} + F_{k,n+2}\mathbf{j} + F_{k,n+3}\mathbf{k},$$

where $F_{k,n}$ is n-th term of the k-th Fibonacci sequence defined by

$$F_{k,0} = 0, \quad F_{k,1} = 1 \quad \text{and} \quad F_{k,n+1} = kF_{k,n} + F_{k,n-1} \qquad \text{for } n > 0.$$

Ramírez also anticipated a formula related to Catalan's identity for the k-Fibonacci quaternions. However, Polatlı and Kesim [4] showed that Ramírez's conjecture was incorrect, and then the authors proved the following Catalan's identity

$$D_{k,n-r}D_{k,n+r} - D_{k,n}{}^2 = (-1)^{n-r+1}(2F_{k,r}D_{k,r} - L_{k,2}F_{k,2r}\mathbf{k}), \tag{6}$$

where $L_{k,2}$ is the second term of the k-th Lucas sequence, which was defined by Falcón [8] as follows

$$L_{k,0} = 2, \quad L_{k,1} = k \quad \text{and} \quad L_{k,n+1} = kL_{k,n} + L_{k,n-1} \qquad \text{for } n > 0.$$

For more detailed information on the generalizations of the Fibonacci quaternions, the references in [10, 11] can be seen.

Today, there are many multiplicative identities for the usual Fibonacci numbers. Two of their most famous are Catalan's and Gelin-Cesáro identities. Within the scope of the theory of quaternions, we update the notation introduced by Fairgrieve and Gould [9] to the Fibonacci quaternions in the form of

$$P_n = P_n(a_i, b_i, r) = \prod_{i=1}^{r} Q_{n+a_i} - \prod_{i=1}^{r} Q_{n+b_i},$$

where $r \geqslant 1$, a_i and b_i are any integers. It is called Product Difference Fibonacci Quaternion Identity (PDFQI) of order r. Depending on the choice of r, a_i, b_i and n, PDFQI reduces to some multiplicative formulae investigated by Iyer [6]. Note that it can be very difficult to calculate multiplicative identities for the Fibonacci quaternions because the set of quaternions is non-commutative.

Based on the current literature, it is clear that the statements, that is Catalan's and Gelin-Cesáro identities, involving the difference of second- and fourth-order products for the Fibonacci and Lucas quaternions have yet to be studied. To fill this gap, in this paper, we present two important properties for the Fibonacci and Lucas quaternions, which are the Gelin-Cesáro identities, followed by Catalan's identity for the Lucas quaternions.

2. Main Results

Here, we present the results of our investigation.

Lemma 1
Let p and q be any quaternions. Then, we obtain

$$pq = qp + 2\tilde{p} \times \tilde{q} \tag{7}$$

and

$$p^2 = 2p_0 p - [N(p)]^2, \tag{8}$$

where $N(p)$ is the norm of p, and the symbol "$\times$" denotes the cross product over the set of $\mathbb{H}$.

Proof. The proof is immediate by employing the multiplication rule in (1).

Lemma 2
Let n and r be any integers. Then, we have

$$L_{2n+r} = 5F_n F_{n+r} + (-1)^n L_r.$$

Proof. By Binet's formulae of the Fibonacci and Lucas numbers, we can write

$$F_n F_{n+r} = \frac{\alpha^n - \beta^n}{\alpha - \beta} \frac{\alpha^{n+r} - \beta^{n+r}}{\alpha - \beta} = \frac{\alpha^n \alpha^{n+r} - \alpha^n \beta^{n+r} - \beta^n \alpha^{n+r} + \beta^n \beta^{n+r}}{5}$$
$$= \frac{\alpha^{2n+r} + \beta^{2n+r} - \alpha^n \beta^n (\alpha^r + \beta^r)}{5}$$
$$= \frac{L_{2n+r} - (-1)^n L_r}{5}.$$

Thus, we complete the proof.

We present the fundamental results of this study below.

THEOREM 3
Let Q_n be any Fibonacci quaternion. For any positive integer n, we have

$$Q_{n-2}Q_{n-1}Q_{n+1}Q_{n+2} - Q_n{}^4 = 1 + 4[(-1)^{n-1}F_{n-1}(F_{n+3}\lambda_1 - 3F_{n+4}\lambda_2) + \lambda_3],$$

where

$$\lambda_1 = \boldsymbol{i} - 5\boldsymbol{j} - 4\boldsymbol{k}, \quad \lambda_2 = \boldsymbol{i} + 2\boldsymbol{j} - 3\boldsymbol{k}, \quad \lambda_3 = 3 - 14\boldsymbol{i} - 19\boldsymbol{j} + 12\boldsymbol{k}. \tag{9}$$

Proof. Substituting $k = 1$ into (6), we obtain Catalan's identity for the usual Fibonacci quaternions as follows

$$Q_{n-r}Q_{n+r} - Q_n{}^2 = (-1)^{n-r+1}(2F_r Q_r - 3F_{2r}\mathbf{k}).$$

Hence, for $r = 1$ and $r = 2$, respectively, we obtain (see Halici [5]),

$$Q_{n-1}Q_{n+1} = Q_n{}^2 + (-1)^n(2Q_1 - 3\mathbf{k}) \tag{10}$$

and

$$Q_{n-2}Q_{n+2} = Q_n{}^2 + (-1)^{n-1}(2Q_2 - 9\mathbf{k}).$$

Therefore,

$$\begin{aligned}Q_{n-1}&Q_{n+1}Q_{n-2}Q_{n+2} \\ &= [Q_n{}^2 + (-1)^n(2Q_1 - 3\mathbf{k})][Q_n{}^2 + (-1)^{n-1}(2Q_2 - 9\mathbf{k})].\end{aligned} \tag{11}$$

Now we consider both sides of equation (11) separately. Using (7) we obtain

$$Q_{n-1}Q_{n+1}Q_{n-2}Q_{n+2} = Q_{n-2}Q_{n-1}Q_{n+1}Q_{n+2} + 2(\tilde{Q}_* \times \tilde{Q}_{n-2})Q_{n+2},$$

where $Q_* = Q_{n-1}Q_{n+1}$. With the aid of (8) and (10) we get

$$Q_* = 2F_n Q_n - N(Q_n)^2 + (-1)^n(2Q_1 - 3\mathbf{k}).$$

Hence, we obtain

$$\begin{aligned}Q_{n-1}&Q_{n+1}Q_{n-2}Q_{n+2} \\ &= Q_{n-2}Q_{n-1}Q_{n+1}Q_{n+2} \\ &\quad + 2(2F_n\tilde{Q}_n \times \tilde{Q}_{n-2} + (-1)^n(2\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}) \times \tilde{Q}_{n-2})Q_{n+2}.\end{aligned} \tag{12}$$

Considering the definition of the cross product, we can write

$$\tilde{Q}_n \times \tilde{Q}_{n-2} = (-1)^n(\mathbf{i} + \mathbf{j} - \mathbf{k})$$

and

$$(2\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}) \times \tilde{Q}_{n-2} = (F_n + 4F_{n-1})\mathbf{i} - (2F_n - F_{n-1})\mathbf{j} + (2F_n - 4F_{n-1})\mathbf{k}.$$

Substituting the last statements into (12) and making some mathematical arrangements, we obtain

$$
\begin{aligned}
Q_{n-1}Q_{n+1}Q_{n-2}Q_{n+2} \\
= Q_{n-2}Q_{n-1}Q_{n+1}Q_{n+2} \\
+ 2(-1)^n((3F_{n+1} + F_{n-1})\mathbf{i} + F_{n-1}\mathbf{j} - 4F_{n-1}\mathbf{k})Q_{n+2}.
\end{aligned}
\tag{13}
$$

Now we consider the right-hand side of (11). We get

$$
\begin{aligned}
[Q_n{}^2 + (-1)^n(2Q_1 - 3\mathbf{k})][Q_n{}^2 + (-1)^{n-1}(2Q_2 - 9\mathbf{k})] \\
= Q_n{}^4 + (-1)^n(2Q_1 - 3\mathbf{k})Q_n{}^2 + Q_n{}^2(-1)^{n-1}(2Q_2 - 9\mathbf{k}) \\
+ (-1)^n(2Q_1 - 3\mathbf{k})(-1)^{n-1}(2Q_2 - 9\mathbf{k}) \\
= Q_n{}^4 + (-1)^n[(2Q_1 - 3\mathbf{k})Q_n{}^2 - Q_n{}^2(2Q_2 - 9\mathbf{k})] \\
- (2Q_1 - 3\mathbf{k})(2Q_2 - 9\mathbf{k}).
\end{aligned}
\tag{14}
$$

From (13) and (14), we can write

$$
\begin{aligned}
Q_{n-2}Q_{n-1}Q_{n+1}Q_{n+2} - Q_n{}^4 \\
= (-1)^n[-2((3F_{n+1} + F_{n-1})\mathbf{i} + F_{n-1}\mathbf{j} - 4F_{n-1}\mathbf{k})Q_{n+2} \\
+ (2Q_1 - 3\mathbf{k})Q_n{}^2 - Q_n{}^2(2Q_2 - 9\mathbf{k})] + 31 + 2\mathbf{i} - 30\mathbf{j} - 4\mathbf{k}.
\end{aligned}
$$

We let Δ denote the term in brackets. After the use of (1) and very extensive mathematical operations, we obtain

$$
\begin{aligned}
\Delta = -2[9(F_{n-1}F_{n+1} - F_n{}^2) + (15F_{n-1}{}^2 + 5F_{n-1}F_n - 29F_n{}^2)\mathbf{i} \\
- (33F_{n-1}{}^2 + 67F_{n-1}F_n + 23F_n{}^2)\mathbf{j} \\
+ 2(6F_{n-1}{}^2 + 20F_{n-1}F_n + 13F_n{}^2)\mathbf{k}].
\end{aligned}
\tag{15}
$$

Recall that Cassini's identity for the usual Fibonacci number is

$$
F_{n-1}F_{n+1} - F_n{}^2 = (-1)^n.
\tag{16}
$$

Considering equations (4) and (16), we can rearrange (15) in the following form

$$
\begin{aligned}
\Delta = -2F_{n-1}[(15F_{n-1} + 5F_n - 29F_{n+1})\mathbf{i} - (33F_{n-1} + 67F_n + 23F_{n+1})\mathbf{j} \\
+ 2(6F_{n-1} + 20F_n + 13F_{n+1})\mathbf{k}] + 2(-1)^n(9 + 29\mathbf{i} + 23\mathbf{j} - 26\mathbf{k}).
\end{aligned}
$$

Applying the recurrence relation in (4) to the last equation, we complete the proof.

The next theorem presents Catalan's identity of Lucas quaternions.

Theorem 4
Let n and r be any integers. Then, we have

$$
K_{n-r}K_{n+r} - K_n{}^2 = 5(-1)^{n-r}F_r[2Q_r - 3L_r\boldsymbol{k}].
$$

Proof. By Binet's formula of the Lucas quaternions, we can write

$$K_{n-r}K_{n+r} - K_n{}^2 = (\tilde{\alpha}\alpha^{n-r} + \tilde{\beta}\beta^{n-r})(\tilde{\alpha}\alpha^{n+r} + \tilde{\beta}\beta^{n+r})$$
$$- (\tilde{\alpha}\alpha^n + \tilde{\beta}\beta^n)(\tilde{\alpha}\alpha^n + \tilde{\beta}\beta^n)$$
$$= \alpha^{n-r}\beta^{n-r}[\tilde{\alpha}\tilde{\beta}\beta^{2r} + \tilde{\beta}\tilde{\alpha}\alpha^{2r} - \alpha^r\beta^r(\tilde{\alpha}\tilde{\beta} + \tilde{\beta}\tilde{\alpha})].$$

In addition, the following equations can be proved

$$\tilde{\alpha}\tilde{\beta} = 2\tilde{\beta} + 3\sqrt{5}\mathbf{k} \quad \text{and} \quad \tilde{\beta}\tilde{\alpha} = 2\tilde{\alpha} - 3\sqrt{5}\mathbf{k}.$$

Hence, we can write

$$K_{n-r}K_{n+r} - K_n{}^2$$
$$= (-1)^{n-r}[(2\tilde{\beta} + 3\sqrt{5}\mathbf{k})\beta^{2r} + (2\tilde{\alpha} - 3\sqrt{5}\mathbf{k})\alpha^{2r}$$
$$- (-1)^r(2\tilde{\beta} + 3\sqrt{5}\mathbf{k} + 2\tilde{\alpha} - 3\sqrt{5}\mathbf{k})]$$
$$= (-1)^{n-r}[2(\tilde{\alpha}\alpha^{2r} + \tilde{\beta}\beta^{2r}) - 3\sqrt{5}(\alpha^{2r} - \beta^{2r})\mathbf{k} - 2(-1)^r(\tilde{\beta} + \tilde{\alpha})]$$
$$= (-1)^{n-r}[2K_{2r} - 15F_{2r}\mathbf{k} - 2(-1)^r K_0].$$

By Lemma 2,

$$K_{2r} = L_{2r} + L_{2r+1}\mathbf{i} + L_{2r+2}\mathbf{j} + L_{2r+3}\mathbf{k}$$
$$= 5F_r(F_r + F_{r+1}\mathbf{i} + F_{r+2}\mathbf{j} + F_{r+3}\mathbf{k}) + (-1)^r(L_0 + L_1\mathbf{i} + L_2\mathbf{j} + L_3\mathbf{k})$$
$$= 5F_rQ_r + (-1)^r K_0.$$

As a result, we get

$$K_{n-r}K_{n+r} - K_n{}^2 = (-1)^{n-r}[2K_{2r} - 15F_{2r}\mathbf{k} - 2(-1)^r K_0]$$
$$= (-1)^{n-r}[10F_rQ_r + 2(-1)^r K_0 - 15F_{2r}\mathbf{k} - 2(-1)^r K_0]$$
$$= 5(-1)^{n-r}[2F_rQ_r - 3F_rL_r\mathbf{k}]$$
$$= 5F_r(-1)^{n-r}[2Q_r - 3L_r\mathbf{k}],$$

and the result follows.

Note that for $r = 1$ in Theorem 4, we obtain the following result, which is Cassini's identity of the Lucas quaternions.

COROLLARY 5
Let n be any integer. Then we have

$$K_{n-1}K_{n+1} - K_n{}^2 = 5(-1)^{n-1}[2Q_1 - 3\mathbf{k}]. \tag{17}$$

Now we present the Gelin-Cesáro identity of the Lucas quaternions.

THEOREM 6
Let n be any positive integer. Then, the following equation is satisfied

$$K_{n-2}K_{n-1}K_{n+1}K_{n+2} - K_n{}^4 = 25 + 20[(-1)^n L_{n-1}(L_{n+3}\lambda_1 - 3L_{n+4}\lambda_2) + 5\lambda_3],$$

where λ_1, λ_2 and λ_3 were defined in (9).

Proof. To prove Theorem 6, we proceed like in Theorem 3. First, we substitute $r = 2$ in Theorem 4 to obtain

$$K_{n-2}K_{n+2} - K_n{}^2 = 5(-1)^n[2Q_2 - 9\mathbf{k}]. \tag{18}$$

Considering (17) and (18), we can write

$$
\begin{aligned}
K_{n-1}&K_{n+1}K_{n-2}K_{n+2} \\
&= [K_n{}^2 + 5(-1)^{n-1}[2Q_1 - 3\mathbf{k}]][K_n{}^2 + 5(-1)^n[2Q_2 - 9\mathbf{k}]].
\end{aligned} \tag{19}
$$

We will compute both sides of (19) separately and start with the left-hand side. Using (7), (8) and (17), we obtain

$$
\begin{aligned}
K_{n-1}K_{n+1}K_{n-2}K_{n+2} &= K_{n-2}K_{n-1}K_{n+1}K_{n+2} \\
&+ 2(2L_n\tilde{K}_n \times \tilde{K}_{n-2} + 5(-1)^{n-1}(2\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}) \times \tilde{K}_{n-2})K_{n+2}.
\end{aligned} \tag{20}
$$

Applying the definition of the cross product to (20), we have

$$\tilde{K}_n \times \tilde{K}_{n-2} = 5(-1)^{n-1}(\mathbf{i} + \mathbf{j} - \mathbf{k})$$

and

$$(2\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}) \times \tilde{K}_{n-2} = (L_n + 4L_{n-1})\mathbf{i} - (2L_n - L_{n-1})\mathbf{j} + (2L_n - 4L_{n-1})\mathbf{k}.$$

As a result, we obtain

$$
\begin{aligned}
K_{n-1}K_{n+1}K_{n-2}K_{n+2} &= K_{n-2}K_{n-1}K_{n+1}K_{n+2} \\
&+ 10(-1)^{n-1}((3L_{n+1} + L_{n-1})\mathbf{i} + L_{n-1}\mathbf{j} - 4L_{n-1}\mathbf{k})K_{n+2}.
\end{aligned} \tag{21}
$$

Expanding the right-hand side of (19) yields

$$
\begin{aligned}
[K_n{}^2 &+ 5(-1)^{n-1}[2Q_1 - 3\mathbf{k}]][K_n{}^2 + 5(-1)^n[2Q_2 - 9\mathbf{k}]] \\
&= K_n{}^4 + 5(-1)^n(K_n{}^2(2 + 4\mathbf{i} + 6\mathbf{j} + \mathbf{k}) - (2 + 2\mathbf{i} + 4\mathbf{j} + 3\mathbf{k})K_n{}^2) \\
&+ 25(31 + 2\mathbf{i} - 30\mathbf{j} - 4\mathbf{k}).
\end{aligned} \tag{22}
$$

If we combine (21) and (22) and use (1), we obtain

$$K_{n-2}K_{n-1}K_{n+1}K_{n+2} - K_n{}^4 = 5(-1)^n\nabla + 25(31 + 2\mathbf{i} - 30\mathbf{j} - 4\mathbf{k}),$$

where

$$
\begin{aligned}
\nabla &= 2[9(L_{n-1}L_{n+1} - L_n{}^2) + (15L_{n-1}{}^2 + 5L_{n-1}L_n - 29L_n{}^2)\mathbf{i} \\
&- (33L_{n-1}{}^2 + 67L_{n-1}L_n + 23L_n{}^2)\mathbf{j} + 2(6L_{n-1}{}^2 + 20L_{n-1}L_n + 13L_n{}^2)\mathbf{k}].
\end{aligned}
$$

Using the Cassini's identity of Lucas numbers given by

$$L_n{}^2 - L_{n-1}L_{n+1} = 5(-1)^n,$$

we can write

$$
\begin{aligned}
\nabla &= 2L_{n-1}[(15L_{n-1} + 5L_n - 29L_{n+1})\mathbf{i} - (33L_{n-1} + 67L_n + 23L_{n+1})\mathbf{j} \\
&+ 2(6L_{n-1} + 20L_n + 13L_{n+1})\mathbf{k}] + 10(-1)^n(9 + 29\mathbf{i} + 23\mathbf{j} - 26\mathbf{k}).
\end{aligned} \tag{23}
$$

Applying (5) into (23) with some algebraic arrangements, we complete the proof.

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Department of Mathematics
Kastamonu University
Turkey
E-mail: ahmetdasdemir37@gmail.com